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Instability of a vortex sheet leaving a right-angled wedge

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We examine the dynamics of a semi-infinite vortex sheet attached not to a semi-infinite plate but instead to a rigid right-angled wedge, with the sheet aligned along one of its edges. Our approach to this problem, which was suggested by David Crighton, accords well with the fundamental ethos of Crighton's work, which was characterized by 'the application of rigorous mathematical approximations to fluid mechanical idealizations of practically relevant problems' (Ffowcs Williams, *Annu. Rev. Fluid Mech.*, vol. 34, 2002, pp. 37–49). The resulting linearised unsteady potential flow is forced by an oscillatory dipole in the uniform stream passing along the top of the wedge, while there is stagnant fluid in the remaining quadrant. Spatial instability is considered according to well-established methods: causality is enforced by allowing the frequency to become temporarily complex. The essentially quadrant-type geometry replaces the usual Wiener–Hopf technique by the Mellin transform. The core difficulty is that a first-order difference equation of period 4 requires a solution of period unity. As a result, the complex fourth roots $(\pm 1 \pm \mathrm{i})$ of -4 appear in the complementary function. The Helmholtz instability wave is excited and requires careful handling to obtain explicit results for the amplitude of the instability wave.

Key words: instability, shear layers, wakes/jets

1. Introduction

Orszag & Crow (1970) considered the unsteady potential flow past the edge of a splitter plate with incompressible uniform flow on one side and stagnant fluid on the other side in the absence of external forcing. This simplest of inhomogeneous flows exhibits a concentrated vortex sheet in an unbounded inviscid fluid. They sought spatial growth, in contrast to the temporal growth more commonly sought in traditional stability theory, and left open the choice of Kutta condition, which determines how the sheet is attached to the plate. Crighton (1972) studied the compressible version of Orszag & Crow's problem and showed that spatial growth is unavoidable if a causal solution is required. In order to apply causality, Jones & Morgan (1972) had considered an initial value problem in which linear theory is applied to acoustic radiation incident on a moving stream. The Helmholtz instability

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of an infinite vortex sheet must be present, along with an edge-scattered instability. Crighton & Leppington (1974), in the context of scattering of a point source of sound by a subsonic splitter plate, showed that causality can be enforced by allowing the frequency to be temporarily complex-valued. Use of the full-Kutta condition, in which the sheet has zero slope at its attachment, is thereby established. Daniels (1978) confirmed this choice by including a triple-deck structure to demonstrate that a full-Kutta condition solution can be matched to a viscous inner solution but a no-Kutta condition cannot do likewise. Crighton (1985) reviewed the application of Kutta conditions to a much wider range of flow problems than considered here. Peake (1994) considered another model problem, namely the flow resulting from the interaction between the trailing edge of a supersonic splitter plate and sound waves incident on the trailing edge from upstream. This is essentially the supersonic version of Jones & Morgan's subsonic study. More recently, Samanta & Freund (2008) noted that accepted practice is to use a full-Kutta condition, for which Bechert & Pfizenmaier (1975) provided experimental support. Rabchuk (2000) identified trailing edge receptivity as disturbances converted into flow instabilities at the trailing edge, where the pressure is singular, in a mixing layer or wake. Triple-deck theory achieves theoretical consistency and gives clear physical justification for the unsteady Kutta condition, which accounts implicitly for viscosity but is inappropriate for the flow structure.

The splitter plate geometry permits solution via the Fourier transform. A natural question to consider is the effect of the local geometry of the trailing edge on the flow characteristics. The usual Kutta condition suggests that any sharp (non-reentrant) corner should lead to similar exponential growth but in the absence of other solutions to the problem, quantitative results are not known. The driving force behind the original work of Orszag & Crow (1970) and subsequently of Crighton (1972) and Crighton & Leppington (1974) was fundamentally the instability of the vortex sheet. The critical contribution of Crighton (1972) was the explanation of the spatial growth. As a result we focus here on the instability. The full solution is obtained from the present formulation, but is not pursued computationally, as this would be lengthy and unmanageable. In addition the solution would be dominated by the instability, so that the extra work would be irrelevant. The finite-thickness unbounded shear layer is inviscidly unstable, with a short-wavelength cutoff. Hence the unphysical short-wavelength instability of the vortex sheet (Kelvin–Helmholtz instability) is not a problem and the present calculation is useful.

The paper is structured as follows. Section 2 formulates the problem: a uniform inviscid stream on one side of a rigid right-angled wedge is separated by a vortex sheet from stagnant inviscid fluid in the fourth quadrant. Unsteady potential flow is forced by an oscillatory doublet placed in the stream alongside the rigid quadrant. The resultant pressure forcing at the sheet is translated, by use of Mellin transforms, into a second-order functional difference equation system of period 2, which is equivalent to a first-order equation of period 4. The corresponding simpler equations for the semi-infinite plate have periods 2 and unity respectively and, for illustrative purposes, are solved in § 3 by each of two distinct methods. The second of these forms the basis of the 90° trailing edge solution given in § 4. Its construction requires both the solution of the homogeneous difference equation of period 4 and the embedding of an inhomogeneous solution of period unity. The former uses the Barnes double Gamma function (Barnes 1899), ably described by Lawrie & King (1994), whose relevant results are listed in appendix A. The embedding exploits the fourth roots $(\pm 1 \pm \mathrm{i})$ of -4 to derive suitable functions of period unity. The four unknown constants are determined by eliminating poles within the strip of regularity. Causality is enforced by selecting a particular phase of the temporarily complex-valued frequency, whence the Helmholtz instability is shown to be present. Finally § 5 describes our conclusions. Some mathematical details are presented in appendix A.

2. Formulation

Consider incompressible, inviscid fluid in the three-quadrant region $r > 0$, $-\pi/2 < \theta < \pi$ bounded by rigid walls, and suppose that a vortex sheet at $\theta = 0$ is created by an imposed streaming flow $U\boldsymbol{e}_x$ in the half-space $y > 0$ (region 1, with region 2 corresponding to $y < 0$). Assume that the perturbed flow

$$U\boldsymbol{e}_x + \nabla\phi_1(x, y)\mathrm{e}^{\mathrm{i}kUt} \quad (-\infty < x < \infty, y > 0), \tag{2.1a}$$
$$\nabla\phi_2(x, y)\mathrm{e}^{\mathrm{i}kUt} \quad (0 < x < \infty, y < 0), \tag{2.1b}$$

is such that the amplitudes of the oscillatory perturbations are small enough to allow linearisation of their governing equations. Then the continuity equation yields

$$\nabla^2\phi_1 = 0, \quad \nabla^2\phi_2 = 0, \tag{2.2a,b}$$

in regions (1) and (2) respectively, the wall conditions are

$$\frac{\partial\phi_1}{\partial y}(x, 0) = 0\,(x < 0), \quad \frac{\partial\phi_2}{\partial x}(0, y) = 0\,(y < 0), \tag{2.3a,b}$$

the pressure is continuous when

$$\mathrm{i}k\phi_1(x, 0) + \frac{\partial\phi_1}{\partial x}(x, 0) = \mathrm{i}k\phi_2(x, 0) \quad (x > 0), \tag{2.4}$$

and the kinematic conditions, involving the interface displacement $\eta(x)\mathrm{e}^{\mathrm{i}kUt}$, are

$$\mathrm{i}kU\eta + U\frac{\partial\eta}{\partial x} = \frac{\partial\phi_1}{\partial y}(x, 0), \quad \mathrm{i}kU\eta = \frac{\partial\phi_2}{\partial y}(x, 0) \quad (x > 0). \tag{2.5a,b}$$

If the forcing disturbance is described by the potential ψ_1 in region (1), then the potentials

$$\phi_1(x, y) + \psi_1(x, y) + \psi_1(x, -y) \quad (y > 0), \tag{2.6a}$$
$$\phi_2(x, y) \quad (y < 0), \tag{2.6b}$$

are such that the wall conditions are satisfied and the resulting kinematic condition reduces to the homogeneous form (2.5). The resulting pressure fields exhibit a discontinuity proportional to

$$\left(\mathrm{i}k + \frac{\partial}{\partial x}\right) 2\psi_1(x, 0), \tag{2.7}$$

from which we deduce that only pressure jump forcing need be considered here. The 'image' structure is made apparent by the acoustic solution given by Crighton & Leppington (1974) for the semi-infinite plate.

