Internal gravity waves, boundary integral equations and radiation conditions

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Abstract

Three-dimensional time-harmonic internal gravity waves are generated by oscillating a bounded object (or by scattering from a fixed object) in a stratified fluid. Energy is found in conical wave beams: the problem is to calculate the wave fields for an object of arbitrary shape. An integral formula for the pressure is derived, using a reciprocal theorem and a Green's function. The boundary integrals are singular: their integrands are infinite along a certain curve (not just at a point) on the boundary, and this happens even when the field point is off the boundary (but within one of the conical wave beams). This is very different to the situation with classical potential theory (Laplace's equation) or linear acoustics (Helmholtz's equation), and is a consequence of the hyperbolic nature of the governing partial differential equation. The boundary integrals are identified as single-layer and double-layer potentials. A method is given for calculating the far field of these potentials. It is verified by comparing with known solutions for spherical objects.

Key words: Boundary integral equations; internal waves; radiation conditions; asymptotics

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1 Introduction

Internal gravity waves are generated by oscillating bodies in density stratified fluids. For a uniform stratification, giving a constant Brunt–Väisälä frequency, N, the significant wave motion is confined to beams forming a "Saint Andrew's cross" (in two dimensions), as shown in famous images obtained by Mowbray and Rarity [19]: for reprints, see [13, p. 44], [14, p. 314] or [25, p. 668]; the last of these also shows waves generated by a large oscillating cylinder. Internal gravity waves can also be generated by the scattering of the barotropic tide in the oceans [7], and are then known as the baroclinic tide.

The governing equations are well known. For three-dimensional time-harmonic motions (frequency ω) of an incompressible inviscid fluid with no rotation, the pressure p solves

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \frac{\omega^2}{N^2 - \omega^2} \frac{\partial^2 p}{\partial z^2} = 0, \tag{1.1}$$

where z is the vertical coordinate and $0 < \omega < N$. Equation (1.1) is a hyperbolic partial differential equation and it is to be solved subject to boundary and far-field conditions. The boundary conditions are clear: prescribed normal velocity on rigid boundaries and zero pressure on free surfaces. The far-field conditions are less clear, but they have been reviewed thoroughly by Voisin [28]. He identifies several approaches for imposing "radiation conditions". One is to require causality in the time domain, which then implies certain analyticity conditions in the complex ω-plane. This approach was used by Pierce [22], Hurley [10] and others, and it will be used later in Section 5. Another approach is to look at the waves themselves, requiring that they be outgoing: in linear acoustics, this would be recognised as the Sommerfeld radiation condition. However, for internal gravity waves, the phase velocity is perpendicular to the group velocity: physically, we may expect energy to travel away from the source at the group velocity, and this could be stated as a radiation condition. A difficulty with such a condition is that energy is a quadratic quantity: it does not seem obvious that linear combinations of such outgoing-energy solutions will also be outgoing.

There are several papers on the generation of internal gravity waves by spheres. The main approach has been as follows: start with Eq. (1.1) when $\omega > N$ (so that Eq. (1.1) is elliptic), scale the z-coordinate so that a boundary-value problem for Laplace's equation exterior to a spheroid is obtained; solve this problem by separation of variables; finally, effect the Pierce–Hurley analytic continuation to obtain the solution for $\omega < N$. See, for example, [9,12,1,28,5,30]. In Appendix A, we use this approach for three problems (pulsating sphere, vertical oscillations and a combination of these two modes), reviewing and extending previous work.

For more complicated body geometries, it is natural to try developing methods that use boundary integral equations, methods that have proved to be very effective for potential flow problems and for acoustic scattering problems [17]. Sturova [26], working in two dimensions, starts by writing

$$p(x, z) = \int_S \mu(x', z')\mathcal{G}(x', z'; x, z)\,\mathrm{d}S(x', z') \tag{1.2}$$

for points (x, z) in the fluid, where S is the surface of the body (a cylinder) and $\mathcal{G}$ is an appropriate Green's function (fundamental solution). Then, application of the boundary condition on S yields an integral equation for the function μ.

Equation (1.2) defines a *single-layer potential*. Similar representations were first used in the context of internal waves by Robinson [23], who considered a thin vertical barrier in a finite-depth ocean and constructed $\mathcal{G}$ so as to satisfy boundary conditions at $z = 0$ and $z = H$. Similar methods have been used for barriers [15,20] and for other two-dimensional bottom topographies [21,2,4]. All of these papers use representations of the stream function as a single-layer potential, leading to a first-kind integral equation. Analogous representations using double-layer potentials (involving the normal derivative of $\mathcal{G}$) could be used. Similar approaches could be developed for three-dimensional problems.

For time-dependent problems, with prescribed initial conditions, the situation is a little simpler: by causality, there can be no motion far away. There is an extensive Russian literature on such problems, using a variety of layer potentials. See, for example, [24] and papers by Gabov and his collaborators; we mention two [6,11] in which three-dimensional problems are analysed.

It is implicit when using representations such as (1.2) that any linear combination of radiating Green's functions (constructed by the Pierce–Hurley method) is itself radiating. In linear acoustics, this is true: in that context, single-layer and double-layer potentials always generate fields that satisfy the Sommerfeld radiation condition, for any choice of the function μ. However, in the context of internal gravity waves, we do not have a precise condition to impose on p. For this reason, we give a method for estimating the far field: it is not straightforward, but we verify that it gives the correct results for two sphere problems (as presented in Appendix A).

We start the paper by setting up the governing equations in Section 2. We derive a general reciprocal theorem, connecting two time-harmonic pressure fields, in Section 3; this permits fluid rotation. The reciprocal theorem is used in Section 4 to obtain representation formulas (in the absence of rotation): these give the pressure in the fluid in terms of boundary integrals over S of p, the normal velocity and a Green's function, G. The Pierce–Hurley analytic continuation of G is discussed in Section 5.

The boundary integrals are singular: their integrands are infinite along a certain curve (not just at a point) on the boundary S, and this happens even when the field point is off the boundary (but within one of the conical wave beams). This is very different to the situation with classical potential theory (Laplace's equation) or linear acoustics (Helmholtz equation), and is a consequence of the hyperbolic nature of (1.1) when $0 < \omega < N$.

The analysis of the far field is given in Section 7. The main idea is to write the boundary integrals as a double integral over a region $\mathcal{E}$ in the $\Theta\Phi$-plane, where Θ and Φ are certain spherical polar coordinates. This unusual choice is made because the singularities occur along the straight line $\Theta = \theta_c$ (which passes through $\mathcal{E}$) and so they can be handled by one-dimensional calculations. (The Φ integrations are benign.) The angle θ_c is defined by $\omega = N \cos\theta_c$. In addition, as the observation point recedes to infinity within the wave beams, the domain $\mathcal{E}$ shrinks so that approximations can be made. Eventually, expressions for the far fields of single-layer and double-layer potentials are obtained. Some consequences of these results are given in Section 8, with concluding remarks in Section 9.

The main contributions of the paper are as follows. First, there is the general reciprocal theorem (Section 3), relating pressure and velocity fields. Next, there are the integral representations in terms of single-layer and double-layer potentials (Section 4); these will provide a basis for the development of boundary integral methods. Then, a new method is given (Section 7) for calculating the far-field behaviour of layer potentials. (Most of the details of this mathematical technique are relegated to an appendix.) The method is applied to specific problems for spheres, and some observations on energy transport are made.

2 Mathematical formulation

We take the ocean to be a variable density fluid rotating with uniform frequency about the vertical axis. We model this situation with the Boussinesq equations [3, Section 11.2], [27, Section 2.4.2]. In their linearized form, they are as follows:

$$\partial \boldsymbol{v}/\partial t + \boldsymbol{f} \times \boldsymbol{v} = -\operatorname{grad} p + b\hat{\mathbf{z}}, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{v} = 0, \quad \partial b/\partial t + N^2 w = 0. \tag{2.2}$$

Here, we have Cartesian coordinates $Oxyz$, with z pointing upwards; $\hat{\mathbf{z}}$ is a unit vector in the z-direction. The velocity is $\boldsymbol{v} = (u, v, w)$ and $\boldsymbol{f} = (0, 0, f)$ is a given constant vector; f is the Coriolis frequency. The excess pressure is $\rho_0 p$, where ρ_0 is the constant background density. The buoyancy frequency,

$N(z)$, is positive and b is the buoyancy.

