Solution VI

1 The first term

$$f_0 = 1 - x$$

satisfies the boundary conditions. The second term comes from

$$f_1'' = \frac{f_0}{f_0'} x^m = \frac{x-1}{x^m},$$

so that for $m \leq 0$

$$f_1 = \frac{x^{3-m}}{(3-m)(2-m)} - \frac{x^{2-m}}{(2-m)(1-m)} + \frac{2x}{(3-m)(2-m)(1-m)}.$$

This gives a solution that satisfies both boundary conditions. i.e. a regular perturbation problem. For $m = 1$, the same approach gives

$$f_1 = \frac{x^2 - x}{2} - x \log x,$$

satisfying both boundary conditions, so again a regular perturbation problem (presumably this is true for $m \leq 1$). For $m = 2$, the correction term becomes

$$f_1 = \log x + x \log x + a_1(1 - x),$$

satisfying the condition at $x = 1$. It cannot satisfy the condition at the origin. Obtaining an outer solution in the form shown gives

$$f_0 = a_0(1 - x), \qquad f_1 = a_1(1 - x), \qquad f_2 = a_0^2[\log x + x \log x + a_2(1 - x)].$$

In the inner variable the governing equation is

$$F_{XX} - \frac{F F_X}{X^2} = 0, \qquad F(0) = 1.$$

We see that $F_0 = 1$ is a solution to

$$F_{0XX} - \frac{F_0 F_{0X}}{X^2} = 0, \qquad F_0(0) = 1.$$

(One can argue that we must have $F_{0X} = 0$ at the origin using l'Hopital's rule.) The next two terms both satisfy the same equation (given here for F_1):

$$F_{1XX} - \frac{F_{1X}}{X^2} = 0, \qquad F_1(0) = 0.$$

Hence, separating variables and applying the boundary condition,

$$F_1 = B_1 \int_0^X e^{-1/u}\, du, \qquad F_2 = B_2 \int_0^X e^{-1/u}\, du.$$

Now match using van Dyke's rule

$$f^{(0,0)} = a_0 = F^{(0,0)} = 1.$$

Next $f^{(1,1)} = f^{(1,1)}$ gives

$$1 - \epsilon X + \epsilon \log \epsilon^{-1} a_1 + \epsilon[\log \epsilon X + a_2] = 1 + \epsilon(B_1 \log \epsilon^{-1} + B_2)\left[\frac{x}{\epsilon} - \log \frac{x}{\epsilon} + \gamma - 1\right].$$

This leads to 0, $B_1 = 0$, $B_2 = -1$, $a_1 = 1$ and $a_2 = 1 - \gamma$. See Hinch § 5.2. For $m = 0$, the equation can be integrated once to give

$$f' - \frac{1}{2}\epsilon f^2 = -D.$$

We see that the right-hand side is negative, since $f(1) = 0$ and f is decreasing at $x = 1$. Now separate variables and obtain

$$f = \frac{2B}{\epsilon} \tanh B(a - x).$$

From the boundary condition at $x = 1$, $a = 1$. The other boundary condition gives the transcendental relation $2B \tanh B = \epsilon$. An approximate solution can be obtained since $\tanh B \sim D$ for small B, so that $B \sim (\epsilon/2)^{1/2}$. We see that

$$f \sim \frac{2B}{\epsilon} B(1 - x) \sim 1 - x,$$

consistent with the regular perturbation expansion. For $m = 1$, the equation is equidimensional, so make the change of variable $t = \log x$. Solving gives

$$f = -\frac{1}{\epsilon} - \frac{2A}{\epsilon} \tanh A(\log x - a).$$

Applying the boundary condition at $x = 0$ gives $A = (1 + \epsilon)/2$, while the boundary condition at $x = 1$ leads to the relation $2A \tanh Aa = 1$, which can be solved for a, leading to

$$f = -\frac{1}{\epsilon} - (1 + \epsilon^{-1}) \tanh \left(\frac{2 \log x}{1 + \epsilon} - \tanh^{-1}(1 + \epsilon^{-1}) \right).$$

This can be expanded to give

$$f = 1 - x - \epsilon x \left(-\log x + \frac{1 - x}{2} \right) + \cdots,$$

which is consistent with the regular pertubation solution found earlier.