For a convenient and fundamental forcing, upstream of the vortex sheet, we set

$$\psi_1(x, y) = \frac{Ur_0}{2\pi k} \frac{(x+x_0)\cos\chi + (y-y_0)\sin\chi}{(x+x_0)^2 + (y-y_0)^2} \quad (x_0, y_0 > 0), \tag{2.8}$$

in which the angle χ allows the dipole at $(-x_0, y_0) = r_0(-\cos\xi_0, \sin\xi_0)$ to be arbitrarily oriented. Then the flow is forced by the pressure condition

$$\begin{aligned}\phi_2(x, 0) - \phi_1(x, 0) - \frac{1}{\mathrm{i}k}\frac{\partial\phi_1}{\partial x}(x, 0) \;&=\; \frac{Ur_0}{2\pi k}\left(1 + \frac{1}{\mathrm{i}k}\frac{\partial}{\partial x_0}\right) \\ &\quad \times\left[\frac{\mathrm{e}^{\mathrm{i}\chi}}{x+x_0-\mathrm{i}y_0} + \frac{\mathrm{e}^{-\mathrm{i}\chi}}{x+x_0+\mathrm{i}y_0}\right] \quad (x>0). \end{aligned} \tag{2.9}$$

In view of (2.2) and the given geometry, define the usual Mellin transform

$$\Phi(s, \theta) = \int_0^\infty \phi(r, \theta) r^{s-1}\,\mathrm{d}r, \tag{2.10}$$

where $0 < \mathrm{Re}(s) < 1$ is required for the transform to exist. Then the transformed conditions at $\theta = 0$ are

$$\frac{\partial\Phi_1}{\partial\theta}(s, 0) = \frac{\partial\Phi_2}{\partial\theta}(s, 0) - \frac{s}{\mathrm{i}k}\frac{\partial\Phi_2}{\partial\theta}(s-1, 0), \tag{2.11a}$$

$$\Phi_2(s, 0) - \Phi_1(s, 0) + \frac{s-1}{\mathrm{i}k}\Phi_1(s-1, 0) = \frac{Ur_0}{\pi}\frac{\Gamma(s)}{k^s}F(s), \tag{2.11b}$$

where, according to (2.9) and with $\mathrm{Re}(s) < 1$,

$$\begin{aligned} F(s) \;&=\; \left(1 + \frac{1}{\mathrm{i}k}\frac{\partial}{\partial x_0}\right)\frac{\Gamma(1-s)}{2}\left[\frac{\mathrm{e}^{\mathrm{i}\chi}}{(kr_0\mathrm{e}^{-\mathrm{i}\xi_0})^{1-s}} + \frac{\mathrm{e}^{-\mathrm{i}\chi}}{(kr_0\mathrm{e}^{\mathrm{i}\xi_0})^{1-s}}\right] \\ &=\; \frac{\Gamma(1-s)}{2(kr_0)^{2-s}}\left\{\mathrm{e}^{\mathrm{i}[\chi+(2-s)\xi_0]}[k(x_0-\mathrm{i}y_0) - \mathrm{i}(s-1)]\right. \\ &\qquad \left. + \,\mathrm{e}^{-\mathrm{i}[\chi+(2-s)\xi_0]}[k(x_0+\mathrm{i}y_0) - \mathrm{i}(s-1)]\right\}. \end{aligned} \tag{2.12}$$

Dimensional coordinates x, y, r are retained so that the Mellin transforms depend on k.

Laplace's equation (2.2) requires Φ_1, Φ_2 to be linear combinations of $\cos s\theta$, $\sin s\theta$ and then the conditions at the rigid boundaries are satisfied by writing

$$\Phi_1 \;=\; Ur_0\frac{\Gamma(s)}{\pi k^s}\alpha(s)\frac{\cos s(\pi-\theta)}{\cos s\pi}, \tag{2.13a}$$

$$\Phi_2 \;=\; Ur_0\frac{\Gamma(s)}{\pi k^s}\gamma(s)\frac{\cos s(\pi+\theta) + \cos s\theta}{\cos s\pi}, \tag{2.13b}$$

with the Mellin transform of the interface displacement given by

$$N(s) = \int_0^\infty \eta(x)x^{s-1}\,\mathrm{d}x = \mathrm{i}r_0\frac{\Gamma(s)}{\pi k^s}\gamma(s-1)\tan s\pi. \tag{2.14}$$

So the interface conditions (2.11a) yield, in matrix form,

$$\begin{bmatrix}\gamma(s-1)\\ \alpha(s-1)\end{bmatrix} - \mathrm{i}\begin{bmatrix}1 & 1\\ -(1+\sec s\pi) & 1\end{bmatrix}\begin{bmatrix}\gamma(s)\\ \alpha(s)\end{bmatrix} = \mathrm{i}F(s)\begin{bmatrix}0\\ 1\end{bmatrix}, \tag{2.15}$$

in which the matrix has period 2 as a function of s. Therefore we have to solve a second-order functional difference equation system of period 2, which is equivalent to a first-order system of period 4.

To identify the required strip of regularity, first note that the velocity components have Mellin transforms

$$\int_0^\infty \frac{\partial \phi_j}{\partial r}(r,\theta) r^{s-1}\,\mathrm{d}r = -(s-1)\Phi_j(s-1,\theta), \tag{2.16a}$$

$$\int_0^\infty \frac{1}{r}\frac{\partial \phi_j}{\partial \theta}(r,\theta) r^{s-1}\,\mathrm{d}r = \frac{\partial \Phi_j}{\partial \theta}(s-1,\theta). \tag{2.16b}$$

In particular, the interface transform is given by

$$N(s) = \frac{1}{\mathrm{i}kU}\frac{\partial \Phi_2}{\partial \theta}(s-1,0). \tag{2.17}$$

These functions involve $\gamma(s-1)$ and $\alpha(s-1)$, whose regularity in $0 < \mathrm{Re}(s) < 1$ is achieved by ensuring the regularity of $\gamma(s)$ and $\alpha(s)$ in $-1 < \mathrm{Re}(s) < 0$.

3. The semi-infinite plate

We use this simpler problem first considered by Orszag & Crow (1970) to demonstrate features and structure of the solution that we construct below for the rigid quadrant problem. With region (2) extended to $\theta = -\pi$, the second wall condition is replaced by

$$\frac{\partial \phi_2}{\partial y}(x,0) = 0 \quad (x<0). \tag{3.1}$$

The conditions at the rigid plate $\theta = \pm\pi$ are satisfied by modifying the second condition in (2.13b) and writing

$$\Phi_1 = Ur_0\frac{\Gamma(s)}{\pi k^s}\alpha(s)\frac{\cos s(\pi-\theta)}{\cos s\pi}, \quad \Phi_2 = Ur_0\frac{\Gamma(s)}{\pi k^s}\gamma(s)\frac{\cos s(\pi+\theta)}{\cos s\pi}, \tag{3.2a,b}$$

whence the interface conditions yield (2.14) and, in matrix form,

$$\begin{bmatrix}\gamma(s-1)\\ \alpha(s-1)\end{bmatrix} - \mathrm{i}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\gamma(s)\\ \alpha(s)\end{bmatrix} = \mathrm{i}F(s)\begin{bmatrix}0\\ 1\end{bmatrix}. \tag{3.3}$$

The eigenvectors of the constant matrix facilitate the rearrangement of (3.3) as a pair of disjoint first-order equations,

$$\alpha(s-1) \pm \mathrm{i}\gamma(s-1) - (\mathrm{i}\mp 1)[\alpha(s) \pm \mathrm{i}\gamma(s)] = \mathrm{i}F(s). \tag{3.4}$$