The basic unknowns are u, v, w, p and b. Eliminating u, v and b gives

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial w}{\partial z} = \frac{\partial}{\partial t}\nabla_H^2 p, \tag{2.3}$$

$$\left(\frac{\partial^2}{\partial t^2} + N^2\right) w = -\frac{\partial^2 p}{\partial z\,\partial t}, \tag{2.4}$$

where $\nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian. From these, a single equation for w can be obtained [3, eqn (11.14)], but we shall not need it.

The local energy is defined by $E = E_{\text{ke}} + E_{\text{pe}}$ with $E_{\text{ke}} = \frac{1}{2}\rho_0 \boldsymbol{v}\cdot\boldsymbol{v}$ (kinetic energy) and $E_{\text{pe}} = \frac{1}{2}\rho_0 b^2/N^2$ (potential energy). We have $\partial E/\partial t = -\operatorname{div}\mathbf{I}$, where $\mathbf{I} = \rho_0 p\boldsymbol{v}$ is known as the energy transport vector (recall that $\rho_0 p$ is the excess pressure). Integrating over a fixed volume V, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V E\,\mathrm{d}V = -\int_{\mathcal{S}} \mathbf{I}\cdot\boldsymbol{n}\,\mathrm{d}S,$$

where $\mathcal{S}$ is the boundary of V and $\boldsymbol{n}$ is the unit outward normal to $\mathcal{S}$.

2.1 Time-harmonic motions

Suppose that $p(x,y,z,t) = \operatorname{Re}\{p(x,y,z)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$, with similar expressions for u, v, w and b. Then, Eqs. (2.1) and (2.4) give

$$(\omega^2 - f^2)u = -\mathrm{i}\omega\frac{\partial p}{\partial x} + f\frac{\partial p}{\partial y}, \tag{2.5}$$

$$(\omega^2 - f^2)v = -\mathrm{i}\omega\frac{\partial p}{\partial y} - f\frac{\partial p}{\partial x}, \tag{2.6}$$

$$(\omega^2 - f^2)w = -\mathrm{i}\omega\Upsilon\frac{\partial p}{\partial z}, \tag{2.7}$$

where

$$\Upsilon(z) = \frac{\omega^2 - f^2}{\omega^2 - N^2(z)}. \tag{2.8}$$

We are interested in frequencies ω satisfying $f^2 < \omega^2 < N^2$ so that $\Upsilon < 0$. (We also obtain $\Upsilon < 0$ in a homogeneous fluid ($N = 0$) with low-frequency motions ($\omega^2 < f^2$).) If we substitute for w from Eq. (2.7) in the time-harmonic version of Eq. (2.3), we obtain a single equation for p,

$$\nabla_H^2 p + \frac{\partial}{\partial z}\left(\Upsilon(z)\frac{\partial p}{\partial z}\right) = 0. \tag{2.9}$$

When considering energy transport with time-harmonic motions, it is natural to average the intensity $\mathbf{I}$ over a period. Thus, we define

$$\mathbf{I}_{\mathrm{av}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathbf{I}\,\mathrm{d}t = \frac{1}{2}\rho_0\,\mathrm{Re}\,\{p\overline{\boldsymbol{v}}\}\,, \tag{2.10}$$

where the overbar denotes complex conjugation. Note that $\mathrm{div}\,\mathbf{I}_{\mathrm{av}} = 0$.

3 A time-harmonic reciprocal theorem

We start with the divergence theorem, $\int_V \mathrm{div}\,\boldsymbol{u}\,\mathrm{d}V = \int_S \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S$, where $\boldsymbol{u}$ is a continuously differentiable vector field. Put $\boldsymbol{u} = \phi\boldsymbol{w}$:

$$\int_V (\boldsymbol{w}\cdot\mathrm{grad}\,\phi + \phi\,\mathrm{div}\,\boldsymbol{w})\,\mathrm{d}V = \int_S \phi\boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{3.1}$$

Suppose that p is a valid pressure field (p solves Eq. (2.9) in V) and that $\boldsymbol{v}^p$ is the corresponding velocity field (defined by Eqs. (2.5)–(2.7)). Then, as $\mathrm{div}\,\boldsymbol{v}^p = 0$, putting $\boldsymbol{w} = \boldsymbol{v}^p$ in Eq. (3.1) gives

$$\int_S \phi\boldsymbol{v}^p\cdot\boldsymbol{n}\,\mathrm{d}S = -\mathrm{i}T\int_V \left\{\Upsilon\frac{\partial p}{\partial z}\frac{\partial\phi}{\partial z} + \frac{\partial p}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial\phi}{\partial y} - \frac{f}{\mathrm{i}\omega}\left(\frac{\partial p}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial\phi}{\partial y}\right)\right\}\mathrm{d}V, \tag{3.2}$$

where $T = \omega/(\omega^2 - f^2)$. If we suppose that ϕ is a valid pressure field, and then interchange p and ϕ in Eq. (3.2), we obtain

$$\int_S p\boldsymbol{v}^\phi\cdot\boldsymbol{n}\,\mathrm{d}S = -\mathrm{i}T\int_V \left\{\Upsilon\frac{\partial p}{\partial z}\frac{\partial\phi}{\partial z} + \frac{\partial p}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial\phi}{\partial y} + \frac{f}{\mathrm{i}\omega}\left(\frac{\partial p}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial\phi}{\partial y}\right)\right\}\mathrm{d}V. \tag{3.3}$$

Subtracting Eq. (3.3) from Eq. (3.2) gives

$$\int_S \left(\phi\boldsymbol{v}^p - p\boldsymbol{v}^\phi\right)\cdot\boldsymbol{n}\,\mathrm{d}S = \frac{2f}{\omega^2 - f^2}\int_V \left(\frac{\partial p}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial\phi}{\partial y}\right)\mathrm{d}V. \tag{3.4}$$

Now, suppose that, in Eq. (3.1), we take $\phi = p$ and

$$\boldsymbol{w} = \frac{2f}{\omega^2 - f^2}\left(-\frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial x}, 0\right) = \boldsymbol{w}^\phi, \tag{3.5}$$

say; as $\mathrm{div}\,\boldsymbol{w}^\phi = 0$, the result is

$$\int_S p\boldsymbol{w}^\phi\cdot\boldsymbol{n}\,\mathrm{d}S = \frac{2f}{\omega^2 - f^2}\int_V \left(\frac{\partial p}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial\phi}{\partial y}\right)\mathrm{d}V.$$

Subtracting this result from Eq. (3.4) gives

$$\int_{\mathcal{S}} \left(\phi \boldsymbol{v}^p - p\boldsymbol{u}^\phi\right) \cdot \boldsymbol{n}\, \mathrm{d}S = 0, \tag{3.6}$$

where $\boldsymbol{u}^\phi = \boldsymbol{v}^\phi + \boldsymbol{w}^\phi$. Thus,

$$(\omega^2 - f^2)\boldsymbol{u}^\phi = \left(-\mathrm{i}\omega\frac{\partial\phi}{\partial x} - f\frac{\partial\phi}{\partial y}, -\mathrm{i}\omega\frac{\partial\phi}{\partial y} + f\frac{\partial\phi}{\partial x}, -\mathrm{i}\omega\Upsilon\frac{\partial\phi}{\partial z}\right),$$

which should be compared with

$$(\omega^2 - f^2)\boldsymbol{v}^p = \left(-\mathrm{i}\omega\frac{\partial p}{\partial x} + f\frac{\partial p}{\partial y}, -\mathrm{i}\omega\frac{\partial p}{\partial y} - f\frac{\partial p}{\partial x}, -\mathrm{i}\omega\Upsilon\frac{\partial p}{\partial z}\right). \tag{3.7}$$

Equation (3.6) is a *reciprocal theorem*, connecting two time-harmonic pressure fields, p and ϕ. Note that Eq. (3.6) involves $\boldsymbol{v}^p \cdot \boldsymbol{n}$, a quantity that is typically prescribed on boundaries. Note also that, in the absence of rotation ($f = 0$), Eq. (3.5) shows that $\boldsymbol{w}^\phi = \boldsymbol{0}$, so that we can then replace $\boldsymbol{u}^\phi$ by $\boldsymbol{v}^\phi$ in Eq. (3.6).