2 There are three regions: local when $\epsilon \sim x$, global where $x = O(1)$ and local when $\epsilon x = O(1)$. Divide the range at δ and M where $\epsilon \ll \delta \ll 1$ and $1 \ll M \ll \epsilon^{-1}$. Then

$$\begin{aligned} I_{L1} &= \int_0^{\delta/\epsilon} \frac{\mathrm{d}u}{(1+u)(1+\epsilon^2 u)} = \int_0^{\delta/\epsilon} \frac{\mathrm{d}u}{1+u}[1 - \epsilon^2 u + O(\epsilon^4 u^2)] \\ &= [\log(1+u)]_0^{\delta/\epsilon} - \epsilon^2[u - \log(1+u)]_0^{\delta/\epsilon} + O(\epsilon^2\delta^2) \\ &= \log\frac{\delta}{\epsilon} + \frac{\epsilon}{\delta} - \epsilon\delta + \epsilon^2 \log\frac{\delta}{\epsilon} + O\left(\frac{\epsilon^2}{\delta^2}, \frac{\epsilon^3}{\delta}, \epsilon^2\delta^2\right) \end{aligned}$$

is the first local contribution. The global contribution is

$$\begin{aligned} I_G &= \int_\delta^M \frac{\mathrm{d}x}{x}\left(1+\frac{\epsilon}{x}\right)(1+\epsilon x)^{-1} = \int_\delta^M \frac{\mathrm{d}x}{x}\left[1 - \frac{\epsilon}{x} - \epsilon x + O\left(\frac{\epsilon^2}{x^2}, \epsilon^2 x^2, \epsilon^2\right)\right] \\ &= \log\frac{M}{\delta} + \frac{\epsilon}{M} - \frac{\epsilon}{\delta} - \epsilon M + \epsilon\delta + O\left(\frac{\epsilon^2}{M^2}, \frac{\epsilon^2}{\delta^2}, \epsilon^2 M^2, \epsilon^2\delta^2, \epsilon^2\log M, \epsilon^2\log\delta\right). \end{aligned}$$

The second local contribution is

$$\begin{aligned} I_{L2} &= \int_{\epsilon M}^\infty \frac{\mathrm{d}v}{(1+v)v}\left(1+\frac{\epsilon^2}{v}\right)^{-1} = \int_{\epsilon M}^\infty \frac{\mathrm{d}v}{(1+v)v}\left[1 - \frac{\epsilon^2}{v} + O\left(\frac{\epsilon^4}{v^2}\right)\right] \\ &= \left[\log\frac{v}{1+v}\right]_{\epsilon M}^\infty - \epsilon^2\left[\log\frac{1+v}{v} - \frac{1}{v}\right]_{\epsilon M}^\infty + O\left(\frac{\epsilon^2}{M^2}\right) \\ &= -\log\epsilon M + \epsilon M - \epsilon^2\log\epsilon M - \frac{\epsilon}{M} + O\left(\epsilon^2 M^2, \epsilon^3 M, \frac{\epsilon^2}{M^2}\right). \end{aligned}$$

Putting this together gives $-2\log\epsilon(1+\epsilon^2)$. Exact solution:

$$\begin{aligned} \int_0^\infty \frac{\mathrm{d}x}{(\epsilon+x)(1+\epsilon x)} &= \int_0^\infty \frac{1}{1-\epsilon^2}\left[\frac{1}{\epsilon+x} - \frac{\epsilon}{1+\epsilon x}\right]\mathrm{d}x = \frac{1}{1-\epsilon^2}\left[\log(\epsilon+x) - \log(1+\epsilon x)\right]_0^\infty \\ &= -2\frac{\log\epsilon}{1-\epsilon^2} = -2\log\epsilon(1+\epsilon^2+\cdots). \end{aligned}$$

3 The function $h(t)$ in the exponent has maxima at $t = n\pi$ for integer n, with

$$h(n\pi) = 0, \quad h'(n\pi) = 0, \quad h''(n\pi) = 2.$$

The maxima have the same value of h, so we add up their contributions and the integral is asymptotic to

$$\sum_{n=1}^\infty \left(\frac{2\pi}{2x}\right)^{1/2} \frac{1}{(n\pi)^2} = \sqrt{\frac{\pi}{x}}\frac{1}{\pi^2}\sum_{n=1}^\infty n^{-2} = \frac{1}{6}\sqrt{\frac{\pi}{x}}.$$

4 Use the double-angle formula to write

$$I = \mathrm{Re} \int_0^\infty \left(\frac{1}{2} \mathrm{e}^{\mathrm{i}x(a+b)t^2} + \frac{1}{2} \mathrm{e}^{\mathrm{i}x[(a-b)t^2 - 2t]} \right) \mathrm{d}t.$$

The first integral can be computed exactly as $\frac{1}{4}\sqrt{\pi/(a+b)x}\mathrm{e}^{\mathrm{i}\pi/4}$. For the second integral, there is a critical point at $t = (a-b)^{-1}$ which is in the range of integration and hence dominates over the endpoint. The standard stationary phase argument gives the contribution $\frac{1}{2}\sqrt{\pi/(a-b)}\mathrm{e}^{-\mathrm{i}x/(a-b)+\mathrm{i}\pi/4}$. Putting this together gives

$$I \sim \frac{1}{4}\sqrt{\frac{\pi}{2(a+b)x}} + \frac{1}{2}\sqrt{\frac{\pi}{(a-b)x}} \cos\left(\frac{x}{a-b} - \frac{\pi}{4} \right).$$

As $a \to b$, the original integral no longer exists and the approximation also blows up.