Particular solutions of the form

$$\alpha(s) \pm \mathrm{i}\gamma(s) = \frac{1}{2\pi}\int_{\nu-\mathrm{i}\infty}^{\nu+\mathrm{i}\infty} F(u) I_\pm(u-s)\,\mathrm{d}u \quad (\nu - 1 < \mathrm{Re}(s) < \nu), \tag{3.5}$$

exist provided that

$$\int_{\nu-1-\mathrm{i}\infty}^{\nu-1+\mathrm{i}\infty} F(u) I_\pm(u-s+1)\,\mathrm{d}u - (\mathrm{i}\mp 1)\int_{\nu-\mathrm{i}\infty}^{\nu+\mathrm{i}\infty} F(u) I_\pm(u-s)\,\mathrm{d}u = 2\pi\mathrm{i}F(s), \tag{3.6}$$

which requires that $I_{\pm}(u-s+1)=(\mathrm{i}\mp 1)I_{\pm}(u-s)$ with $I_{\pm}(u-s)$ regular in the strip $\nu-1<\mathrm{Re}(s)<\nu$ except at $u=s$ where its residue is $(-\mathrm{i}\pm 1)^{-1}$. A suitable choice is $I_{\pm}(v)=\pi(-\mathrm{i}\pm 1)^{v-1}/\sin\pi v$, whence (3.4) has solution, in the strip $-1<\mathrm{Re}(s)<0$,

$$\alpha(s)\pm\mathrm{i}\gamma(s)=C_{\pm}(\mathrm{i}\mp 1)^{-s}+\frac{1}{2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{F(u)}{\sin\pi(u-s)}(-\mathrm{i}\pm 1)^{u-s-1}\,\mathrm{d}u. \tag{3.7}$$

The physically necessary elimination of the poles in Φ_1, Φ_2 at $s=-1/2$ is achieved by setting $\gamma(-1/2)=0=\alpha(-1/2)$. The second interface condition then eliminates the pole in Φ_2 at $s=-3/2$, corresponding to applying the full-Kutta condition (tangential contact of the interface; see Daniels (1978)). Thus, we choose

$$C_{\pm}(\mathrm{i}\mp 1)^{1/2}=-\frac{1}{2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{F(u)}{\cos\pi u}(-\mathrm{i}\pm 1)^{u-1/2}\,\mathrm{d}u. \tag{3.8}$$

For Φ_1 and Φ_2 to be proper Mellin transforms, they must decay as $\mathrm{Im}(s)\to\pm\infty$ within the strip of regularity. This can be achieved, as explained by Crighton & Leppington (1974) for the conjugate time dependence, by temporarily replacing k by $k-\mathrm{i}K$ in the complementary functions and requiring $(k-\mathrm{i}K)(1+\mathrm{i})$ and $(k-\mathrm{i}K)(-1+\mathrm{i})$ to have positive real parts. Inversion and the subsequent setting of K to zero yields the solution. In particular, the term with C_+ generates the exponentially growing Helmholtz potentials ϕ_2^H, ϕ_1^H (Orszag & Crow 1970) given by

$$\phi_1^H=\mathrm{i}Uh\exp[(1-\mathrm{i})k(x+\mathrm{i}y)] \quad (y>0), \tag{3.9a}$$
$$\phi_2^H=Uh\exp[(1-\mathrm{i})k(x-\mathrm{i}y)] \quad (y<0). \tag{3.9b}$$

Equivalently, one can replace k by $k_0\mathrm{e}^{-\mathrm{i}\pi\beta}k$ with $\pi/4<\beta<3\pi/4$ during the inversion and set $k=k_0$ (i.e. $\beta=0$) in the final results.

We now rework the problem as an example of the following result. If the matrix $\boldsymbol{A}(s)$ has period 4 and $\boldsymbol{A}(s)\boldsymbol{A}(s+1)\boldsymbol{A}(s+2)\boldsymbol{A}(s+3)=g(s)\boldsymbol{I}$, where $\boldsymbol{I}$ denotes the identity matrix, then $g(s)$ has period 1 and the system

$$\boldsymbol{x}(s-1)-\boldsymbol{A}(s)\boldsymbol{x}(s)=f(s)\boldsymbol{c}, \tag{3.10}$$

with $\boldsymbol{c}$ a constant vector, has solution

$$\begin{aligned}\boldsymbol{x}(s) \;=\; &\{\boldsymbol{A}(s+1)\boldsymbol{A}(s+2)\boldsymbol{A}(s+3)D(s)+\boldsymbol{A}(s+1)\boldsymbol{A}(s+2)D(s-1)\\ &+\boldsymbol{A}(s+1)D(s-2)+\boldsymbol{I}D(s-3)\}\,\boldsymbol{c},\end{aligned} \tag{3.11}$$

provided $D(s)$ satisfies the difference equation $D(s-4)-g(s)D(s)=f(s)$. Thus (3.3) has a solution of the form

$$\begin{bmatrix}\gamma(s)\\ \alpha(s)\end{bmatrix}=D(s)\begin{bmatrix}-2\mathrm{i}\\ 2\mathrm{i}\end{bmatrix}+D(s-1)\begin{bmatrix}-2\\ 0\end{bmatrix}+D(s-2)\begin{bmatrix}\mathrm{i}\\ \mathrm{i}\end{bmatrix}+D(s-3)\begin{bmatrix}0\\ 1\end{bmatrix}, \tag{3.12}$$

provided that

$$D(s-4)+4D(s)=\mathrm{i}F(s). \tag{3.13}$$

The function $D(s)$, of period 4, has the particular solution, constructed as in (3.7),

$$\frac{1}{32}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}2^{(u-s)/2}\frac{F(u)}{\sin\pi(u-s)/4}\,\mathrm{d}u \quad \text{in} \quad -4<\mathrm{Re}(s)<0. \tag{3.14}$$

The difference equation (3.13) of period 4 has a complementary function that displays the fourth roots of -4. Thus, in the strip $-4 < \mathrm{Re}(s) < 0$,

$$D(s) = \frac{1}{16}\left\{C_1(1+\mathrm{i})^{1-s} + C_2(-1+\mathrm{i})^{1-s} + C_3(-1-\mathrm{i})^{1-s} + C_4(1-\mathrm{i})^{1-s} + \frac{1}{2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} 2^{(u-s)/2}\frac{F(u)}{\sin \pi(u-s)/4}\,\mathrm{d}u\right\}. \tag{3.15}$$

Use of the identity

$$\frac{2^{v/2}}{\sin \pi v/4} = \frac{(1+\mathrm{i})^v + (-1+\mathrm{i})^v + (-1-\mathrm{i})^v + (1-\mathrm{i})^v}{\sin \pi v} \tag{3.16}$$

in (3.15) leads, in the strip $-1 < \mathrm{Re}(s) < 0$, to

$$\begin{bmatrix}\gamma(s)\\ \alpha(s)\end{bmatrix} = -\frac{1}{2}\left\{C_1(1+\mathrm{i})^{-s}\begin{bmatrix}\mathrm{i}\\ 1\end{bmatrix} + C_2(-1+\mathrm{i})^{-s}\begin{bmatrix}-\mathrm{i}\\ 1\end{bmatrix}\right\} + \frac{1}{4}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{F(u)}{\sin \pi(u-s)}\,\mathrm{d}u\left\{(-1-\mathrm{i})^{u-s-1}\begin{bmatrix}\mathrm{i}\\ 1\end{bmatrix} + (1-\mathrm{i})^{u-s-1}\begin{bmatrix}-\mathrm{i}\\ 1\end{bmatrix}\right\}, \tag{3.17}$$

in agreement with (3.7). This result can be obtained by manipulation of the trigonometric functions in

$$\begin{bmatrix}-2\mathrm{i}\\ 2\mathrm{i}\end{bmatrix}\frac{1}{\sin \pi(u-s)/4} + \begin{bmatrix}-2\\ 0\end{bmatrix}\frac{\sqrt{2}}{\sin \pi(u-s+1)/4} + \begin{bmatrix}\mathrm{i}\\ \mathrm{i}\end{bmatrix}\frac{2}{\sin \pi(u-s+2)/4} + \begin{bmatrix}0\\ 1\end{bmatrix}\frac{2\sqrt{2}}{\sin \pi(u-s+3)/4}. \tag{3.18}$$

To obtain this relation, we use the principal branch of the logarithm for the powers of $(\pm 1 \pm \mathrm{i})$.