If we choose $\phi = \overline{p}$ in Eq. (3.2), we obtain

$$\int_{\mathcal{S}} \overline{p}\, \boldsymbol{v}^p \cdot \boldsymbol{n}\, \mathrm{d}S = -\mathrm{i}T \int_V \left(\left|\frac{\partial p}{\partial x}\right|^2 + \left|\frac{\partial p}{\partial y}\right|^2 + \Upsilon \left|\frac{\partial p}{\partial z}\right|^2 + \frac{2f}{\omega}\mathrm{Im}\left\{\frac{\partial p}{\partial x}\frac{\partial \overline{p}}{\partial y}\right\}\right)\mathrm{d}V. \tag{3.8}$$

Taking the real part of this equation, using Eq. (2.10), gives

$$\int_{\mathcal{S}} \mathbf{I}_{\mathrm{av}} \cdot \boldsymbol{n}\, \mathrm{d}S = 0. \tag{3.9}$$

4 An elliptic problem: $\omega > N$

Suppose henceforth that there is no rotation ($f = 0$) and that N is a positive constant. Then, from Eq. (2.8), $\Upsilon = \omega^2/(\omega^2 - N^2)$ is a constant and the (reduced, time-harmonic) pressure p solves

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \Upsilon\,\frac{\partial^2 p}{\partial z^2} = 0. \tag{4.1}$$

The velocity $\boldsymbol{v} = (u, v, w)$ is given in terms of p by Eqs. (2.5)–(2.7); these reduce to

$$u = -\frac{\mathrm{i}}{\omega}\frac{\partial p}{\partial x}, \quad v = -\frac{\mathrm{i}}{\omega}\frac{\partial p}{\partial y}, \quad w = -\frac{\mathrm{i}\Upsilon}{\omega}\frac{\partial p}{\partial z}. \tag{4.2}$$

The boundary condition is that $\boldsymbol{v} \cdot \boldsymbol{n}$ is prescribed, where $\boldsymbol{n}$ is a normal to the boundary. As this boundary condition and Eq. (4.1) both involve derivatives of p, we can assume that $p \to 0$ at infinity. Beyond this, there is also some kind of radiation condition at infinity. The far-field behaviour of p and $\boldsymbol{v}$ will be discussed later.

Physically, we are interested in frequencies satisfying $0 < \omega < N$ ($\Upsilon < 0$). However, we start by supposing that $\omega > N$ ($\Upsilon > 1$), so that Eq. (4.1) is elliptic. It is easy to see that

$$G(x, y, z;\, x_0, y_0, z_0) = \{(x - x_0)^2 + (y - y_0)^2 + \Upsilon^{-1}(z - z_0)^2\}^{-1/2} \tag{4.3}$$

solves Eq. (4.1), where (x_0, y_0, z_0) is a fixed point.

Let $\boldsymbol{v}^G$ denote the velocity field generated by the pressure G, using Eq. (4.2):

$$\boldsymbol{v}^G = \frac{\mathrm{i}}{\omega} \frac{(x - x_0,\ y - y_0,\ z - z_0)}{\{(x - x_0)^2 + (y - y_0)^2 + \Upsilon^{-1}(z - z_0)^2\}^{3/2}}. \tag{4.4}$$

Let S_ε denote the sphere of radius ε, centred at (x_0, y_0, z_0). On S_ε, introduce spherical polar coordinates, $x - x_0 = \varepsilon \sin\theta \cos\varphi$, $y - y_0 = \varepsilon \sin\theta \sin\varphi$ and $z - z_0 = \varepsilon \cos\theta$. Then, with $\boldsymbol{n}$ pointing out of S_ε,

$$\int_{S_\varepsilon} \boldsymbol{v}^G \boldsymbol{\cdot} \boldsymbol{n} \,\mathrm{d}S = \frac{2\pi\,\mathrm{i}}{\omega} \int_0^\pi \frac{\sin\theta \,\mathrm{d}\theta}{(\sin^2\theta + \Upsilon^{-1}\cos^2\theta)^{3/2}} = \frac{4\pi\,\mathrm{i}}{\omega} C(\Upsilon), \tag{4.5}$$

say, where

$$C(\Upsilon) = \int_0^{\pi/2} \frac{\sin\theta \,\mathrm{d}\theta}{(\sin^2\theta + \Upsilon^{-1}\cos^2\theta)^{3/2}} = \int_1^\infty \frac{\xi \,\mathrm{d}\xi}{(\xi^2 - 1 + \Upsilon^{-1})^{3/2}} = \Upsilon^{1/2}$$

and we used the substitution $\xi = \sec\theta$.

Let p and ϕ be pressure fields with velocity fields $\boldsymbol{v}^p$ and $\boldsymbol{v}^\phi$, respectively. Assume that p and ϕ are regular (no singularities) everywhere inside a closed surface, $\mathcal{S}$. Then, from Section 3, we have the reciprocal theorem,

$$\int_{\mathcal{S}} \left(p\boldsymbol{v}^\phi - \phi\boldsymbol{v}^p\right) \boldsymbol{\cdot} \boldsymbol{n} \,\mathrm{d}S = 0. \tag{4.6}$$

We shall use this formula with $\phi = G$ in order to obtain an integral representation for p.

Proceeding in a standard way [17], suppose that there is a bounded rigid object with boundary S. Choose a point P at (x_0, y_0, z_0) in the fluid outside S. Surround P be a small sphere S_ε (as above). Surround S and P by a large sphere, S_R, of radius R. Apply the reciprocal theorem to $p(x, y, z)$ and $G(x, y, z;\, x_0, y_0, z_0)$ in the region bounded by S, S_ε and S_R. The contribution from S_R vanishes as $R \to \infty$; see Section 8. The integration over S_ε picks out the value of p at P. Thus, using Eq. (4.5),

$$\frac{4\pi\,\mathrm{i}}{\omega}C(\Upsilon)p(x_0,y_0,z_0)+\int_S\left(p\boldsymbol{v}^G-G\boldsymbol{v}^p\right)\boldsymbol{\cdot}\boldsymbol{n}\,\mathrm{d}S=0,$$

where $\boldsymbol{n}$ points into the fluid. Hence, as $C(\Upsilon)/\omega=(\omega^2-N^2)^{-1/2}$,

$$p(P)=\frac{\mathrm{i}}{2\pi}(\omega^2-N^2)^{1/2}\int_S\left(p\boldsymbol{v}^G-G\boldsymbol{v}^p\right)\boldsymbol{\cdot}\boldsymbol{n}\,\mathrm{d}S,\quad P\text{ outside }S. \tag{4.7}$$

This gives a formula for the pressure in the fluid in terms of the (unknown) pressure and the (known) normal velocity on S.

When $P\in S$, the left-hand side of Eq. (4.7) is replaced by $\mathcal{A}(P)p(P)$ where $\mathcal{A}(P)$ arises from an integration over a small hemisphere at P; it is calculated in Appendix B. The result is an integral equation for the boundary values of p on S.

As in classical potential theory, define single-layer and double-layer potentials by

$$(\mathrm{S}\mu)(P)=\mathrm{i}(\omega^2-N^2)^{1/2}\int_S\mu G\,\frac{\mathrm{d}S}{4\pi}\quad\text{and}\quad(\mathrm{D}\mu)(P)=\mathrm{i}(\omega^2-N^2)^{1/2}\int_S\mu\boldsymbol{v}^G\boldsymbol{\cdot}\boldsymbol{n}\,\frac{\mathrm{d}S}{4\pi}, \tag{4.8}$$

respectively. Thus, Eq. (4.7) becomes

$$p(P)=\mathrm{D}p-\mathrm{S}(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{n}),\quad P\text{ in the fluid.} \tag{4.9}$$

This formula shows that p can always be written as a combination of single-layer and double-layer potentials. However, a double-layer potential suffices for scattering problems, as we show next.

For scattering problems, we suppose that we have an incident field, p_{in}, satisfying Eq. (4.1) in a region that includes the interior of S. Then, an application of the reciprocal theorem inside S (but retaining the outward-pointing normal) gives

$$\frac{\mathrm{i}}{2\pi}(\omega^2-N^2)^{1/2}\int_S\left(p_{\mathrm{in}}\boldsymbol{v}^G-G\boldsymbol{v}^{p_{\mathrm{in}}}\right)\boldsymbol{\cdot}\boldsymbol{n}\,\mathrm{d}S=0,\quad P\text{ outside }S. \tag{4.10}$$

As S is rigid, $(\boldsymbol{v}^p+\boldsymbol{v}^{p_{\mathrm{in}}})\boldsymbol{\cdot}\boldsymbol{n}=0$ on S. Hence, adding Eqs. (4.7) and (4.10) gives

$$p(P)=(\mathrm{D}p_{\mathrm{tot}})(P),\quad P\text{ in the fluid outside }S, \tag{4.11}$$

where $p_{\mathrm{tot}}=p+p_{\mathrm{in}}$ is the total pressure on S. From here, we can derive a boundary integral equation for p_{tot} on S.

