Solutions I

1 We showed in class that the pendulum oscillates if $0 < El/g < 2$, with the maximum angle θ_* being given by $\cos\theta_* = 1 - El/g$ with $0 < \theta_* < \pi$. The period is the time it takes the pendulum to swing from 0 to the maximum value θ_*, back and around to $-\theta_*$ and finally to the origin again. By symmetry this is 4 times the time it takes to reach θ_* from the origin, during which time $\dot\theta \geq 0$. Hence

$$T = 4\int_0^{\theta_*} \frac{\mathrm{d}\theta}{[2E + 2(\cos\theta - 1)g/l]^{1/2}} = 4\sqrt{\frac{l}{g}}\int_0^{\theta_*} \frac{\mathrm{d}\theta}{[2(\cos\theta - \cos\theta_*)]^{1/2}}.$$

This shows that θ_* is a convenient non-dimensional parameter. Small amplitude means small θ_*; then in the integral $\theta \ll 1$, so that we can expand the cosines. This leads to the non-dimensional period

$$\sqrt{\frac{g}{l}}T \sim 4\int_0^{\theta_*} \frac{\mathrm{d}\theta}{\sqrt{\theta_*^2 - \theta^2}} = 4\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^2}} = 2\pi,$$

as expected, after making the change of variable $\theta = \theta_* u$. (The answer should be a number in this limit, since E and θ_* are related.) The general integral is simple to evaluate numerically if one pays attention to the inverse square-root singularity (Matlab's `integral` function does fine). Alternatively one can write

$$\sqrt{\frac{g}{l}}T = 4\int_0^{\theta_*} \frac{\mathrm{d}\theta}{[\sin^2(\theta_*/2) - \sin^2(\theta/2)]^{1/2}} = 4\int_0^{\pi/2} \frac{\mathrm{d}u}{\sqrt{1-k^2\sin^2 u}} = 4K(k),$$

making the substitution $\sin(\theta/2) = \sin u \sin(\theta_*/2)$, where $k = \sin(\theta_*/2)$ and $K(k)$ is the complete elliptical integral of the first kind with modulus k (or alternatively parameter $m = k^2$). Figure 1 shows the non-dimensional period as a function of θ_*.

2 The conditions on $f(x)$ correspond to a double zero of $f(x)$, so that the graph $y = f(x)$ is tangent to the y-axis. Vanishing higher derivatives correspond to higher zeros, with the curve being tangent if the highest vanishing derivative is odd. The function $f(x) = x^3 - x^2$ is a cubic, with a double zero at 0 and a simple zero at 1. We find $f'(x) = 3x^2 - 2x$, so that $f'(0) = 0$ and $f'(1) = 1$. Hence the fixed point 0 is semi-stable and the fixed point 1 is unstable, as can be seen from the graph. The local behavior near the origin is given by $\dot x = -x^2$, with solution

$$x(t) = \frac{1}{t + x_0^{-1}}.$$

where x_0 is the initial value of $x(t)$. For $x_0 < 0$, $x(t)$ blows up to $-\infty$ in finite time, while for $x_0 > 0$, $x(t)$ approaches the origin. This is consistent with a semi-stable fixed point