Computing the solution numerically, e.g. for $\eta(x)$, is a two-step process. First we obtain C_1 and C_2 by enforcing the regularity condition $\gamma(-1/2) = \alpha(-1/2) = 0$ in (3.17): this gives two equations. Then $\eta(x)$ can be found by computing the inverse Mellin transform of (2.14) using (3.17) to obtain $\gamma(s-1)$. We shall not carry out this programme here. Instead we concentrate on the exponentially growing contribution: this is the most interesting physically and also requires care in its calculation.

The homogeneous terms in $\eta(x)$ from (3.17) are related to the following Mellin transform (which does not seem to appear in standard references):

$$\Gamma(s)\tan s\pi = \int_0^\infty \frac{2}{\sqrt{\pi}}F_D(\sqrt{r})r^{s-1}\,\mathrm{d}r, \tag{3.19}$$

in which $F_D(z) \equiv \mathrm{e}^{-z^2}\int_0^z \mathrm{e}^{t^2}\,\mathrm{d}t$ is Dawson's integral (Olver *et al.* 2010). The solution can also be written in terms of error functions of imaginary arguments, as in Orszag & Crow (1970, (4.4)). The growing homogeneous contribution to $\eta(x)$ comes from

$$\frac{(1-\mathrm{i})r_0}{2\pi}C_2[k(-1+\mathrm{i})]^{-s}\Gamma(s)\tan s\pi. \tag{3.20}$$

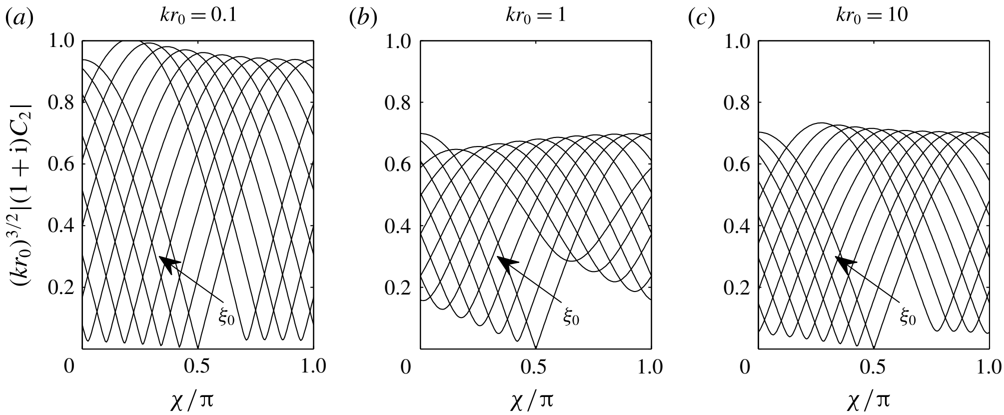

FIGURE 1. Scaled amplitude $(kr_0)^{-3/2}|(1+\mathrm{i})C_2|$. The range of ξ_0 is $(0, \pi/2)$, with ξ_0 increasing in the direction of the arrow.

This does not have an inverse Mellin transform. However, the analytic continuation procedure of Crighton & Leppington (1974) corresponds to taking the inverse Mellin transform formally, using the standard scaling transformation for inverse Mellin transforms of $F_D(s)a^{-s}$, which yields

$$\eta(x) \sim \frac{(1-\mathrm{i})r_0}{\pi^{3/2}} C_2 F_D(\sqrt{(-1+\mathrm{i})kx}), \tag{3.21}$$

which grows exponentially downstream along the vortex sheet. The dominant behaviour of $\eta(x)$ for large x can be obtained as

$$\eta(x) \sim \frac{(1+\mathrm{i})r_0}{2\pi} C_2 \mathrm{e}^{(1-\mathrm{i})kx}. \tag{3.22}$$

The amplitude of the Helmholtz mode is a function of k, U, r_0, ξ_0 and χ. Of these variables, U can be scaled out since the problem is linear, while kr_0 is the only non-dimensional length. There are hence three parameters: kr_0 and ξ_0 giving the polar coordinates of the dipole from the edge of the wedge and χ measuring the orientation of the dipole with respect to the horizontal. Figure 1 shows plots of the scaled amplitude of the Helmholtz mode, $(kr_0)^{-3/2}|(1+\mathrm{i})C_2|$, as a function of χ for certain values of kr_0 and $0 \leqslant \xi_0 \leqslant \pi/2$. The scaling $(kr_0)^{-3/2}$ for both small and large values of kr_0 comes from the dominant singularity of the integrand in (3.17), which is at $1/2$ for $kr_0 \ll 1$ and at $-1/2$ for $kr_0 \gg 1$. One can show that, up to the kr_0 factor, the integrals in (3.17) for large kr_0 differ by a factor of $\sqrt{2}$ compared to those for small kr_0. This gives a factor of $\sqrt{2}$ difference between figures 1(*a*) and 1(*c*). The amplitude of the mode is a smoothly varying function with a minimum and maximum for each value of ξ_0. For $\xi_0 = 0$, the amplitude actually vanishes at $\chi = \pi/2$.

4. The right-angled wedge

4.1. *Construction of solution*

The repeated use of (2.15) yields

$$\begin{bmatrix} \gamma(s-2) \\ \alpha(s-2) \end{bmatrix} - \begin{bmatrix} \sec s\pi & -2 \\ 2 & -\sec s\pi \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \alpha(s) \end{bmatrix} = \begin{bmatrix} -F(s) \\ \mathrm{i}F(s-1) - F(s) \end{bmatrix}, \tag{4.1}$$

which is a second-order system of period 2. We obtain a first-order equation of period 4 by observing that the matrix in (2.15) allows a solution of the form

$$\begin{aligned}\begin{bmatrix}\gamma(s)\\ \alpha(s)\end{bmatrix} &= D(s)(2-\sec s\pi)\begin{bmatrix}-\mathrm{i}\\ \mathrm{i}\end{bmatrix} + D(s-1)\begin{bmatrix}-2\\ -\sec s\pi\end{bmatrix}\\ &\quad + D(s-2)\begin{bmatrix}\mathrm{i}\\ \mathrm{i}\end{bmatrix} + D(s-3)\begin{bmatrix}0\\ 1\end{bmatrix},\end{aligned} \tag{4.2}$$

provided that

$$D(s-4) + (4-\sec^2 s\pi)D(s) = \mathrm{i}F(s). \tag{4.3}$$

We seek to reduce (4.3) to the form of (3.13) and hence introduce a function $H(s)$ that satisfies the difference equation

$$H(s-4) = (1-\tfrac{1}{4}\sec^2 s\pi)H(s). \tag{4.4}$$

It is shown in apendix A that a solution of (4.4) in integral form is given by

$$H(s) = \exp\left\{2\int_0^\infty \frac{\sinh\dfrac{t}{4}s\sinh^2\dfrac{t}{48}}{\sinh\dfrac{t}{2}\sinh\dfrac{t}{8}}\frac{\mathrm{d}t}{t}\right\}, \tag{4.5}$$

valid and analytic for $-7/3 < \mathrm{Re}(s) < 7/3$, which simply demonstrates that $H(s)H(-s) = 1$ and $H(s) \to \mathrm{e}^{\pm\mathrm{i}\pi/144}$ as $\mathrm{Im}(s) \to \pm\infty$ within this strip. See appendix A for the evaluation of the integral in (4.5). The pole structure of $H(s)$ is conveniently identified from its constituent Gamma functions which are shown in appendix A to yield

$$H(s) = \prod_{m=0}^{\infty}\left[\frac{\Gamma\left(\dfrac{5}{8}+\dfrac{s}{4}+\dfrac{m}{4}\right)}{\Gamma\left(\dfrac{5}{8}-\dfrac{s}{4}+\dfrac{m}{4}\right)}\right]^2 \frac{\Gamma\left(\dfrac{2}{3}-\dfrac{s}{4}+\dfrac{m}{4}\right)\Gamma\left(\dfrac{7}{12}-\dfrac{s}{4}+\dfrac{m}{4}\right)}{\Gamma\left(\dfrac{2}{3}+\dfrac{s}{4}+\dfrac{m}{4}\right)\Gamma\left(\dfrac{7}{12}+\dfrac{s}{4}+\dfrac{m}{4}\right)}. \tag{4.6}$$