5 Analytic continuation and the radiation condition

The calculations in Section 4 assume that $\omega > N$, but we are interested in solutions with $0 < \omega < N$. To find these, we effect analytic continuation with respect to ω: Voisin [28] refers to this as the Pierce–Hurley method. The main idea is to impose causality in the time domain, which means there should be no motion before a disturbance is excited. As we have used a time-dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$, causality implies that there should be no singularities or branch cuts in the upper half of the complex ω-plane.

Start with $(\omega^2 - N^2)^{1/2}$. Put cuts emanating from $\omega = \pm N$, going vertically downwards. Then, as $(\omega^2 - N^2)^{1/2}$ must be real and positive when ω is real and greater than N, we find that

$$(\omega^2 - N^2)^{1/2} = \begin{cases} +\sqrt{\omega^2 - N^2}, & \omega > N, \\ \mathrm{i}\sqrt{N^2 - \omega^2}, & -N < \omega < N, \\ -\sqrt{\omega^2 - N^2}, & \omega < -N. \end{cases} \tag{5.1}$$

In particular, on this branch of the square-root, we do not have an even function of ω, a fact that is emphasised by Voisin [28]. Then, for single-layer potentials, we need G, defined by Eq. (4.3). Using spherical polar coordinates,

$$x_0 - x = R\sin\Theta\cos\Phi, \quad y_0 - y = R\sin\Theta\sin\Phi, \quad z_0 - z = R\cos\Theta, \tag{5.2}$$

we have

$$G = \frac{1}{R(\sin^2\Theta + \Upsilon^{-1}\cos^2\Theta)^{1/2}} = \frac{\omega}{R(\omega^2 - N^2\cos^2\Theta)^{1/2}}, \tag{5.3}$$

giving branch points at $\omega = \pm N|\cos\Theta|$, with cuts extending downwards.

As we are interested in using G when $0 < \omega < N$, we define an angle θ_{c} by $\omega = N\cos\theta_{\mathrm{c}}$ with $0 < \theta_{\mathrm{c}} < \pi/2$. Then, from Eqs. (4.8) and (5.3), we can write the basic single-layer potential as

$$(\mathrm{S}\mu)(x_0, y_0, z_0) = \int_S \mu(x, y, z)\mathcal{M}(\Theta)\,\frac{\mathrm{d}S(x, y, z)}{4\pi R}, \tag{5.4}$$

where

$$\mathcal{M}(\Theta) = \begin{cases} \dfrac{-\omega\sqrt{N^2 - \omega^2}}{\sqrt{\omega^2 - N^2\cos^2\Theta}} = \dfrac{-N\cos\theta_{\mathrm{c}}\sin\theta_{\mathrm{c}}}{\sqrt{\cos^2\theta_{\mathrm{c}} - \cos^2\Theta}}, & |\cos\Theta| < \cos\theta_{\mathrm{c}}, \\ \dfrac{\mathrm{i}\omega\sqrt{N^2 - \omega^2}}{\sqrt{N^2\cos^2\Theta - \omega^2}} = \dfrac{\mathrm{i}N\cos\theta_{\mathrm{c}}\sin\theta_{\mathrm{c}}}{\sqrt{\cos^2\Theta - \cos^2\theta_{\mathrm{c}}}}, & \cos\theta_{\mathrm{c}} < |\cos\Theta|. \end{cases} \tag{5.5}$$

We proceed similarly for double-layer potentials, starting with an appropriate branch for $(\omega^2 - N^2)^{-3/2}$ and $\boldsymbol{v}^G \cdot \boldsymbol{n}$, with $\boldsymbol{v}^G$ given by Eq. (4.4). Thus,

$$\boldsymbol{v}^G \cdot \boldsymbol{n} = \frac{\mathrm{i}\,\mathcal{N}(\Theta,\Phi)}{\omega R^2(\sin^2\Theta + \Upsilon^{-1}\cos^2\Theta)^{3/2}} = \frac{\mathrm{i}\omega^2\mathcal{N}(\Theta,\Phi)}{R^2(\omega^2 - N^2\cos^2\Theta)^{3/2}},$$

where

$$\mathcal{N}(\Theta,\Phi) = \mathcal{N}(\Theta,\Phi;Q) = -(n_1\cos\Phi + n_2\sin\Phi)\sin\Theta - n_3\cos\Theta \qquad (5.6)$$

and $\boldsymbol{n}(Q) = (n_1, n_2, n_3)$ is the outward normal at $Q = (x, y, z) \in S$. Then, from Eq. (4.8), we can write the double-layer potential as

$$(\mathrm{D}\mu)(x_0, y_0, z_0) = \int_S \mu(x,y,z)\mathcal{N}(\Theta,\Phi)\mathcal{D}(\Theta)\,\frac{\mathrm{d}S(x,y,z)}{4\pi R^2}, \qquad (5.7)$$

where

$$\mathcal{D}(\Theta) = \begin{cases} \dfrac{-\mathrm{i}\omega^2\sqrt{N^2-\omega^2}}{(\omega^2 - N^2\cos^2\Theta)^{3/2}} = \dfrac{-\mathrm{i}\cos^2\theta_\mathrm{c}\sin\theta_\mathrm{c}}{(\cos^2\theta_\mathrm{c} - \cos^2\Theta)^{3/2}}, & |\cos\Theta| < \cos\theta_\mathrm{c}, \\ \dfrac{+\omega^2\sqrt{N^2-\omega^2}}{(N^2\cos^2\Theta - \omega^2)^{3/2}} = \dfrac{+\cos^2\theta_\mathrm{c}\sin\theta_\mathrm{c}}{(\cos^2\Theta - \cos^2\theta_\mathrm{c})^{3/2}}, & \cos\theta_\mathrm{c} < |\cos\Theta|. \end{cases} \qquad (5.8)$$

Examining Eqs. (5.5) and (5.8), we see that $\mathcal{M}(\Theta)$ and $\mathcal{D}(\Theta)$ are singular at $\Theta = \theta_\mathrm{c}$ and at $\Theta = \pi - \theta_\mathrm{c}$, so we must investigate when these singularities arise and how to handle them.

The singularities in $\mathcal{M}(\Theta)$ are integrable but $\mathcal{D}(\Theta)$ has non-integrable singularities. In detail, for $\Theta \simeq \theta_\mathrm{c}$, $\cos^2\theta_\mathrm{c} - \cos^2\Theta \simeq (\Theta - \theta_\mathrm{c})\sin 2\theta_\mathrm{c}$. As we shall want to integrate with respect to Θ, for fixed θ_c, we define $(\Theta - \theta_\mathrm{c})^\nu$ in the complex Θ-plane, with a cut going downwards from $\Theta = \theta_\mathrm{c}$, taking real positive values when Θ is real with $\Theta > \theta_\mathrm{c}$; here, ν is a parameter. Then, $(\Theta - \theta_\mathrm{c})^\nu = \mathrm{e}^{\mathrm{i}\nu\pi}(\theta_\mathrm{c} - \Theta)^\nu$ when Θ is real with $\Theta < \theta_\mathrm{c}$. The choice of cut ensures that we have agreement with Eqs. (5.5) and (5.8) when $\Theta \simeq \theta_\mathrm{c}$, and then we can write

$$\mathcal{M}(\Theta) \simeq -(N/2)\sqrt{\sin 2\theta_\mathrm{c}}\,(\Theta - \theta_\mathrm{c})^{-1/2}, \quad \mathcal{D}(\Theta) \simeq -(\mathrm{i}/4)\sqrt{2\cot\theta_\mathrm{c}}\,(\Theta - \theta_\mathrm{c})^{-3/2} \qquad (5.9)$$

for complex Θ near θ_c.

