Solution IV

1 Skip the naive expansion. The leading-order solution is $x_0 = A(T)\mathrm{e}^{\mathrm{i}t} + c.c.$ At $O(\epsilon)$, one finds

$$x_{1tt} + x_1 + 2x_{0tT} + (4\cos\alpha t)x_0 = 0.$$

Secular terms are of the form $\mathrm{e}^{\pm\mathrm{i}t}$, so look at the last term:

$$2(\mathrm{e}^{\mathrm{i}\alpha t} + \mathrm{e}^{-\mathrm{i}\alpha t})(A\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{\mathrm{i}t}) = 2A[\mathrm{e}^{\mathrm{i}(1+\alpha)t} + \mathrm{e}^{\mathrm{i}(1-\alpha)t}] + 2A^*[\mathrm{e}^{\mathrm{i}(-1+\alpha)t} + \mathrm{e}^{\mathrm{i}(-1-\alpha)t}].$$

Secular terms are hence possible for $\alpha = -2, 0, 2$. For a real equation, $\alpha = -2$ is the same as $\alpha = 2$. The equation can be solved exactly for $\alpha = 0$ and gives $A\mathrm{e}^{\mathrm{i}\omega t}$ + c.c with $\omega = \sqrt{1+4\epsilon}$: no growth. For $\alpha = 2$, the amplitude equation is

$$2\mathrm{i}A_T + 2A^* = 0$$

Writing $A = u + \mathrm{i}v$ gives $\mathrm{i}u_T - v_T + u - \mathrm{i}v = 0$, i.e.

$$u_T = v, \qquad v_T = u.$$

Hence $u = a\mathrm{e}^T + b\mathrm{e}^{-T}$ and $v = a\mathrm{e}^T - b\mathrm{e}^{-T}$. Except for very special initial conditions, solutions grow exponentially with T.

2 This is MMS. The $O(1)$ equation gives

$$x_0 = A(T)\sin[\tau + \varphi(T)], \qquad A(0) = 1, \quad \varphi(0) = 0.$$

The $O(\epsilon)$ equation is

$$x_{1\tau\tau} + x_1 + 2x_{0\tau T} - \frac{1}{2 + x_{0\tau}} = 0.$$

Substituting in gives

$$x_{1\tau\tau} + x_1 + 2[A_T\cos(\tau+\varphi) - A\varphi_T\sin(\tau+\varphi))] - \frac{1}{2 + A\cos(\tau+\varphi)} = 0.$$

Now integrate against $\cos(\tau+\varphi)$ and $\sin(\tau+\varphi)$ over one period. The second integration gives $\varphi_T = 0$, so φ vanishes identically. Assuming that $0 < A < 2$, the second gives

$$2A_T\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta - \int_0^{2\pi}\frac{\cos\theta}{2 + A\cos\theta}\,\mathrm{d}\theta = 2\pi A_T - 2\pi\frac{\sqrt{4-A^2}-2}{A\sqrt{4-A^2}} = 0.$$

We have hence obtained the following equation for A:

$$A_T = \frac{\sqrt{4-A^2}-2}{A\sqrt{4-A^2}}.$$

This is not nice to solve in closed form but we can just use a phase line analysis. It is evident that $A_T < 0$ for $0 < A < 2$ (there is a singularity at the origin). Hence since we start at $A = 1$, the solution decays and also stays in the range $(0, 2)$. For small A, the equation becomes $A_T \approx -A/8$, so $A \propto \mathrm{e}^{-T/8}$ in that limit (the proportionality constant is not 1). For the bonus part, either write $z = \mathrm{e}^{\mathrm{i}\theta}$ and use contour integration with $\cos\theta = (z+z^{-1})/2$ on the unit circle, or use the change of variable $t = \tan(\theta/2)$.