This defines $H(s)$ in the whole complex s-plane with $H(0) = 1$. ‘Shifted’ values of $H(s)$ are obtained by noting, from (4.6) and the relation $\Gamma(z)\Gamma(1-z) = \pi\csc\pi z$, that

$$\frac{H(s+1/2)}{H(s-1/2)} = \frac{1+\cos s\dfrac{\pi}{2}}{\cos\dfrac{\pi}{12}+\cos s\dfrac{\pi}{2}}, \quad \frac{H(s+1)}{H(s-1)} = \frac{\left(\dfrac{1}{\sqrt{2}}+\cos s\dfrac{\pi}{2}\right)^2}{\left(\dfrac{\sqrt{3}}{2}+\cos s\dfrac{\pi}{2}\right)\left(\dfrac{1}{2}+\cos s\dfrac{\pi}{2}\right)}. \tag{4.7}$$

Since the solution (4.2) includes values of $D(s)$ in the strip $-4 < \mathrm{Re}(s) < 0$ when $-1 < \mathrm{Re}(s) < 0$, we note that $H(s+2)$ is regular in $-13/3 < \mathrm{Re}(s) < 1/3$ and arrange (4.3) in the standard form

$$\frac{D(s-4)}{H(s-2)} + 4\frac{D(s)}{H(s+2)} = \frac{\mathrm{i}F(s)}{H(s-2)}, \tag{4.8}$$

because $H(s)$ is a solution of (4.4). We mimic the solution (3.15) of (3.13) in deducing that, in the strip $-4 < \mathrm{Re}(s) < 0$,

$$\begin{aligned}\frac{D(s)}{H(s+2)} &= \frac{1}{16}\left\{C_1(1+\mathrm{i})^{1-s} + C_2(-1+\mathrm{i})^{1-s} + C_3(-1-\mathrm{i})^{1-s} + C_4(1-\mathrm{i})^{1-s}\right.\\ &\quad \left. + \frac{1}{2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} 2^{(u-s)/2}\frac{F(u)\,\mathrm{d}u}{H(u-2)\sin\pi(u-s)/4}\right\}. \end{aligned} \tag{4.9}$$

Then the substitution of (4.9) into (4.2) yields, in the strip $-1 < \mathrm{Re}(s) < 0$,

$$\begin{aligned}&\begin{bmatrix}\gamma(s)\\ \alpha(s)+\gamma(s)\end{bmatrix} = \frac{1}{16}\\
&\times\left\{C_1(1+\mathrm{i})^{1-s}\begin{bmatrix}-\mathrm{i}(2-\sec s\pi)H(s+2) - 2(1+\mathrm{i})H(s+1) - 2H(s)\\ -(2+\sec s\pi)(1+\mathrm{i})H(s+1) - 4H(s) + 2(-1+\mathrm{i})H(s-1)\end{bmatrix}\right.\\
&+ C_2(-1+\mathrm{i})^{1-s}\begin{bmatrix}-\mathrm{i}(2-\sec s\pi)H(s+2) - 2(-1+\mathrm{i})H(s+1) + 2H(s)\\ -(2+\sec s\pi)(-1+\mathrm{i})H(s+1) + 4H(s) + 2(1+\mathrm{i})H(s-1)\end{bmatrix}\\
&+ C_3(-1-\mathrm{i})^{1-s}\begin{bmatrix}-\mathrm{i}(2-\sec s\pi)H(s+2) - 2(-1-\mathrm{i})H(s+1) - 2H(s)\\ -(2+\sec s\pi)(-1-\mathrm{i})H(s+1) - 4H(s) + 2(1-\mathrm{i})H(s-1)\end{bmatrix}\\
&+ C_4(1-\mathrm{i})^{1-s}\begin{bmatrix}-\mathrm{i}(2-\sec s\pi)H(s+2) - 2(1-\mathrm{i})H(s+1) + 2H(s)\\ -(2+\sec s\pi)(1-\mathrm{i})H(s+1) + 4H(s) + 2(-1-\mathrm{i})H(s-1)\end{bmatrix}\\
&+ \frac{1}{2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{F(u)\,\mathrm{d}u}{H(u-2)\sin\pi(u-s)}\\
&\times\left((1+\mathrm{i})^{u-s}\begin{bmatrix}-\mathrm{i}(2-\sec s\pi)H(s+2) + 2(1+\mathrm{i})H(s+1) - 2H(s)\\ (2+\sec s\pi)(1+\mathrm{i})H(s+1) - 4H(s) - 2(-1+\mathrm{i})H(s-1)\end{bmatrix}\right.\\
&+ (-1+\mathrm{i})^{u-s}\begin{bmatrix}-\mathrm{i}(2-\sec s\pi)H(s+2) + 2(-1+\mathrm{i})H(s+1) + 2H(s)\\ (2+\sec s\pi)(-1+\mathrm{i})H(s+1) + 4H(s) - 2(1+\mathrm{i})H(s-1)\end{bmatrix}\\
&+ (-1-\mathrm{i})^{u-s}\begin{bmatrix}-\mathrm{i}(2-\sec s\pi)H(s+2) + 2(-1-\mathrm{i})H(s+1) - 2H(s)\\ (2+\sec s\pi)(-1-\mathrm{i})H(s+1) - 4H(s) - 2(1-\mathrm{i})H(s-1)\end{bmatrix}\\
&\left.\left. + (1-\mathrm{i})^{u-s}\begin{bmatrix}-\mathrm{i}(2-\sec s\pi)H(s+2) + 2(1-\mathrm{i})H(s+1) + 2H(s)\\ (2+\sec s\pi)(1-\mathrm{i})H(s+1) + 4H(s) - 2(-1-\mathrm{i})H(s-1)\end{bmatrix}\right)\right\}.\end{aligned} \tag{4.10}$$

4.2. *Behaviour for large* Im(s)

In the limits $\mathrm{Im}(s) \to \pm\infty$, $\sec s\pi \to 0$ and $H(s) \sim \mathrm{e}^{\pm \mathrm{i}\pi/144}$ and only values of u near s are significant in the integral. Thus (4.10) reduces to (3.17), as expected, because (4.2) and (4.3) then reduce to (3.3) and (3.13) respectively. The vectors multiplying C_3, C_4 in (4.10) are $\mathbf{0}$ at leading order.

It is readily shown from (4.7) that, with $s = \xi + \mathrm{i}\eta$,

$$\frac{H(s+1)}{H(s)} = 1 + O(\mathrm{e}^{-|\eta|\pi/2}) \quad \text{as } |\eta| \to \infty \tag{4.11}$$

in the strip of regularity. Set $k = k_0\mathrm{e}^{-\mathrm{i}\beta}$ and note that

$$\Gamma(\xi + \mathrm{i}\eta) = O(|\eta|^{\xi-1/2}\mathrm{e}^{-|\eta|\pi/2}) \quad \text{as } |\eta| \to \infty \tag{4.12}$$

(Olver *et al.* 2010). Then, for exponential decay,

$$\frac{\Gamma(s)}{k^s(1+\mathrm{i})^s} = O[|\eta|^{\xi-1/2}\mathrm{e}^{-(\beta-\pi/4)\eta}\mathrm{e}^{-|\eta|\pi/2}] \text{ requires } -\frac{\pi}{4} < \beta < \frac{3\pi}{4}, \tag{4.13a}$$

$$\frac{\Gamma(s)}{k^s(-1+\mathrm{i})^s} = O[|\eta|^{\xi-1/2}\mathrm{e}^{-(\beta-3\pi/4)\eta}\mathrm{e}^{-|\eta|\pi/2}] \text{ requires } \frac{\pi}{4} < \beta < \frac{5\pi}{4}, \tag{4.13b}$$

$$\frac{\Gamma(s)\mathrm{e}^{-|\eta|\pi/2}}{k^s(-1-\mathrm{i})^s} = O[|\eta|^{\xi-1/2}\mathrm{e}^{-(\beta+3\pi/4)\eta}\mathrm{e}^{-|\eta|\pi}] \text{ requires } -\frac{7\pi}{4} < \beta < \frac{\pi}{4}, \tag{4.13c}$$

$$\frac{\Gamma(s)\mathrm{e}^{-|\eta|\pi/2}}{k^s(1-\mathrm{i})^s} = O[|\eta|^{\xi-1/2}\mathrm{e}^{-(\beta+\pi/4)\eta}\mathrm{e}^{-|\eta|\pi}] \text{ requires } -\frac{5\pi}{4} < \beta < \frac{3\pi}{4}, \tag{4.13d}$$

and the choice $\beta = \pi/4$ ensures exponential decay in two cases, but in the other two relies on algebraic decay due to the additional $|\eta|^{\xi-1/2}$ factor with $-1 < \xi < 0$ in the strip of regularity.