6 Geometry and singularities

To proceed, we partition the fluid domain into several regions. First, choose an origin O inside S. Suppose that S can be enclosed by a sphere, S_a, of

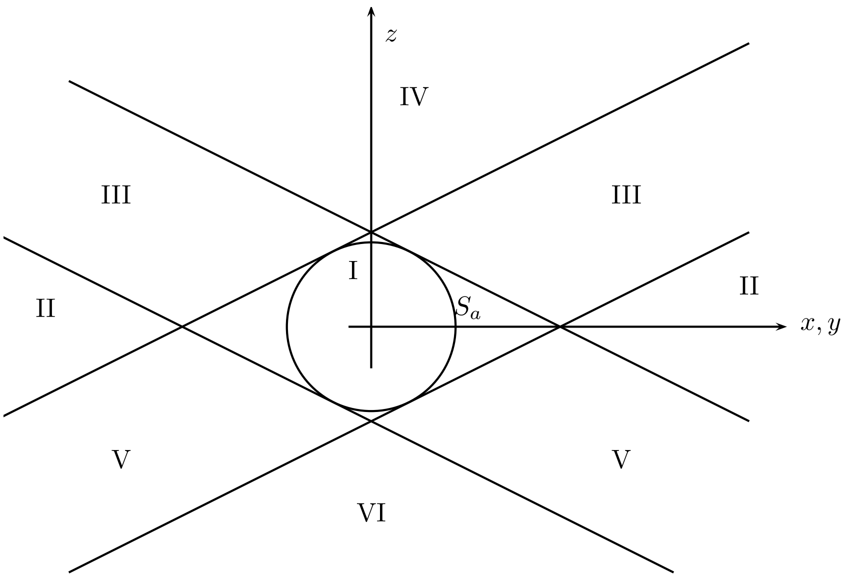

Fig. 1. The body (scatterer or vibrator) is located inside the sphere, S_a. Regions III and V are the conical wave beams bounded by characteristic cones. The pressure decays rapidly in Regions II, IV and VI. Similar figures can be found in other papers, such as Fig. 1 of [1] and Fig. 10 of [28].

radius a, centred at O (Fig. 1). Define cylindrical polar coordinates (ϱ, ϕ, z) at O. Define spherical polar coordinates (r, θ, ϕ) at O, with $\theta = 0$ as the positive z-axis. A source at $r = 0$ would propagate energy along the upper cone, $\theta = \theta_c$ ($z = \varrho \cot \theta_c$) and along the lower cone, $\theta = \pi - \theta_c$.

Define a thick conical shell of thickness $2a$ with surfaces given by $z = \varrho \cot \theta_c \pm a \csc \theta_c$. This defines the upper wave beam (Region III in Fig. 1). The lower wave beam is defined by $z = -\varrho \cot \theta_c \pm a \csc \theta_c$ (Region V). The object S lies in the intersection of the two conical wave beams.

The observation point is P at (x_0, y_0, z_0). The integration point is Q at (x, y, z). Using Eq. (5.2), we have

$$\{(x - x_0)^2 + (y - y_0)^2\} \cos^2 \theta_c - (z - z_0)^2 \sin^2 \theta_c = R^2(\cos^2 \theta_c - \cos^2 \Theta).$$

Thus, the singularities in Eqs. (5.5) and (5.8) occur when

$$(z - z_0)^2 = \{(x - x_0)^2 + (y - y_0)^2\} \cot^2 \theta_c. \tag{6.1}$$

This defines a double cone in xyz-space with apex at P.

If P is not inside one of the conical wave beams (so P is in Regions II, IV or VI), the double cone does not intersect S: there are no singularities in the integrations over S in Eqs. (5.4) and (5.7). Specifically, we have $\theta_c < \Theta < \pi - \theta_c$ when P is in Region II, $0 \le \Theta < \theta_c$ when P is in Region IV, and $\pi - \theta_c < \Theta \le \pi$

when P is in Region VI. Thus, $|\cos\Theta| < \cos\theta_c$ in Region II and $\cos\theta_c < |\cos\Theta|$ in Regions IV and VI.

Suppose now that P is inside the upper wave beam, Region III. Then, the lower half of the double cone, Eq. (6.1),

$$z - z_0 = -\sqrt{(x-x_0)^2 + (y-y_0)^2}\,\cot\theta_c,$$

will intersect S in a closed curve $\mathcal{C}$. (For simplicity, assume that there is just one of these curves; this will be the case if the object is convex, for example.) Thus, as Q is at (x, y, z), we see that the singularities in the integration over S in Eq. (4.11) occur at all points Q on the curve $\mathcal{C}$. This curve is characterised as being where $\Theta = \theta_c$.

A similar construction can be made when P is inside the lower wave beam, Region V, using the upper half of the double cone, Eq. (6.1). The singularity corresponds to $\Theta = \pi - \theta_c$.

We emphasise that there are singularities in the boundary integrals over S even when P is not on S (but is in Region III or V). This is very different from classical potential theory, for example, where typical boundary integrals only contain singularities when the field point P is on the boundary, and then the singularity is at P, not along a curve on S. All this is a consequence of the hyperbolic nature of the governing partial differential equation.

7 The far field

The observation point P, at (x_0, y_0, z_0), has cylindrical polar coordinates (ϱ_0, ϕ_0, z_0) and spherical polar coordinates (r_0, θ_0, ϕ_0). We estimate the pressure field as $r_0 \to \infty$. The results are different depending on whether P is outside or inside the wave beams.

7.1 The far field outside the wave beams

As already noted, within Regions II, IV and VI, $\mathcal{M}$ and $\mathcal{D}$ are finite. Thus, Eqs. (5.4), (5.7) and (4.9) show that $\mathrm{S}\mu = O(r_0^{-1})$, $\mathrm{D}\mu = O(r_0^{-2})$ and $p = O(r_0^{-1})$, respectively, as $r_0 \to \infty$. For the scattering problem, Eq. (4.11) shows that p is smaller: $p = O(r_0^{-2})$ as $r_0 \to \infty$. In the same limit, Eq. (5.2) shows that $\mathcal{N}(\Theta, \Phi) \sim \mathcal{N}(\theta_0, \phi_0)$.

7.2 The far field within the wave beams

To quantify the fields inside the beams, it is convenient to introduce additional sets of coordinates. Following [29], introduce two sets of conical polar coordinates, (x_+^0, ϕ_0, z_+^0) and (x_-^0, ϕ_0, z_-^0), with

$$\varrho_0 = r_0 \sin\theta_0 = x_\pm^0 \cos\theta_\mathrm{c} \pm z_\pm^0 \sin\theta_\mathrm{c}, \quad z_0 = r_0 \cos\theta_0 = \mp x_\pm^0 \sin\theta_\mathrm{c} + z_\pm^0 \cos\theta_\mathrm{c}. \tag{7.1}$$

Inverting, $x_\pm^0 = \mp z_0 \sin\theta_\mathrm{c} + \varrho_0 \cos\theta_\mathrm{c}$ and $z_\pm^0 = z_0 \cos\theta_\mathrm{c} \pm \varrho_0 \sin\theta_\mathrm{c}$. So, P is on the upper cone ($\theta = \theta_\mathrm{c}$) when $x_+^0 = 0$ and P is on the lower cone ($\theta = \pi - \theta_\mathrm{c}$) when $x_-^0 = 0$. Moreover, P is within the upper conical beam when $|x_+^0| < a$ with P receding to infinity as $x_-^0 \to \infty$. Similarly, P is within the lower conical beam when $|x_-^0| < a$ with P receding to infinity as $x_+^0 \to \infty$.

Consider the far-field behaviour within the upper conical wave beam, Region III. In that region, $|x_+^0| < a$, $|X_\pm| < a$ and $|Y| < a$ but $z_+^0 \to \infty$; here, $X_\pm$ and Y are defined by Eqs. (C.1)–(C.3). In what follows, we focus on Region III, and so we simplify notation slightly, and write $\sigma_0 \equiv x_+^0$ and $\zeta_0 \equiv z_+^0$. To calculate the radiated field, we suppose that the waves are generated by vibrations of the spherical surface, S_a, represented by single-layer and double-layer potentials over S_a. Using a sphere will permit explicit and detailed calculations, and comparisons with earlier work on oscillating spheres. Also, integral representations such as Eq. (4.9) (with integrations over S_a) hold outside S_a, with the actual object, S, inside S_a.

