Solutions V

1 From class,

$$y = \frac{1+x}{x} - \epsilon \frac{(1-x)(1+3x)}{2x^3}.$$

In the inner region with $x = \epsilon^{1/2}\xi$ and $y = \epsilon^{-1/2}Y$, the governing equation becomes

$$(\xi + Y)Y_\xi + Y = \epsilon^{1/2}.$$

The equation for the leading-order term can be integrated to give

$$\xi Y_0 + \frac{1}{2}Y_0^2 = A,$$

where A is a constant. Hence

$$Y_0 = -\xi \pm \sqrt{\xi^2 + 2A}.$$

Van Dyke's rule gives

$$y^{(0,0)} = \frac{\epsilon^{-1/2}}{\xi}, \qquad Y^{(0,0)} = -\epsilon^{-1/2}x \pm \epsilon^{-1/2}x\left(1 + \frac{\epsilon A}{x^2}\right).$$

Hence we take the plus sign and $A = 1$.

2 Start with the outer solution. At leading order, we find

$$xy_0' + y_0 = 2x$$

with solution $y_0 = x + Ax^{-1}$. One can look for a boundary layer near $x = 1$: the resulting equation can be solved, but the solution becomes singular within the boundary layer. So there is a boundary layer near the origin, and $A = 0$. The next equation is

$$xy_1' + y_1 = -y_0'' - y_0^3 = -x^3, \qquad y_1(1) = 0,$$

with solution $y_1 = \frac{1}{4}(x^{-1} - x^3)$. We see that the expansion becomes disordered when $x = O(\epsilon^{1/2})$. In that region, $y_0 \sim \epsilon^{1/2}$, but the boundary condition at the origin is $O(1)$, so it's probably best not to rescale y near the origin and write $y = Y$. Write $x = \epsilon^{1/2}X$ in the boundary layer. This leads to

$$Y_{XX} + XY_x + \epsilon^{1/2}Y_X^3 + Y + \epsilon Y^3 = 2\epsilon^{1/2}X.$$

The leading-order problem is

$$Y_{0XX} + XY_{0X} + Y_0 = 0, \qquad Y_0(0) = -1.$$

This equation can be integrated once to give

$$Y_{0X} + XY_0 = A,$$

where A is a constant. Now consider van Dyke's rule: the outer solution gives

$$y^{(0,\cdot)} = \epsilon^{1/2} X.$$

Hence $y^{(0,0)} = Y^{(0,0)} = 0$. This means that the limit of $Y_0(X)$ for large X is zero. Hence the constant value $Y_{0X} + XY_0$ vanishes for large X, and $A = 0$. We now integrate the resulting equation and apply the boundary condition. The result is $Y_0 = -\mathrm{e}^{-X^2/2}$. We expand in powers of $\epsilon^{1/2}$, so at the next order,

$$Y_{1XX} + XY_{1X} + Y_1 = 2X - Y_{0X}^3 = 2X + X^3 \mathrm{e}^{-3X^2/2}, \qquad Y_1(0) = 0.$$

We can integrate again, giving

$$Y_{1X} + XY_1 = X^2 - \frac{3X^2+2}{9}\mathrm{e}^{-3X^2/2} + B.$$

Writing $Y_1 = X + F$ gives

$$1 + F_X + XF = -\frac{3X^2+2}{9}\mathrm{e}^{-3X^2/2} + B,$$

Our goal is to find B simply. For large X, the form of $y(x)$ indicates that F will be an expansion in negative powers of X, i.e. $F = F_0 + F_1 X^{-1} + \cdots$, so that $F_X = -F_1 X^{-2} + \cdots$. Plug this into the above relation and find $F_0 = 0$ and $1 + F_1 = B$. Now Van Dyke's rule gives

$$y^{(1,1)} = \epsilon^{1/2} X + \frac{\epsilon^{1/2}}{4X} = Y^{(1,1)} = x + \frac{\epsilon F_1}{x}.$$

Hence $F_1 = 1/4 = B - 1$. We can now solve for F

$$F = -\mathrm{e}^{-X^2/2} \int_0^X \frac{3u^2+2}{9}\mathrm{e}^{-u^2}\,\mathrm{d}u + \frac{1}{4}\mathrm{e}^{-X^2/2}\int_0^u \mathrm{e}^{u^2/2}\,\mathrm{d}u.$$

Finally

$$Y_1 = X - \frac{7\pi^{1/2}\mathrm{e}^{X^2}\,\mathrm{erf}\,X - 6X}{36}\mathrm{e}^{-3X^2/2} + \frac{1}{4}\mathrm{e}^{-X^2/2}\int_0^u \mathrm{e}^{u^2/2}\,\mathrm{d}u$$

(avoiding using error functions of imaginary argument).