4.3. *Determination of the coefficients*

The result (4.5) confirms that $H(s+2)$, $H(s+1)$, $H(s)$ and $H(s-1)$ have the common strip of analyticity $-4/3 < \mathrm{Re}(s) < 1/3$ and therefore the only pole in (4.10) arises from $\sec s\pi$. Inspection of the Mellin transforms, given by (2.13*a*), (2.13*b*), shows that $\gamma(s)\sec s\pi$ and $\alpha(s)\sec s\pi$ are regular at $s=-1/2$, as required to eliminate terms of order $r^{1/2}$ from the small-r expansions of ϕ_2 and ϕ_1. As in the semi-infinite plate case, the first equation in (2.15) then implies that $\gamma(-3/2)=0$ and hence, according to (2.14), satisfies the full-Kutta condition by proscribing an $x^{1/2}$ term in $\eta(x)$.

The poles at $s=-1/2$ are eliminated by choice of the arbitrary constants, $C_j(1 \leqslant j \leqslant 4)$ in (4.9) by demanding, in (4.2),

$$\left.\begin{aligned} D(-1/2) = 0 = D(-3/2), \quad & \frac{1}{\pi}D'(-1/2) + D(-5/2) = 0, \\ \frac{1}{\pi}D'(-3/2) = 2\mathrm{i}D(-5/2) + D(-7/2). & \end{aligned}\right\} \tag{4.14}$$

The first two of these conditions facilitate the evaluation of the derivatives without knowledge of $H'(3/2), H'(1/2)$. It is readily deduced from (4.7) that

$$\left.\begin{aligned} H(1/2) = \sec\frac{\pi}{24} \quad & H(-1/2) = \cos\frac{\pi}{24}, \\ H(3/2) = \sec\frac{\pi}{12}\sec\frac{\pi}{24} \quad & H(-3/2) = \cos\frac{\pi}{12}\cos\frac{\pi}{24}. \end{aligned}\right\} \tag{4.15}$$

For algebraic brevity, write (4.9) as

$$\begin{aligned} \frac{D(s)}{H(s+2)} &= B_1(1+\mathrm{i})^{-(s+1/2)} + B_2(-1+\mathrm{i})^{-(s+1/2)} + B_3(-1-\mathrm{i})^{-(s+1/2)} \\ &\quad + B_4(1-\mathrm{i})^{-(s+1/2)} + 2^{-(s+1/2)/2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{G(u)\,\mathrm{d}u}{\sin\pi(u-s)/4}, \end{aligned} \tag{4.16}$$

where

$$[C_1, C_2, C_3, C_4] = 16\left[B_1(1+\mathrm{i})^{-3/2}, B_2(-1+\mathrm{i})^{-3/2}, B_3(-1-\mathrm{i})^{-3/2}, B_4(1-\mathrm{i})^{-3/2}\right], \tag{4.17}$$

and

$$F(u) = 32 \times 2^{-(u+1/2)/2}G(u)H(u-2). \tag{4.18}$$

Then substitution of (4.15), (4.16) into (4.14) gives

$$B_1+B_2+B_3+B_4+\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{G(u)\,\mathrm{d}u}{\sin\pi(u+1/2)/4}=0, \tag{4.19}$$

$$\begin{aligned}&B_1(1+\mathrm{i})+B_2(-1+\mathrm{i})+B_3(-1-\mathrm{i})+B_4(1-\mathrm{i})\\&\quad+2^{1/2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{G(u)\,\mathrm{d}u}{\sin\pi(u+3/2)/4}=0,\end{aligned} \tag{4.20}$$

$$\begin{aligned}&2\cos\frac{\pi}{12}\left[B_1-B_2+B_3-B_4-\mathrm{i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{G(u)\,\mathrm{d}u}{\cos\pi(u+1/2)/4}\right]\\&=\frac{1}{4}\sec^2\frac{\pi}{24}\left[B_1+3B_2-3B_3-B_4\right.\\&\quad\left.+\mathrm{i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{G(u)\,\mathrm{d}u}{\sin\pi(u+1/2)/4}\cot\pi(u+1/2)/4\right],\end{aligned} \tag{4.21}$$

$$\begin{aligned}&4\left[-B_1+B_2-B_3+B_4+\mathrm{i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{G(u)\,\mathrm{d}u}{\cos\pi(u+1/2)/4}\right]\\&\quad+2\cos\frac{\pi}{12}\left[B_1(-1+\mathrm{i})+B_2(1+\mathrm{i})+B_3(1-\mathrm{i})+B_4(-1-\mathrm{i})\right.\\&\quad\left.+2^{1/2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{G(u)\,\mathrm{d}u}{\cos\pi(u+3/2)/4}\right]\\&=\frac{1}{4}\sec^2\frac{\pi}{24}\left[B_1(1-\mathrm{i})+3B_2(1+\mathrm{i})+3B_3(1-\mathrm{i})+B_4(1+\mathrm{i})\right.\\&\quad\left.+2^{1/2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{G(u)\,\mathrm{d}u}{\sin\pi(u+3/2)/4}\cot\pi(u+3/2)/4\right].\end{aligned} \tag{4.22}$$

4.4. *Calculation of the exponentially growing contribution*

Once again we focus on the exponentially growing part of the solution, which comes from the terms containing $(-1\pm\mathrm{i})^{-s}$ in (4.10). However, the term in C_3 is not needed because it vanishes to leading order as described in § 4.2. The computation of the displacement is more involved than for the half-plane case because of the presence of the function $H(s)$. First (4.19)–(4.22) are solved for B_1, B_2, B_3 and B_4. Note that the required integrals exist when k is real. Then C_2 is obtained and used in (4.10) to obtain $\gamma(s)$. This is then used in the expression for $N(s)$, which is inverse Mellin transformed. The condition $\beta=\pi/4$ is necessary for the exponentially growing terms to be Mellin transform. While we could carry out the inverse transform analytically for the semi-infinite plate, this is no longer possible here. To avoid having to use β in the evaluation of the inverse Mellin transform, we proceed as follows.

We write for the Mellin transform of the growing homogeneous solution involving C_2

$$\gamma_H(s-1)=-\frac{\mathrm{i}}{8}C_2(-1+\mathrm{i})^{-s}H(s)[4-4\mathrm{i}+l(s)], \tag{4.23}$$

where

$$l(s)=-2+2\mathrm{i}-\mathrm{i}(2+\sec s\pi)\frac{H(s+1)}{H(s)}+2r_-(s)\frac{H(s-1)}{H(s)} \tag{4.24}$$

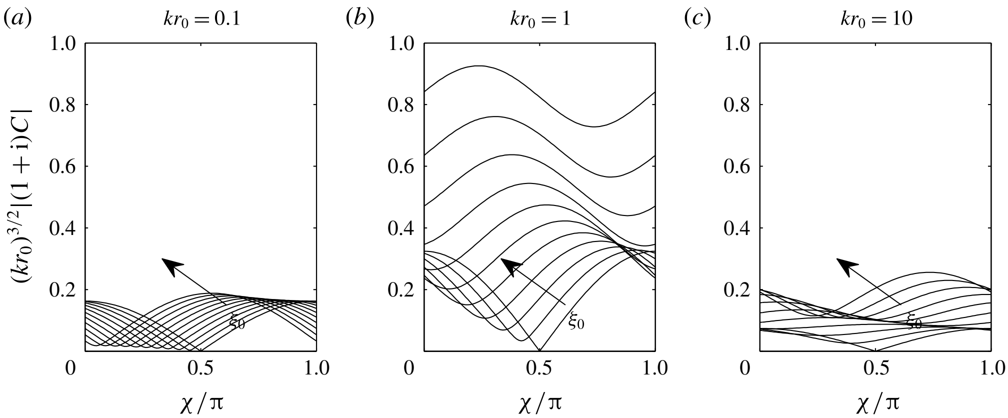

FIGURE 2. Scaled amplitude $(kr_0)^{-3/2}|(1+\mathrm{i})C_2|$. The range of ξ_0 is $(0, \pi/2)$, with ξ_0 increasing in the direction of the arrow.