So, we consider a single-layer potential, $\mathrm{S}\mu$, and a double-layer potential, $\mathrm{D}\mu$, with integrations over S_a. We parametrise S_a using Θ and Φ to locate points $Q = (x, y, z)$ on S_a, leading to double integrals over a certain domain $\mathcal{E}$ in the $\Theta\Phi$-plane. Points around the perimeter of $\mathcal{E}$ correspond to points of contact of tangent lines from P to S_a; see, for example, [28, Fig. 9]. The integrands are singular along the coordinate line $\Theta = \theta_\mathrm{c}$, a straight line that passes through $\mathcal{E}$. The domain $\mathcal{E}$ shrinks to a point as P recedes to infinity. These facts simplify the computation of the far field. The details of the calculation can be found in Appendix C.

7.3 Radiation by a single-layer potential

Consider the single-layer potential, defined by Eq. (5.4). Parametrising S using Θ and Φ gives

$$(\mathrm{S}\mu)_\pm = \frac{a}{4\pi} \int_{\mathcal{E}} \mu_\pm(\Theta, \Phi) \mathcal{M}(\Theta) \frac{\mathcal{Q}_\pm^2}{\sqrt{\Delta}} \sin\Theta \frac{\mathrm{d}\Theta\,\mathrm{d}\Phi}{R_\pm}$$

where the $\pm$ refers to the two sides of S, and the two contributions must be summed to obtain $\mathrm{S}\mu$. In the far field, $R_\pm \sim r_0$ and $r_0 \to \infty$. In this limit, $\mathcal{E}$ shrinks onto the point $(\Theta, \Phi) = (\theta_\mathrm{c}, \phi_0)$, so we can approximate.

From Eq. (C.9), as $\zeta_0 \sim r_0$, we have $\mathcal{Q}_\pm \sim -r_0$. Then, from Eq. (C.17),

$$\Delta(\Theta, \Phi) \simeq [\Phi_+(\Theta) - \Phi][\Phi - \Phi_-(\Theta)]\, r_0^2 \sin^2\theta_\mathrm{c}.$$

Hence,

$$\mathrm{S}\mu = (\mathrm{S}\mu)_+ + (\mathrm{S}\mu)_- \sim \frac{a}{2}\int_{\Theta_-}^{\Theta_+} \mu_\mathrm{S}(\Theta)\mathcal{M}(\Theta)\,\mathrm{d}\Theta, \tag{7.2}$$

where

$$\mu_\mathrm{S}(\Theta) = \frac{1}{2\pi}\int_{\Phi_-}^{\Phi_+} \frac{\{\mu_+(\Theta,\Phi) + \mu_-(\Theta,\Phi)\}\,\mathrm{d}\Phi}{\sqrt{[\Phi_+(\Theta) - \Phi][\Phi - \Phi_-(\Theta)]}}; \tag{7.3}$$

if $\mu_+ = \mu_- =$ constant, $\mu_\mathrm{S} = \mu_\pm$, exactly.

Then, we expand $\mu_\mathrm{S}(\Theta)$ as a Taylor series about $\Theta = \theta_\mathrm{c}$; if the Taylor coefficient of $(\Theta - \theta_\mathrm{c})^n$ is $c_n(r_0)$, we suppose that $m_n^\mathrm{S} = \lim_{r_0\to\infty} c_n/r_0^n$ exists and write

$$\mu_\mathrm{S}(\Theta) \simeq \sum_{n=0}^{\infty} m_n^\mathrm{S} r_0^n (\Theta - \theta_\mathrm{c})^n. \tag{7.4}$$

(Note that, in Region III, r_0 is large and $|\Theta - \theta_\mathrm{c}|$ is small but their product is $O(a)$.) Substituting in Eq. (7.2) followed by use of Eq. (5.9) shows that the remaining integrals are of the form

$$\begin{aligned}\int_{\Theta_-}^{\Theta_+} (\Theta - \theta_\mathrm{c})^{\nu-1}\,\mathrm{d}\Theta &= \frac{1}{\nu}\left\{(\Theta_+ - \theta_\mathrm{c})^\nu - \mathrm{e}^{\mathrm{i}\nu\pi}(\theta_\mathrm{c} - \Theta_-)^\nu\right\} \\ &\sim \nu^{-1} r_0^{-\nu}\left\{(a + \sigma_0)^\nu - \mathrm{e}^{\mathrm{i}\nu\pi}(a - \sigma_0)^\nu\right\},\end{aligned} \tag{7.5}$$

using Eq. (C.18). (The branch in the complex Θ-plane is described above Eq. (5.9).)

Thus, formally, we obtain the far-field estimate

$$\mathrm{S}\mu \sim \frac{aN}{2\sqrt{r_0}}\sqrt{\sin 2\theta_\mathrm{c}}\sum_{n=0}^{\infty}\frac{m_n^\mathrm{S}}{2n+1}\mathcal{F}_n(\sigma_0), \tag{7.6}$$

where

$$\mathcal{F}_n(x) = \mathrm{i}(-1)^n(a - x)^{n+1/2} - (a + x)^{n+1/2}. \tag{7.7}$$

We emphasise, first, that the coefficients m_n^S in Eqs. (7.4) and (7.6) can depend on the lateral coordinate σ_0 and on the azimuthal angle ϕ_0. Second, Eq. (7.6) is not a far-field *expansion*: every term in the series must be retained in order to obtain the leading-order estimate (the quantity multiplying $r_0^{-1/2}$).

Consider the double-layer potential, defined by Eq. (5.7). Parametrising S using Θ and Φ, and then proceeding as for $\mathrm{S}\mu$ gives

$$(\mathrm{D}\mu)_\pm = \frac{a}{4\pi}\int_{\mathcal{E}} \mu_\pm(\Theta,\Phi)\mathcal{N}_\pm(\Theta,\Phi)\mathcal{D}(\Theta)\frac{\mathcal{Q}_\pm^2}{\sqrt{\Delta}}\sin\Theta\frac{\mathrm{d}\Theta\,\mathrm{d}\Phi}{R_\pm^2}. \tag{7.8}$$

Let us evaluate $\mathcal{N}_\pm$. On S_a, $\boldsymbol{n} = (x,y,z)/a$. Then, using Eqs. (5.6), (C.1), (C.2) and (C.3),

$$a\mathcal{N}(\Theta,\Phi) = (X_+S - Z_+C)\cos\Theta - \{(X_+C + Z_+S)c + Ys\}\sin\Theta.$$

We have $X_+ = \tilde{X} + \sigma_0$ and $Z_+ = \tilde{Z} + \zeta_0$; $\tilde{Z}$ is given by Eq. (C.8), and $\tilde{X}$ and Y are given by Eq. (C.11). Hence, substitution gives $a\mathcal{N} = -Q_3 - \mathcal{Q}_\pm = \mp\sqrt{\Delta}$, using Eqs. (C.9) and (C.10), and then Eq. (7.8) reduces to

$$(\mathrm{D}\mu)_\pm = \mp\frac{1}{4\pi}\int_{\mathcal{E}} \mu_\pm(\Theta,\Phi)\mathcal{D}(\Theta)\,\mathcal{Q}_\pm^2\sin\Theta\frac{\mathrm{d}\Theta\,\mathrm{d}\Phi}{R_\pm^2}. \tag{7.9}$$

As in Section 7.3, we approximate. Thus, as $(\mathcal{Q}_\pm/R_\pm)^2 \sim 1$ and $\sin\Theta \sim S$,

$$\mathrm{D}\mu = (\mathrm{D}\mu)_+ + (\mathrm{D}\mu)_- \sim -\frac{a}{4r_0}\int_{\Theta_-}^{\Theta_+} \mu_\mathrm{D}(\Theta)\mathcal{D}(\Theta)\,\mathrm{d}\Theta,$$

where

$$\mu_\mathrm{D}(\Theta) = \frac{r_0S}{\pi a}\int_{\Phi_-}^{\Phi_+}\{\mu_+(\Theta,\Phi) - \mu_-(\Theta,\Phi)\}\,\mathrm{d}\Phi. \tag{7.10}$$

Next, expand $\mu_\mathrm{D}(\Theta)$ about $\Theta = \theta_\mathrm{c}$, as we did with μ_S (see Eq. (7.4)):

$$\mu_\mathrm{D}(\Theta) \simeq \sum_{n=0}^{\infty} m_n^\mathrm{D} r_0^n(\Theta - \theta_\mathrm{c})^n. \tag{7.11}$$

Then, using Eq. (5.9) and Eq. (7.5) with $\nu = n - 1/2$, we obtain the estimate

$$\mathrm{D}\mu \sim -\frac{\mathrm{i}a}{8\sqrt{r_0}}\sqrt{2\cot\theta_\mathrm{c}}\sum_{n=0}^{\infty}\frac{m_n^\mathrm{D}}{2n-1}\mathcal{F}_{n-1}(\sigma_0), \tag{7.12}$$

where $\mathcal{F}_n$ is defined by Eq. (7.7). Again, m_n^D can depend on σ_0 and ϕ_0. See also the remarks below Eq. (7.7).