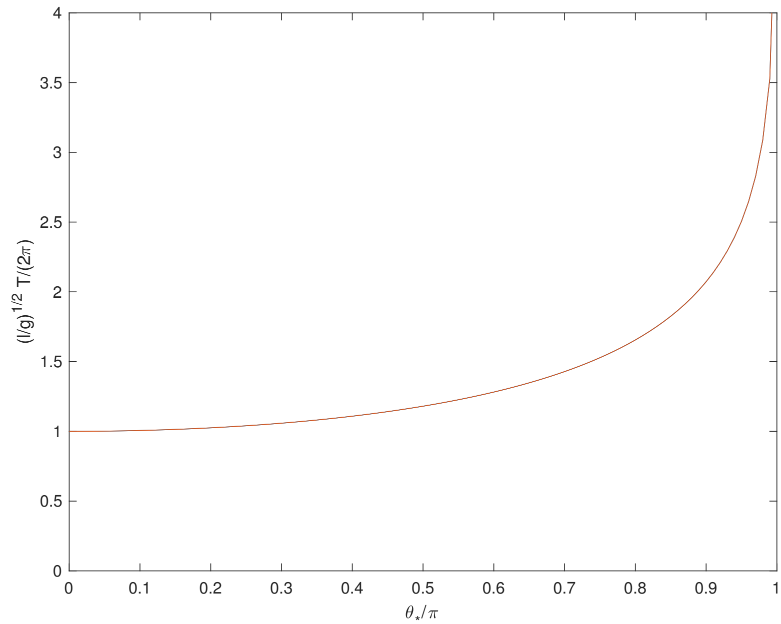

Figure 1: Period of pendulum.

with $\dot{x} < 0$, but is only the approximate behavior since the approximation breaks down as $|x(t)|$ becomes large. The exact solution can be obtained by separation of variables and partial fractions:

$$\frac{\mathrm{d}x}{x^3 - x^2} = \mathrm{d}x\left(\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}\right) = \mathrm{d}t.$$

This can be integrated to give

$$\log\frac{(x-1)x_0}{(x_0-1)x} + \frac{1}{x} - \frac{1}{x_0} = t.$$

If $f(x)$ becomes $x^3 - x^2 - \delta$, then the fixed point at 1 shifts a little, while the fixed point at 0 either vanishes if $\delta < 0$ or splits into two fixed points if $\delta > 0$ (the location of new fixed points can be obtained approximately using perturbation theory; see later in the course). Hence there is a quantitative change in the dynamics (or structure) of the semi-stable fixed point for a small change in δ. This is the origin of the general semi-stable fixed points are not structurally stable. (For higher odd zeros, the argument is not quite so clear.)

3 The condition $x_* = f(x_*)$ corresponds graphically to an intersection of the curve $y = f(x)$ with the straight line $y = x$. Writing $x_n = x_* + \xi_n$ and linearizing gives

$$\xi_{n+1} \simeq f'(x_n)\xi_n,$$

with solution $\xi_n = [f'(x_n)]^n \xi_0$. It's important to view ξ_n as a variable that evolves under a mapping. It grows in magnitude if $|f'(x_n)| > 1$ and decays to 0 if $|f'(x_n)| < 1$. Graphically this means that if the slope is less than 1 in magnitude, the fixed point is stable,

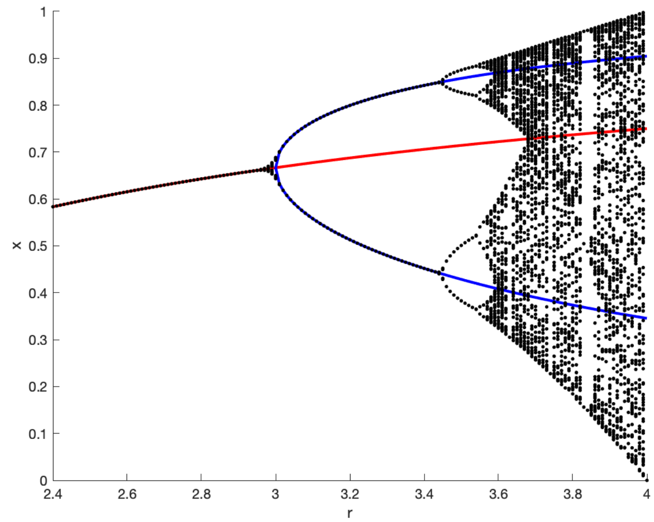

Figure 2: Dots: iterates $x_{101}, \ldots, x_{200}$ starting from $x_1 = 1/2$. Red curve: $1 - r^{-1}$ (fixed point of the map); blue curve: fixed points of the iterated map $f(f(x_n))$ for $r > 3$. Some transients are visible near $r = 3$.