decays as $\mathrm{Im}(s) \to \infty$. Our aim is to separate the inverse Mellin transform of $N(s)$ into two parts: one that can be done analytically and the other which has a Mellin transform for $\beta = 0$. Since $H(s) \sim \mathrm{e}^{\pm \mathrm{i}\pi/144}$ as $\mathrm{Im}(s) \to \pm\infty$, we can decompose $H(s)\tan s\pi$ as

$$\begin{aligned} H(s)\tan s\pi &= [H(s)\tan s\pi - \cos(\pi/144)\tan s\pi + \sin(\pi/144)] \\ &\quad + \cos(\pi/144)\tan s\pi - \sin(\pi/144) = H_1(s) + H_2(s). \end{aligned} \tag{4.25}$$

As $\mathrm{Im}(s) \to \pm\infty$, $H_1(s)$ decays exponentially. The Mellin transform of the exponentially growing part of the interface displacement reduces to two parts:

$$N_H(s) = N(s) = \frac{2r_0}{\pi k^s} C_2(-1+\mathrm{i})^{-s}\Gamma(s)[H_1(s) + H_2(s)][4 - 4\mathrm{i} + l(s)]. \tag{4.26}$$

The growing part comes from the $H_2(s)(4-4\mathrm{i})$ term and can be carried out by hand, as in (3.22), giving

$$\eta(x) \sim \frac{2r_0}{\pi} C_2(4-4\mathrm{i}) \left[\cos(\pi/144)\frac{2}{\sqrt{\pi}} F_D(\sqrt{(-1+\mathrm{i})kx}) - \sin(\pi/144)\mathrm{e}^{(1-\mathrm{i})kx}\right]. \tag{4.27}$$

Figure 2 shows plots of the scaled amplitude of the Helmholtz mode, $(kr_0)^{-3/2}|(1+\mathrm{i})C_2|$, as a function of χ for certain values of kr_0 and $0 \leqslant \xi_0 \leqslant \pi/2$. The scaling $(kr_0)^{-3/2}$ persists for both small and large values of kr_0. The amplitude of the mode is once again smoothly varying, vanishing at $\chi = \pi/2$ when $\xi_0 = 1$, but varies more than for the semi-infinite plate. In particular the similarity between small and large kr_0 has gone, since $H(s)$ differs between $s = -1/2$ and $s = 1/2$. For large kr_0, the largest amplitude is found for values of χ around $3\pi/4$.

5. Conclusion

We have obtained the linearised evolution of the vortex sheet behind a right-angled wedge using the Mellin transform. This leads to a difference equation that can be solved using an appropriate integral representation. The fourth roots of -4 lead to complementary functions whose amplitudes are fixed by regularity conditions for the Mellin transform corresponding to the full-Kutta condition at the corner. For the semi-infinite plate, the amplitude of the single Helmholtz mode for the vortex sheet displacement can be obtained in analytic form: applying the necessary Mellin shift theorem involves an implicit use of the analytic continuation procedure discussed by Crighton & Leppington (1974). For the right-angled wedge, this procedure requires breaking up the Mellin transform into different contributions, one of which has a similar form and can be computed explicitly. In both cases, the amplitude of the growing mode is a function of the governing parameter of the problem (vanishing when the dipole has vertical orientation and lies along the x-axis), with the form $\exp[(1-\mathrm{i})kx]$ for $x \gg 1$.

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Appendix A. Homogeneous solution

The bracketed function in (4.4) has the factorization

$$1-\frac{1}{4}\sec^2 s\pi = \frac{\cos 2s\pi + \dfrac{1}{2}}{2\cos^2 s\pi} = \frac{\cos\left(s-\dfrac{1}{6}\right)\pi\cos\left(s+\dfrac{1}{6}\right)\pi}{\cos^2 s\pi}. \tag{A 1}$$

Equations of this type are solved by Lawrie & King (1994) in terms of the Barnes double Gamma function (Barnes 1899) defined by

$$G(z+1,\delta) = \Gamma\left(\frac{z}{\delta}\right)G(z,\delta), \quad G(1,\delta)=1. \tag{A 2}$$

For earlier examples, see Williams (1959) and Lawrie (1990). Thus

$$\sin\left(\frac{\pi z}{\delta}\right) = \frac{\pi G(z,\delta)G(\delta-z,\delta)}{G(1+z,\delta)G(1+\delta-z,\delta)}, \tag{A 3a}$$

$$\cos\left(\frac{\pi z}{\delta}\right) = \frac{\pi G(\delta/2+z,\delta)G(\delta/2-z,\delta)}{G(1+\delta/2+z,\delta)G(1+\delta/2-z,\delta)}. \tag{A 3b}$$

Then, with $s=4S$, $E(s)=f(S)$, the difference equation (4.4) may be written as

$$\frac{f(S-1)}{f(S)} = \left[\frac{G\left(\dfrac{9}{8}+S,\dfrac{1}{4}\right)G\left(\dfrac{9}{8}-S,\dfrac{1}{4}\right)}{G\left(\dfrac{1}{8}+S,\dfrac{1}{4}\right)G\left(\dfrac{1}{8}-S,\dfrac{1}{4}\right)}\right]^2 \times\frac{G\left(\dfrac{1}{6}+S,\dfrac{1}{4}\right)G\left(\dfrac{1}{6}-S,\dfrac{1}{4}\right)G\left(\dfrac{1}{12}+S,\dfrac{1}{4}\right)G\left(\dfrac{1}{12}-S,\dfrac{1}{4}\right)}{G\left(\dfrac{7}{6}+S,\dfrac{1}{4}\right)G\left(\dfrac{7}{6}-S,\dfrac{1}{4}\right)G\left(\dfrac{13}{12}+S,\dfrac{1}{4}\right)G\left(\dfrac{13}{12}-S,\dfrac{1}{4}\right)}, \tag{A 4}$$

from which it is easy to deduce that $E(s-2)$ is a multiple, of period 4 in s, of

$$H(s)=\left[\frac{G\left(\frac{5}{8}-\frac{s}{4},\frac{1}{4}\right)}{G\left(\frac{5}{8}+\frac{s}{4},\frac{1}{4}\right)}\right]^2\frac{G\left(\frac{2}{3}+\frac{s}{4},\frac{1}{4}\right)G\left(\frac{7}{12}+\frac{s}{4},\frac{1}{4}\right)}{G\left(\frac{2}{3}-\frac{s}{4},\frac{1}{4}\right)G\left(\frac{7}{12}-\frac{s}{4},\frac{1}{4}\right)}. \tag{A 5}$$

Lawrie & King (1994) give an integral representation for $\ln G(z,\delta)$ which enables (A 5) to be reduced to (4.5). Of broader help is a Barnes-derived formula for $G(z,\delta)$ which yields

$$\frac{G(\beta-z,\delta)}{G(\beta+z,\delta)}=\mathrm{e}^{-2z[A(\delta)+\beta B(\delta)]}\frac{\Gamma(\beta+z)}{\Gamma(\beta-z)}\Pi_{m=1}^{\infty}\frac{\Gamma(\beta+z+m\delta)}{\Gamma(\beta-z+m\delta)}\mathrm{e}^{-2z[\psi(m\delta)+\beta\psi'(m\delta)]}. \tag{A 6}$$

The functions $A(\delta)$, $B(\delta)$ and the Digamma function ψ cancel in (A 5) and the resulting ratios of double Gamma functions yield (4.6).