7.5 Pulsating sphere

On the sphere, $\boldsymbol{v} \cdot \boldsymbol{n} = U_0$, a constant. From the known exact solution, the pressure on the sphere is also constant, $p = p_0$. Then, from Eq. (4.9),

$$p(P) = \mathrm{D}p - \mathrm{S}(\boldsymbol{v} \cdot \boldsymbol{n}) = \mathrm{D}p_0 - \mathrm{S}U_0, \quad P \text{ in the fluid.}$$

Comparison with the results in Section 7.4 shows that $\mu_\pm = p_0$ so that $\mu_\mathrm{D} = 0$: the double-layer contribution in the far field is negligible. Comparison with the results in Section 7.3 shows that $\mu_\pm = U_0 = \mu_\mathrm{S} = m_0^\mathrm{S}$, and then Eq. (7.6) gives, to leading order,

$$p(P) \sim -\frac{aNU_0}{2\sqrt{r_0}}\sqrt{\sin 2\theta_\mathrm{c}}\,\mathcal{F}_0(\sigma_0), \tag{7.13}$$

where

$$\mathcal{F}_0(x) = \mathrm{i}\sqrt{a-x} - \sqrt{a+x} = -\sqrt{2a}\exp\{-\tfrac{1}{2}\mathrm{i}\arccos(x/a)\}. \tag{7.14}$$

The far-field estimate, Eq. (7.13), agrees precisely with the known exact solution: see [29, Eq. (4.47)], [28, Eq. (8.25)] (where a time dependence of $\mathrm{e}^{+\mathrm{i}\omega t}$ is used) and Eq. (A.5).

7.6 Vertically oscillating rigid sphere

On the sphere, $\boldsymbol{v} \cdot \boldsymbol{n} = W_0 z/a$ for some constant W_0. From the known exact solution, $p = \mathcal{P}z/a$, where $\mathcal{P}$ is a complex constant given by Eq. (A.7). Then, from Eq. (4.9),

$$p(P) = \mathrm{D}p - \mathrm{S}(\boldsymbol{v} \cdot \boldsymbol{n}) = \mathcal{P}\mathrm{D}\mu - W_0\mathrm{S}\mu, \quad \mu = z/a, \quad P \text{ in the fluid.} \tag{7.15}$$

To estimate the far field, we start by calculating μ_S and μ_D. We have

$$\begin{aligned} a\mu_\pm &= z = Z_+C - X_+S = \zeta_0 C - \sigma_0 S + \tilde{Z}C - \tilde{X}S = \zeta_0 C - \sigma_0 S + \mathcal{Q}_\pm \cos\Theta \\ &= \zeta_0(C\sin\Theta - cS\cos\Theta)\sin\Theta - \sigma_0(S\sin\Theta + cC\cos\Theta)\sin\Theta \pm \sqrt{\Delta}\cos\Theta \\ &\sim r_0 S(\Theta - \theta_\mathrm{c}) - \sigma_0 S \pm r_0 CS\sqrt{(\Phi_+ - \Phi)(\Phi - \Phi_-)}. \end{aligned}$$

From Eq. (7.3), we integrate with respect to Φ to obtain

$$a\mu_\mathrm{S}(\Theta) = r_0 S(\Theta - \theta_\mathrm{c}) - \sigma_0 S,$$

a linear function of Θ. Hence comparison with Eq. (7.4) shows that $m_0^{\mathrm{S}} = -\sigma_0 S/a$, $m_1^{\mathrm{S}} = S/a$ and $m_n^{\mathrm{S}} = 0$ for $n \geq 2$. Similarly, Eq. (7.10) gives

$$(a^2/C)\mu_{\mathrm{D}}(\Theta) = r_0^2[\Phi]^2(S/2)^2 = r_0^2(\Theta_+ - \Theta)(\Theta - \Theta_-),$$

a quadratic in Θ, using Eq. (C.19) for $[\Phi]$. Hence, comparison with Eq. (7.11) gives

$$(a^2/C)m_0^{\mathrm{D}} = a^2 - \sigma_0^2, \quad (a^2/C)m_1^{\mathrm{D}} = 2\sigma_0, \quad (a^2/C)m_2^{\mathrm{D}} = -1$$

and $m_n^{\mathrm{D}} = 0$ for $n \geq 3$.

Examination of Eqs. (7.6) and (7.12) shows that we require $\mathcal{F}_{\pm 1}$ and $\mathcal{F}_0$. From Eq. (7.7),

$$\mathcal{F}_{-1}(x) = \frac{\mathrm{i}}{\sqrt{a^2 - x^2}}\,\mathcal{F}_0(x) \quad \text{and} \quad \mathcal{F}_1(x) = \left(2x + \mathrm{i}\sqrt{a^2 - x^2}\right)\mathcal{F}_0(x), \tag{7.16}$$

with $\mathcal{F}_0$ given by Eq. (7.14).

From Eq. (7.6),

$$\mathrm{S}\mu \sim \frac{aN}{6\sqrt{r_0}}\sqrt{\sin 2\theta_{\mathrm{c}}}\,\left\{3m_0^{\mathrm{S}}\mathcal{F}_0 + m_1^{\mathrm{S}}\mathcal{F}_1\right\} = -\frac{NS}{12\sqrt{r_0}}\sqrt{\sin 2\theta_{\mathrm{c}}}\,\mathcal{F}_0^3(\sigma_0). \tag{7.17}$$

From Eq. (7.12),

$$\begin{aligned}\mathrm{D}\mu &\sim -\frac{\mathrm{i}a}{24\sqrt{r_0}}\sqrt{2\cot\theta_{\mathrm{c}}}\,\left\{-3m_0^{\mathrm{D}}\mathcal{F}_{-1} + 3m_1^{\mathrm{D}}\mathcal{F}_0 + m_2^{\mathrm{D}}\mathcal{F}_1\right\}\\ &= -\frac{\mathrm{i}C}{12a\sqrt{r_0}}\sqrt{2\cot\theta_{\mathrm{c}}}\,\mathcal{F}_0^3(\sigma_0).\end{aligned} \tag{7.18}$$

Combining Eqs. (7.15), (7.17) and (7.18) gives

$$p \sim p_\infty(a/r_0)^{1/2}\exp\left(-\tfrac{3}{2}\mathrm{i}\arccos(\sigma_0/a)\right) \tag{7.19}$$

where

$$p_\infty = (\mathrm{i}/3)\mathcal{P}C\sqrt{C/S} - (a/3)W_0 NS\sqrt{SC}. \tag{7.20}$$

From Eqs. (A.6) and (A.7), W_0 and $\mathcal{P}$ are related: $aW_0NS^2 = \mathrm{i}C\mathcal{B}_0(Q_1 + C^{-2})$ and $\mathcal{P} = \mathcal{B}_0 Q_1$. Substitution in Eq. (7.20) gives $p_\infty = -(\mathrm{i}/3)\mathcal{B}_0/\sqrt{SC}$, in precise agreement with the known exact solution, Eq. (A.8).

Inspection of Eqs. (7.17) and (7.18) shows that $\mathrm{S}\mu$ and $\mathrm{D}\mu$ are both approximately constant multiples of $\mathcal{F}_0^3$. Thus, for the vertically oscillating sphere, we can write $p = \mathrm{S}\mu_1$ or $p = \mathrm{D}\mu_2$, where μ_1 and μ_2 are certain constant multiples of z. For example, a short calculation gives

$$\mu_1 = \mathcal{B}_0 z/(\mathrm{i}a^2 N S^2 C) = -W_0(z/a)[1 + C^2 Q_1(\mathrm{i}/c)]^{-1},$$

and this agrees with [30, Eq. (2.10)].

8 Energy and radiation conditions

There are two kinds of far-field behaviour to discuss. First, we must check that the fields decay fast enough to ensure that the surface integral over the large sphere, S_R, vanishes as its radius $R \to \infty$. Outside the wave beams (Regions II, IV and VI in Fig. 1), the decay is rapid (Section 7.1), so standard potential-theory estimates give the result. Within the beams, p decays slowly, as $R^{-1/2}$, but the total area of the beam cross-sections also grows slowly (it is $8a\pi R \sin\theta_\mathrm{c}$) and so simple estimates suffice.