while if the slope is greater than 1 in magnitude, the fixed point is unstable. There is a characteristic 'spiderweb' picture as the iteration converges or diverges. There was a typo in the logistic map, which I mean to write as

$$x_{n+1} = rx_n(1 - x_n),$$

but this didn't get updated in the final posted version. As a result the values for r in the bonus part of the question were problematic; none of you pointed this out. The logistic map takes $(0, 1)$ to itself if $0 \leq r \leq 4$, since the maximum of $x(1 - x)$ is $1/4$. The fixed points come from solving the quadratic $x = rx(1 - x)$, which has roots at 0 and $1 - r^{-1}$. The stability of these roots can be determined by looking at $f'(x) = r(1 - 2x)$, so that $f'(0) = r$ and $f'(1 - r^{-1}) = 2 - r$. The origin is hence unstable for $r > 1$. The larger fixed point exists for $r > 1$ and is stable for $|2 - r| < 1$, i.e. $1 < r < 3$. The second iterate has fixed points when $xn + 2 = x_n$, that is at $x = r^2x(1 - x)(1 - rx + rx^2)$. This quartic has roots at 0 and $1 - r^{-1}$, as before, and also at $1/2 + 1/(2r) \pm (1 - 2/r - 3/r^2)^{1/2}/2$. This has real roots for $r > 3$. Figure 2 shows the iterates for $100 < n \leq 200$ starting from $x_1 = 1/2$. The loss of stability of the different iterates is visible. More complicated are the apparently chaotic regions for $r > 3.56995$ or so. See the literature.

4 There are four fixed points: $(0, 0)$, $(2, 2)$, $(0, 1)$ and $(-1, 2)$. The matrix of derivatives is

$$Df = \begin{pmatrix} (1 - 2x)/3 & (-1 + 2y)/3 \\ 2 - y & -x \end{pmatrix}.$$

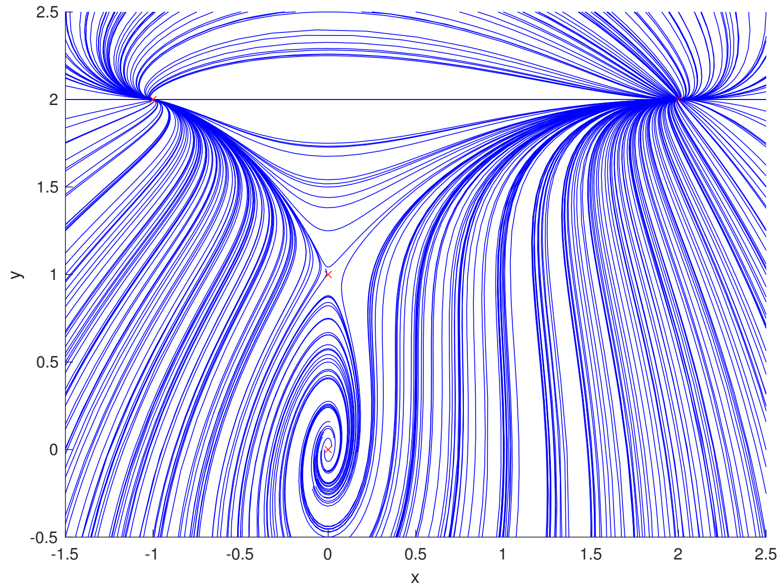

Figure 3: Phase plane for **4**.

At $(0,0)$ we have

$$A = \begin{pmatrix} 1/3 & -1/3 \\ 2 & 0 \end{pmatrix},$$

This has eigenvalues $(1 \pm i\sqrt{23})/6$, so an unstable focus going anticlockwise (check behavior on axes). At $(2,2)$ we have

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}.$$

This has eigenvalue -1 corresponding to eigenvector $(1,0)$ and eigenvector -2 corresponding to eigenvector $(-1,1)$, so a stable node tangential to $(1,0)$. At $(0,1)$ we have

$$A = \begin{pmatrix} 1/3 & 1/3 \\ 1 & 0 \end{pmatrix}.$$

This has positive eigenvalue $(1+\sqrt{13})/6$ corresponding to approximate eigenvector $(0.61, 0.79)$ and negative eigenvector $(1-\sqrt{13})/6$ corresponding to approximate eigenvector $(-0.40, 0.92)$, so a saddle. At $(-1,2)$ we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This has a repeated eigenvalue of 1 and is the canonical degenerate unstable node with eigenvector $(1,-1)$. Combining these four local behaviors gives the phase plane shown in Figure 3.