From (4.5), with $|\mathrm{Re}(s)|<7/3$,

$$\ln H(s)=\int_0^\infty\frac{\sinh\frac{t}{4}s\left(\cosh\frac{t}{24}-1\right)}{\sinh\frac{t}{2}\sinh\frac{t}{8}}\frac{\mathrm{d}t}{t}=PV\int_0^\infty\frac{\exp\left(\frac{t}{4}s\right)\left(\cosh\frac{t}{24}-1\right)}{2\sinh\frac{t}{2}\sinh\frac{t}{8}}\frac{\mathrm{d}t}{t}, \tag{A 7}$$

which is an odd function of s. For $\mathrm{Im}(s)\geqslant 0$, close the contour in the upper half-plane with an indentation at $t=0$, whence (A 7) gives

$$\ln H(s)=\frac{\mathrm{i}\pi}{144}+\mathrm{i}\pi\sum_{n=1}^{\infty}\text{residues at }2n\mathrm{i}\pi(4n-3,4n-2,4n-1,4n). \tag{A 8}$$

We now evaluate the different residues in turn. First

$$\begin{aligned}\mathrm{i}\pi\sum_{n=1}^{\infty}R(4n-3,4n-1) &= \frac{1}{2\mathrm{i}}\left\{S\left[\exp\left(\frac{\mathrm{i}\pi}{2}(s+1/6)\right)\right]+S\left[\exp\left(\frac{\mathrm{i}\pi}{2}(s-1/6)\right)\right]\right.\\ &\qquad\left.-2S\left[\exp\left(\frac{\mathrm{i}\pi s}{2}\right)\right]\right\},\end{aligned} \tag{A 9}$$

where (for $|\alpha|<1$)

$$\begin{aligned}S(\alpha) &= \sqrt{2}\sum_{n=1}^{\infty}(-1)^n\left[\frac{\alpha^{4n-3}}{4n-3}+\frac{\alpha^{4n-1}}{4n-1}\right]\\ &= \frac{\mathrm{i}}{2}\ln\left(\frac{1-\alpha^2+\mathrm{i}\alpha\sqrt{2}}{1-\alpha^2-\mathrm{i}\alpha\sqrt{2}}\right)=-\arctan\left(\frac{\alpha\sqrt{2}}{1-\alpha^2}\right).\end{aligned} \tag{A 10}$$

Then

$$
\begin{aligned}
&\mathrm{i}\pi\sum_{n=1}^{\infty}R(4n-3,4n-1)\\
&=\frac{1}{4}\ln\left[\frac{-\sin\dfrac{\pi}{2}\left(s+\dfrac{1}{6}\right)+\sin\dfrac{\pi}{4}}{-\sin\dfrac{\pi}{2}\left(s+\dfrac{1}{6}\right)-\sin\dfrac{\pi}{4}}\right]+\frac{1}{4}\ln\left[\frac{-\sin\dfrac{\pi}{2}\left(s-\dfrac{1}{6}\right)+\sin\dfrac{\pi}{4}}{-\sin\dfrac{\pi}{2}\left(s-\dfrac{1}{6}\right)-\sin\dfrac{\pi}{4}}\right]\\
&\quad-\frac{1}{2}\ln\left[\frac{-\sin\dfrac{\pi}{2}s+\sin\dfrac{\pi}{4}}{-\sin\dfrac{\pi}{2}s-\sin\dfrac{\pi}{4}}\right] \qquad \text{(A 11)}\\
&=\frac{1}{4}\ln\left[\frac{\sin\dfrac{\pi}{4}-\sin\dfrac{\pi}{2}\left(s+\dfrac{1}{6}\right)}{\sin\dfrac{\pi}{4}+\sin\dfrac{\pi}{2}\left(s+\dfrac{1}{6}\right)}\right]+\frac{1}{4}\ln\left[\frac{\sin\dfrac{\pi}{4}-\sin\dfrac{\pi}{2}\left(s-\dfrac{1}{6}\right)}{\sin\dfrac{\pi}{4}+\sin\dfrac{\pi}{2}\left(s-\dfrac{1}{6}\right)}\right]\\
&\quad-\frac{1}{2}\ln\left[\frac{\sin\dfrac{\pi}{4}-\sin\dfrac{\pi}{2}s}{\sin\dfrac{\pi}{4}+\sin\dfrac{\pi}{2}s}\right], \qquad \text{(A 12)}
\end{aligned}
$$

in which the first form is used for $\mathrm{Im}(s)\to\infty$ and the second for $s\to 0$. Continuing in the same vein

$$
\begin{aligned}
\mathrm{i}\pi\sum_{n=1}^{\infty}R(4n-2) &= \frac{1}{2i}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\mathrm{e}^{\mathrm{i}\pi s(2n-1)}\left[\cos\frac{\pi}{6}(2n-1)-1\right] &\text{(A 13)}\\
&= -\frac{1}{8}\ln\left[\frac{1+\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(s+(1/6))}}{1-\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(s+(1/6))}}\right]-\frac{1}{8}\ln\left[\frac{1+\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(s-(1/6))}}{1-\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(s-(1/6))}}\right]+\frac{1}{4}\ln\left[\frac{1+\mathrm{i}\mathrm{e}^{\mathrm{i}\pi s}}{1-\mathrm{i}\mathrm{e}^{\mathrm{i}\pi s}}\right] &\text{(A 14)}\\
&= -\frac{1}{8}\ln\left[\left(\frac{-\sin\pi s+\sin\dfrac{\pi}{3}}{-\sin\pi s-\sin\dfrac{\pi}{3}}\right)\left(\frac{-\sin\pi s-\sin\dfrac{\pi}{2}}{-\sin\pi s+\sin\dfrac{\pi}{2}}\right)\right] &\text{(A 15)}\\
&= -\frac{1}{8}\ln\left[\left(\frac{\sin\dfrac{\pi}{3}-\sin\pi s}{\sin\dfrac{\pi}{3}+\sin\pi s}\right)\left(\frac{\sin\dfrac{\pi}{2}+\sin\pi s}{\sin\dfrac{\pi}{2}-\sin\pi s}\right)\right], &\text{(A 16)}
\end{aligned}
$$

in which the first form is used for $\mathrm{Im}(s)\to\infty$ and the second for $s\to 0$. A more lengthy calculation at the double poles yields

$$
\begin{aligned}
\mathrm{i}\pi\sum_{n=1}^{\infty}R(8n\mathrm{i}\pi) &= \frac{\mathrm{i}}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2}\mathrm{e}^{2n\mathrm{i}\pi s}\left(\cos\frac{n\pi}{3}-1\right)\\
&\quad+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}\left[\left(s+\frac{1}{6}\right)\mathrm{e}^{2n\mathrm{i}\pi(s+(1/6))}+\left(s-\frac{1}{6}\right)\mathrm{e}^{2n\mathrm{i}\pi(s-(1/6))}-2s\mathrm{e}^{2n\mathrm{i}\pi s}\right]
\end{aligned}
\qquad \text{(A 17)}
$$

$$
\begin{aligned}
&= \frac{\mathrm{i}}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\mathrm{e}^{2n\mathrm{i}\pi s}\left(\cos\frac{n\pi}{3}-1\right)-\frac{\mathrm{i}\pi}{72} \\
&\quad -\frac{s}{4}\ln\left(\frac{\cos 2\pi s+\cos\dfrac{\pi}{3}}{\cos 2\pi s+1}\right)-\frac{1}{24}\ln\left[\frac{\cos\pi\left(s+\dfrac{1}{6}\right)}{\cos\pi\left(s-\dfrac{1}{6}\right)}\right].
\end{aligned}
\tag{A 18}
$$

The first summation in (A 18) has the exact value $\mathrm{i}\pi/144$ at $s=0$ and the last term tends to $\mathrm{i}\pi/72$ as $\mathrm{Im}(s)\to\infty$. When (A 12)–(A 18) are substituted into (A 8), it may be verified that apparent branch points at $s=\pm 1/3$, $\pm 1/2$, $\pm 2/3$, $\pm 4/3$, $\pm 3/2$, $\pm 5/3$ are non-existent, as required for $H(s)$ to be analytic for $-7/3<\mathrm{Re}(s)<7/3$. The leading asymptotic correction to $H(s)$ as $\mathrm{Im}(s)\to\infty$ is the contribution to (A 12) from the residue at $t=2\mathrm{i}\pi$.

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