3 (i) Skip the naive expansion. The leading-order solution is $y_0 = A(X) + B(X)\mathrm{e}^{-x}$ At $O(\epsilon)$, one finds

$$y_{1xx} + 2y_{0xX} + y_{1x} + y_{0X} - y_0^2 = 0.$$

There are two types of secular terms: constant and e^{-X}. This gives two amplitude equations

$$A_X - A^2 = 0, \qquad B_X + 2AB = 0.$$

The boundary conditions give

$$A(0) + B(0) = 0, \qquad A(1) = 1,$$

where a term of the form $\mathrm{e}^{-\epsilon^{-1}}$ has been neglected in the second condition, since it is smaller than all orders in ϵ. The A-equation can be solved first, yielding

$$A = \frac{1}{2-X}.$$

The B-equation then gives

$$B = -\frac{1}{8}(2-X)^2.$$

The MMS solution is then

$$y_{MMS} = \frac{1}{2-\epsilon x} - \frac{1}{8}(2-\epsilon x)^2\mathrm{e}^{-x} + O(\epsilon)$$

uniformly in the domain.
(ii) The outer solution is in the variable X, for which the governing equation is

$$\epsilon y_{XX} + y_X - y^2 = 0.$$

The leading-order solution satisfies $y_{0X} - y_0^2 = 0$ and $y_0(1) = 1$. We have already solved this problem and the answer is

$$y_0 = \frac{1}{2-X}.$$

The leading-order inner solution is in the variable x and satisfies $Y_0'' + Y_0' = 0$ with the condition $Y_0(0) = 0$, so $Y_0 = C(\mathrm{e}^{-x} - 1)$. Matching to the outer solution gives $C = -1/2$ and hence

$$Y_0 = \frac{1}{2}(1 - \mathrm{e}^{-x}).$$

The leading-order uniform solution is

$$y_u = \frac{1}{2 - \epsilon x} - \frac{1}{2}\mathrm{e}^{-x}.$$

The two solutions are very similar. Neglecting the ϵx in the denominator of the second term of y_{MMS} gives y_u. Neither satisfies the boundary condition at $x = \epsilon^{-1}$ exactly, but the error is smaller than all powers of ϵ. Figure 1 shows the solutions for $\epsilon = 0.02$.

