# Solutions II

**1** Transform partially to polar coordinates:

$$\dot{x} = \frac{x}{1+\cos^2\theta}(1-ar^2) - \frac{y}{1+\sin^2\theta},$$
$$\dot{y} = \frac{y}{1+\cos^2\theta}(1-ar^2) + \frac{x}{1+\sin^2\theta}.$$

Hence as $x$ and $y$ tend to zero, $\dot{x}$ and $\dot{y}$ vanish (the denominators are positive and bounded), so the origin is a fixed point. Now use

$$r\dot{r} = x\dot{x} + y\dot{y}, \qquad r^2\dot{\theta} = x\dot{y} - y\dot{x}$$

to obtain

$$\dot{r} = \frac{r(1-ar^2)}{1+\cos^2\theta}, \qquad \dot{\theta} = \frac{1}{1+\sin^2\theta}.$$

For small $r$, $\dot{r}$ is dominated by the $r$ term and is positive. For large $r$, $\dot{r}$ is dominated by the $-ar^3$ term and is hence negative for positive $a$. So trajectories must leave the vicinity of the origin and come in from infinity. By the Poincaré–Bendixson theorem, there must be a periodic orbit somewhere. In fact the periodic orbit is clearly at $r = a^{-1/2}$. Ignoring the denominator in the $\dot{r}$ equation shows that this periodic orbit is stable, meaning that trajectories close to it tend to $r = a^{-1/2}$. To solve exactly, divide the two equations and separate variables:

$$\frac{\mathrm{d}r}{r(1-ar^2)} = \frac{1+\cos^2\theta}{1+\sin^2\theta}\mathrm{d}\theta.$$

On the left-hand side,

$$\int \frac{\mathrm{d}r}{r(1-ar^2)} = \int \left[\frac{1}{r} - \frac{a}{1-ar^2}\right] \mathrm{d}r = \log r - \tfrac{1}{2}\log|1-ar^2| + C = \tfrac{1}{2}\log\frac{r^2}{|1-ar^2|} + C.$$

On the right-hand side, make the substitution $t = \tan\theta$:

$$\int \frac{1+\cos^2\theta}{1+\sin^2\theta}\mathrm{d}\theta = \int \left[\frac{3}{1+\cos^2\theta} - 1\right] \mathrm{d}\theta = \int 3\frac{\mathrm{d}t}{2+t^2} - \theta + D = \frac{3}{\sqrt{2}}\tan^{-1}\frac{\tan\theta}{\sqrt{2}} - \theta + D.$$

This gives

$$r(\theta) = r(\theta_0)\frac{\sqrt{a\pm 1}}{\sqrt{a \pm \mathrm{e}^{-2f(\theta)+2f(\theta_0)}}},$$

where the plus sign is for $1 - ar^2 > 0$ and the minus sign for $1 - ar^2 < 0$. The function $f$ is given by

$$f(\theta) = \frac{3}{\sqrt{2}}\tan^{-1}\frac{\tan\theta}{\sqrt{2}} - \theta.$$

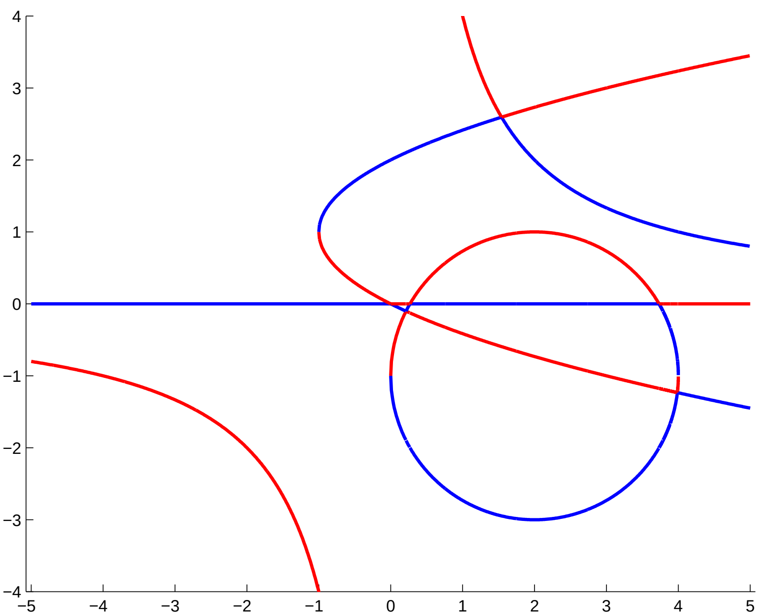


Figure 1: Bifurcation diagram in $(\mu, x)$ plane. Blue curves are stable and red curves are unstable.

