Homework III

1 We write $y(x) = \mathrm{e}^{S(x)}$ and obtain

$$x^2(S'' + S'^2) + 2xS' + x^2 - \nu(\nu+1) = 0.$$

We make the usual assumption that $S'^2 \gg S''$. Then

$$S'^2 \sim -1,$$

which may be integrated to give $S \sim \pm \mathrm{i}x$, satisfying the assumption on S' and S'' since $S'' = 0$. We now seek the next term by writing $S = \pm \mathrm{i}x + C$, where $C \ll x$. This leads to

$$C'' \pm 2\mathrm{i}C' + C'^2 \pm \frac{2\mathrm{i}}{x} + \frac{2C'}{x} - \frac{\nu(\nu+1)}{x^2} = 0.$$

The dominant balance becomes

$$C' \sim -\frac{1}{x}.$$

This gives $C \sim -\log x$. The next term in $S(x)$ will be small, so the controlling behavior is given by

$$y(x) \sim x^{-1}\mathrm{e}^{\pm \mathrm{i}x}.$$

We now calculate

$$j_0(x) = \frac{1}{2}\int_0^\pi \cos(x\cos\theta)\sin\theta\,\mathrm{d}\theta = \frac{1}{2}\left[-\frac{\sin(x\cos\theta)}{x}\right]_0^\pi = \frac{\sin x}{x} = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}x}.$$

This is a linear combination of the two controlling behaviors obtained.

2 Rescale with $x = \epsilon^a X$. The size of the terms is then $2+3a$, $2a$, a and 1 respectively. The three consistent balances occur for $a = 0$, $a = 1$, and $a = -2$. For $a = 0$, we have $x_0^2 + 2x_0 = 0$ so $x_0 = -2$ (the 0 root is the $a = 1$ root). Then $2x_0x_1 + 2x_1 + 1 = 0$, so $x_1 = 1/2$. For $a = 1$, the equation becomes $\epsilon^4 X^3 + \epsilon X^2 + 2X + 1 = 0$. Then we have $2X_0 + 1 = 0$, so $X_0 = -1/2$. Next $X_0^2 + 2X_1 = 0$, so $X_1 = -1/8$. For $a = -2$, the equation becomes $X^3 + X^2 + 2\epsilon^2 X^2 + \epsilon^5 = 0$. Then we have $X_0^3 + X_0^2 = 0$ so $X_0 = -1$ (the two 0 roots are the other roots). The next term is at $O(\epsilon^2)$ with $3X_0^2X_2 + 2X_0X_2 + 2X_0 = 0$, so $X_2 = 2$. The three roots are hence

$$-\epsilon^{-2} + 2, \qquad -2 + \tfrac{1}{2}\epsilon, \qquad -\tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2.$$

3 Let the center of the Earth be O, the point at which the rope leads the Earth's surface A, and the highest point P, with the latter a distance h from the surface. The length of the rope is known, and is made up of twice the arc along the equator and the taut section AP of length d. This may be expressed as

$$2\pi R + l = 2(\pi - \theta)R + 2d,$$

Now from elementary trigonometry,

$$\tan\theta = d/R.$$

Combining the two equations, and noting that $\epsilon = l/R$ is a small parameter, we obtain

$$\epsilon/2 = -\theta + \tan\theta.$$

The left-hand side is small, so θ must also be small. Expanding,

$$\epsilon/2 = -\theta + \theta + \tfrac{1}{3}\theta^3 + \cdots.$$

Hence $\theta = (3\epsilon/2)^{1/3}$ to leading order. The distance from the Earth, h, is given by the trigonometric relation

$$\cos\theta = \frac{R}{R+h} = \frac{1}{1+h/R}.$$

The ratio h/R must also be small so

$$1 - \tfrac{1}{2}\theta^2 + \cdots = 1 - h/R + \cdots.$$

Hence, to leading order,

$$h/R = \tfrac{1}{2}(3\epsilon/2)^{2/3}.$$

Plugging in numbers, $h \approx 5.65$ m.

4 The equation may be rewritten as

$$x^2 = -\log(\epsilon x),$$

and since the left-hand side decreases more slowly than the right-hand side, we try the iteration ($L_1 = \log\frac{1}{\epsilon}$)

$$x_{n+1}^2 = L_1 - \log x_n, \qquad x_0 = L_1^{1/2} = (\log\tfrac{1}{\epsilon})^{1/2}.$$

Then

$$x_1^2 = L_1 - \log L_1^{1/2} = L_1 - \tfrac{1}{2}L_2 = L_1(1 - L_2/2L_1) + \cdots, \qquad L_2 = \log\log\tfrac{1}{\epsilon}.$$

Now $L_2 \ll L_1$, so

$$x_1 \approx L_1^{1/2}(1 - L_2/4L_1).$$

The next iteration gives

$$x_2^2 = L_1 - \log L_1^{1/2} - \log\left(1 - L_2/4L_1\right) = L_1 - \tfrac{1}{2}L_2 + L_2/4L_1 + \cdots$$

so

$$x_2 \approx L_1^{1/2}(1 - L_2/4L_1 + L_2/8L_1^2).$$

In fact, as pointed out by Hinch, one should take both $1/L_1^2$ terms to minimize error. This requires going to the next order:

$$x_3 = L_1^{1/2}(1 - L_2/4L_1 + L_2/8L_1^2 - L_2^2/32L_1^2).$$