3 The equation becomes

$$\epsilon^2 y_{TT} - \epsilon T \mathrm{e}^{-T} y_T + y = 0$$

in T. Inserting an LG approximant gives

$$\phi_{0T}^2 - T\mathrm{e}^{-T}\phi_{0T} + 1 = 0, \qquad \phi_{0TT} + 2\phi_{0T}\phi_{1T} - T\mathrm{e}^{-T}\phi_{1T} = 0.$$

The first equation can be solved to give

$$\phi_0 = \int^T \frac{\tau\mathrm{e}^{-\tau} \pm \mathrm{i}\sqrt{4-\tau^2\mathrm{e}^{-2T}}}{2}\,\mathrm{d}\tau.$$

Once again there are two roots. The equation for ϕ_1 can be solved using an integrating factor, which gives a double integral, and is not nearly as simple as in the standard WKB case. Liouville normal form (LNF) for the general equation $y'' + ay' + by = 0$ comes from writing $y = \mathrm{e}^I z$ and removing the term in z'. One obtains $I' = a/2$, so that

$$I = \frac{1}{2}\int \epsilon T\mathrm{e}^{-T}\,\mathrm{d}T = -\frac{1}{2}(T+1)\mathrm{e}^{-\epsilon t}.$$

Then

$$z'' + \left(b - \frac{a'}{2} - \frac{a^2}{4}\right)z = 0.$$

Here this leads to

$$z_{TT} + \left(1 - \frac{T^2\mathrm{e}^{-2T}}{4} + \frac{\mathrm{e}^{-T}(1-T)}{2}\epsilon\right)z = 0.$$

The WKB solution is not standard because of the term in ϵ. The equations are

$$\phi_{0T}^2 = 1 - \frac{T^2\mathrm{e}^{-2T}}{4}, \qquad \phi_{0TT} + 2\phi_{0T}\phi_{1T} + \frac{\mathrm{e}^{-T}(1-T)}{2} = 0.$$

The first leads to

$$\phi_0 = \pm\frac{\mathrm{i}}{2}\int^T \sqrt{4-\tau^2\mathrm{e}^{-2T}}\,\mathrm{d}\tau.$$

We see that we get the same answer, taking into account the multiplying factor e^I. The same will happen for ϕ_1.

4 The WKB solution that satisfies the connection formulas at $x = -a$ is

$$y \sim (1 - x^4/E)^{-1/4}\cos\left(E^{1/2}\int_{-a}^{x}\sqrt{1-u^4/E}\,\mathrm{d}u - \frac{\pi}{4}\right)$$

where $a = E^{1/4}$. Even solutions require

$$\cos\left(E^{1/2}\int_{-a}^{x}\sqrt{1-u^4/E}\,\mathrm{d}u - \frac{\pi}{4}\right) = \cos\left(E^{1/2}\int_{-a}^{-x}\sqrt{1-u^4/E}\,\mathrm{d}u - \frac{\pi}{4}\right).$$

Now $\cos A = \cos B$ if $A = B + 2n\pi$ or $A = -B + 2n\pi$. Changing variable in the second integral gives

$$E^{1/2}\int_{-a}^{x}\sqrt{1-u^4/E}\,\mathrm{d}u - \frac{\pi}{4} = \pm\left(E^{1/2}\int_{x}^{a}\sqrt{1-u^4/E}\,\mathrm{d}u - \frac{\pi}{4}\right) + 2n\pi.$$

The plus sign is too restrictive. The minus sign leads to

$$E^{1/2}\int_{-a}^{a}\sqrt{1-u^4/E}\,\mathrm{d}u = \frac{\pi}{2} + 2n\pi.$$

For odd solutions, the same procedure leads to

$$E^{1/2}\int_{-a}^{a}\sqrt{1-u^4/E}\,\mathrm{d}u = -\frac{\pi}{2} + 2n\pi.$$

Now change variables inside the integral using $u = E^{1/4}v$ and include both odd and even cases:

$$E^{3/4}\int_{-1}^{1}\sqrt{1-v^4}\,\mathrm{d}v = -\frac{\pi}{2} + n\pi.$$

Hence the eigenvalues are

$$E \sim I^{-4/3}\pi^{4/3}(n-\tfrac{1}{2})^{4/3}$$

for $n = 1, 2, \ldots$, where

$$I = \int_{-1}^{1}\sqrt{1-v^4}\,\mathrm{d}v.$$

(The integral $I = 1.748038369528080\ldots$ is related to the complete integral elliptic integral of the first kind.)