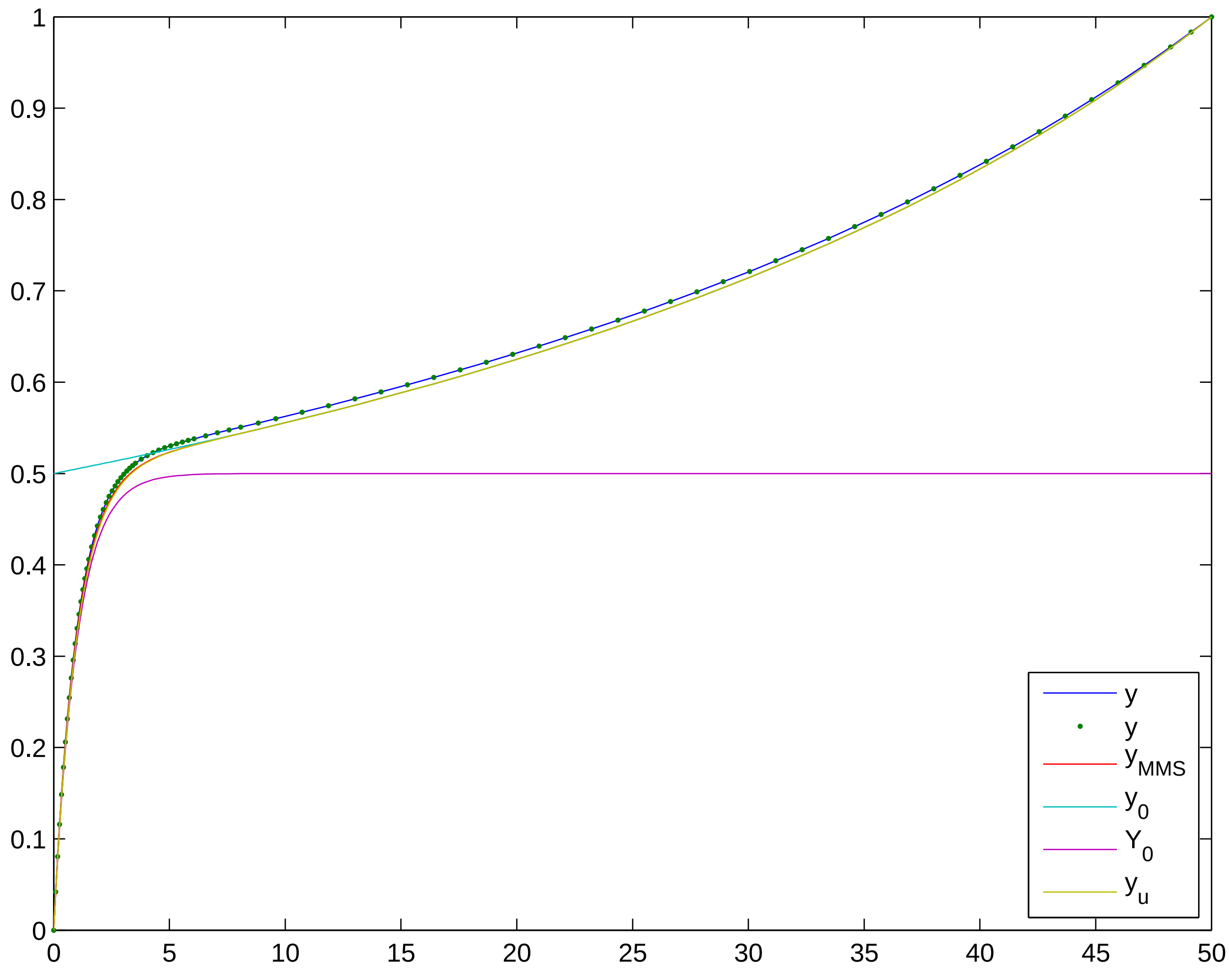

Figure 1: Exact, inner, outer, uniform and MMS solutions for $\epsilon = 0.02$.

4 This is a singular perturbation problem. Given the sign of the y' term, expect the boundary layer to be at 1. The boundary conditions show that $y = O(1)$ near the boundaries so no need to scale y. Writing $x = 1 - \epsilon^{\alpha} X$ shows that $\alpha = 1$ and the equation for the inner solution is

$$Y_{XX} + Y_X + \epsilon^2 (1 - \epsilon X) Y^2 = 2\epsilon(1 - \epsilon X).$$

Solve the leading-order outer problem:

$$-y_0' = 2x, \qquad y(0) = 2,$$

giving $y_0 = 2 - x^2$. The leading-order inner problem is $Y_{0XX} + Y_{0X} = 0$ with $Y(0) = 2$. The solution is $Y_0 = 2 + C(\mathrm{e}^{-X} - 1)$. Matching naively gives $1 = 2 - C$, so the inner solution becomes $Y_0 = 1 + \mathrm{e}^{-X}$. The $O(\epsilon)$ outer solution satisfies

$$y_0'' - y_1' + x y_0^2 = 0, \qquad y_1(0) = 0,$$

so $y_1 = x^6/6 - x^4 + 2x^2 - 2x$. The next inner problem is $Y_{1XX} + Y_{1X} = 2$ with $Y(0) = 1$. The solution is $Y_1 = 2X + 1 + D(\mathrm{e}^{-X} - 1)$. To match, use van Dyke's rule:

$$\begin{aligned}
E_1 H_1 y &= E_1\{2 - (1-\epsilon X)^2 + \epsilon[(1-\epsilon X)^6/6 - (1-\epsilon X)^4 + 2(1-\epsilon X)^2 - 2(1-\epsilon X)]\} \\
&= 1 + \epsilon(2X - 5/6), \\
H_1 E_1 y &= H_1[1 + \mathrm{e}^{(x-1)/\epsilon} + \epsilon\{(2(1-x)/\epsilon + 1 + D(\mathrm{e}^{(x-1)/\epsilon} - 1)\}] \\
&= 3 - 2x + \epsilon(1 - D) = 1 + \epsilon(2X + 1 - D).
\end{aligned}$$

Hence $D = 11/6$ and the outer and inner solutions are

$$y = 2 - x^2 + \epsilon(x^6/6 - x^4 + 2x^2 - 2x) + O(\epsilon^2)$$

and

$$Y = 1 + \mathrm{e}^{-X} + \epsilon(2X - 5/6 + 11\mathrm{e}^{-X}/6) + O(\epsilon^2).$$