**2** The fixed points in the $(\mu, x)$ plane are shown in Figure 1. These points lie along the hyperbolae $H$ given by $\mu x = 4$, the parabola $P$ given by $\mu = 2x - x^2$, the circle $C$ given by $(\mu - 2)^2 + (x+1)^2 = 1$, and the line $L$ given by $x = 0$. There are nine bifurcations points:

1. $(-1, 1)$: saddle-node.
2. $(0, -1)$: saddle-node.
3. $(0, 0)$: transcritical.
4. $(\mu_1, x_1) \approx (0.2132, -0.1015)$: transcritical. Intersection of $P$ and $C$.
5. $(2 - \sqrt{3}, 0)$: transcritical. Intersection of $C$ and $L$.
6. $(\mu_3, x_3) \approx (1.5418, 2.5943)$; transcritical. Intersection of $P$ and $H$.
7. $(2 + \sqrt{3}, 0)$: transcritical. Intersection of $C$ and $L$.
8. $(\mu_2, x_2) \approx (3.9864, -1.2330)$: transcritical. Intersection of $P$ and $C$.
9. $(4, -1)$: saddle-node.

**3** There are three fixed points: $(0,0,0)$ and $(\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$. The matrix of derivatives is

$$Df = \begin{pmatrix} -10 & 10 & 0 \\ 28-w & -1 & -u \\ v & u & -8/3 \end{pmatrix}.$$

At $(0,0)$ we have

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}.$$

This has eigenvalues $(-22.83, 11.83, -8/3)$ with eigenvectors $(-0.61, 0.79, 0)$, $(-0.42, -0.91, 0)$ and $(0,0,1)$. In fact the $w$-equation decouples from the other two. Equivalently the cubic factorizes and the eigenvalues can be obtained in closed form. This is a saddle with one unstable directions and two stable direction. One of the stable directions is out of the $xy$-plane. At $(\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$ we have

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & \mp 6\sqrt{2} \\ \pm 6\sqrt{2} & \pm 6\sqrt{2} & -8/3 \end{pmatrix}.$$

Both choices of sign give eigenvalues $(-13.859, 0.09 \pm 10.19\mathrm{i})$. This corresponds to a stable direction coming in to a weakly unstable focus. [The Lorenz equations are discussed extensively in the literature. For example there's an entire book on them by Sparrow.]

**4** Write $y = \mathrm{e}^{S(x)}$, giving

$$xS'' + x(S')^2 + (b-x)S' - a = 0.$$

For large $x$, assume $S'' \ll (S')^2$ as well as $(S')^2 \gg x^{-1}$ and $S' \gg x^{-1}$. The dominant balance is then

$$(S')^2 \sim S'.$$

This leads to $S' \sim 1$. The other option is $S' \sim 0$, which we leave for later. We find $S \sim x$, which satisfies the assumptions above. Now write $S(x) = x + R(x)$ with $R(x) \ll x$, giving

$$xR'' + x[1 + 2R' + (R')^2] + (b-x)(1+R') - a = 0.$$

Since $R' \ll 1$ and $R''$ is even smaller, the remaining dominant balance is

$$xR' \sim a - b,$$

so that $R \sim (a-b)\log x$. By the usual argument, subsequent terms in $S(x)$ decay for large $x$, so that we have one solution behaving like

$$y \sim x^{a-b}\mathrm{e}^x.$$

The other solution must violate one of the above approximations. The first one to be violated as $S'$ becomes small as $S' \gg x^{-1}$, so we try $S' \sim Ax^{-1}$. The dominant balance is then $A \sim a$. Again, subsequent terms are small, so we have a second solution

$$y \sim x^a.$$

If $a = b$ or $a = 0$, these behaviors still hold, although one should be looking for the next terms rather than trying to balance $R'$ with 0. If $a = 0$, the equation can be solved exactly using an integrating factor. One solution is $y = 1$, corresponding to the fact that $S = 0$ is an exact solution. Another is

$$y = \int^x t^{-b} \mathrm{e}^t \, \mathrm{d}t.$$

This behaves like $x^{-b}\mathrm{e}^x$ for large $x$. If $a = b = 0$, $S = 0$ and $S = x$ are exact solutions, and the solution to the equation is $y = C + D\mathrm{e}^x$.