The second concern is physical: energy transport. Thus, from Eq. (2.10), we consider the vector $\mathbf{I}_\mathrm{av} = \frac{1}{2}\rho_0 \,\mathrm{Re}\,\{p\overline{\boldsymbol{v}}\}$. The velocity field $\boldsymbol{v}$ can be calculated from p using Eq. (4.2). Within the beam, Region III, the velocity in the far field is parallel to the beam. Explicitly, if $\hat{\mathbf{z}}_+^0$ is a unit vector in the direction of increasing $z_+^0 \equiv \zeta_0$ (away from the object), then Voisin [29, Eq. (4.36)] has shown that $\boldsymbol{v} \sim v\hat{\mathbf{z}}_+^0$, where

$$v = \frac{1}{\mathrm{i}N \sin\theta_\mathrm{c}} \frac{\partial p}{\partial \sigma_0}.$$

Now, from Eqs. (7.6) and (7.12), the far-field behaviour of both single-layer and double-layer potentials has the form

$$p \sim r_0^{-1/2} F(\sigma_0, \phi_0), \tag{8.1}$$

where F is a complex-valued function of σ_0 and ϕ_0. Hence,

$$\mathbf{I}_\mathrm{av} = \frac{1}{2}\rho_0 \,\mathrm{Re}\,\{p\overline{\boldsymbol{v}}\} \sim \frac{\rho_0}{2N r_0 \sin\theta_\mathrm{c}} \,\mathrm{Im}\,(\overline{F}F')\, \hat{\mathbf{z}}_+^0, \tag{8.2}$$

where $F' \equiv \partial F/\partial\sigma_0$ and we have used $\mathrm{Re}\,(\mathrm{i}F\overline{F'}) = \mathrm{Im}\,(\overline{F}F')$.

For a pulsating sphere, (7.13) gives $F = \mathcal{F}_0$ (apart from a constant real factor), $F' = \mathcal{F}_0'(\sigma_0) = (\mathrm{i}/2)(a^2 - \sigma_0^2)^{-1/2}\mathcal{F}_0(\sigma_0)$ and $\mathrm{Im}\,(\overline{F}F') = a(a^2 - \sigma_0^2)^{-1/2}$, which is positive. Thus, $\mathbf{I}_\mathrm{av}$ points away from the sphere.

For a vertically oscillating rigid sphere, (7.19) gives $F = \mathcal{F}_0^3$, $F' = 3\mathcal{F}_0^2\mathcal{F}_0'$ and $\mathrm{Im}\,(\overline{F}F') = 12a^3(a^2 - \sigma_0^2)^{-1/2}$, which is positive: again, $\mathbf{I}_{\mathrm{av}}$ points away from the sphere.

These two results suggest that a plausible far-field condition is that $\mathbf{I}_{\mathrm{av}}$ should point away from the radiator, especially given textbook interpretations of the meaning of $\mathbf{I}_{\mathrm{av}}$. For example, Lighthill [14, p. 14] states that $\mathbf{I} \cdot \boldsymbol{n}$ is the rate at which energy is transported in the direction of $\boldsymbol{n}$ across a small plane element that is perpendicular to $\boldsymbol{n}$, per unit area of that element.

However, there are two objections. First, there is some arbitrariness in the definition of $\mathbf{I}_{\mathrm{av}}$: any divergence-free vector can be added to $\mathbf{I}_{\mathrm{av}}$ without violating $\mathrm{div}\,\mathbf{I}_{\mathrm{av}} = 0$. The significance of this observation was noted by Longuet-Higgins [16]. Second, although there is a requirement that $\int_{S_R} \mathbf{I}_{\mathrm{av}} \cdot \boldsymbol{n}\,\mathrm{d}S > 0$ when S_R encloses a radiating object (see (3.9)), it is easy to construct examples where $\mathbf{I}_{\mathrm{av}}$ points towards the object in parts of the wave beams. For example, consider $\mathrm{S}\mu$ with $\mu = 2a + 3z/S$. This is a linear combination of two pieces, namely $2a$ and $3z/S$, each of which generates outgoing $\mathbf{I}_{\mathrm{av}}$. However, their sum does not, as we show next. We have $m_0^{\mathrm{S}} = 2a - 3\sigma_0$, $m_1^{\mathrm{S}} = 3$ and

$$F(\sigma_0) = 3m_0^{\mathrm{S}}\mathcal{F}_0(\sigma_0) + m_1^{\mathrm{S}}\mathcal{F}_1(\sigma_0) = \{6a - 3\sigma_0 + 3\mathrm{i}(a^2 - \sigma_0^2)^{1/2}\}\mathcal{F}_0(\sigma_0),$$

using (7.16). Differentiating,

$$F'(\sigma_0) = (3/2)\mathcal{F}_0(\sigma_0)\{2\mathrm{i}a - 3\mathrm{i}\sigma_0 - 3(a^2 - \sigma_0^2)^{1/2}\}(a^2 - \sigma_0^2)^{-1/2}.$$

Let $\psi_0 = \arccos(\sigma_0/a)$ with $0 < \psi_0 < \pi$, so that $a\mathrm{e}^{\mathrm{i}\psi_0} = \sigma_0 + \mathrm{i}(a^2 - \sigma_0^2)^{1/2}$. Then,

$$F = 3a(2 - \mathrm{e}^{-\mathrm{i}\psi_0})\mathcal{F}_0, \quad F' = (3/2)\mathrm{i}(2 - 3\mathrm{e}^{-\mathrm{i}\psi_0})\mathcal{F}_0 \csc\psi_0$$

and, as $|\mathcal{F}_0|^2 = 2a$ (see (7.14)),

$$\mathrm{Im}\,(\overline{F}F') = 9a^2(7 - 8\cos\psi_0)\csc\psi_0.$$

Evidently, this is negative in part of the wave beam: in that part, $\mathbf{I}_{\mathrm{av}}$ points towards S.

For another (less artificial) example, consider the problem of a sphere that is both pulsating and oscillating vertically. Adding the two known solutions for the constituent problems leads to a solution that may or may not satisfy $\mathbf{I}_{\mathrm{av}} \cdot \hat{\mathbf{z}}_+^0 > 0$, depending on the strengths of the modes. See Section A.3 for details.

9 Discussion

Pressure fields exterior to radiators in stratified fluids may be sought in the form of a single-layer potential, $p = \mathrm{S}\mu$, or as a double-layer potential, $p = \mathrm{D}\mu$. To justify these choices, we have to show that μ can be chosen so as to satisfy the boundary condition and that the far fields are physically meaningful. In the absence of a precise radiation condition, it is worthwhile to be able to calculate the far fields from a knowledge of μ. We have shown how to do this, using a careful asymptotic analysis.

The specification of a radiation condition in the frequency domain is problematic. If we had a plane-wave (Fourier transform) representation, we could select those plane waves with outgoing group velocity. This technique can be used for special geometries [18,20], but it is not general. Another possibility would be to use Pierce–Hurley analytic continuation, but it is unclear how to do that computationally. We have shown that it is not sufficient to check that the energy transport, $\mathbf{I}_{\mathrm{av}}$, points outwards in all directions: imposing this condition would eliminate physically meaningful solutions.

Instead of writing $p = \mathrm{S}\mu$, we could use one of the representations derived in Section 4, involving boundary integrals of p and $\boldsymbol{v} \cdot \boldsymbol{n}$. These have two advantages: the representations are known to be valid (in the sense that if the boundary value problem has a solution, then the solution can be represented as claimed) and they involve physical quantities (as opposed to μ).

A Exact solutions for a sphere

We consider the axisymmetric oscillations of a sphere (radius a). Use cylindrical polar coordinates, r and z. Start with the elliptic problem, $\omega > N$. Following [28], the governing equations for p are Eq. (1.1) with boundary condition $\boldsymbol{v} \cdot \boldsymbol{n} = f$ on the sphere; the latter becomes

$$(\omega^2 - N^2) r \frac{\partial p}{\partial r} + \omega^2 z \frac{\partial p}{\partial z} = \mathrm{i}\omega(\omega^2 - N^2) a f \quad \text{on } R^2 \equiv r^2 + z^2 = a^2. \qquad \text{(A.1)}$$

Introduce "stretched oblate spheroidal coordinates" (see [1, Eq. (2.3)] or [28, Eq. (8.5)]), ξ and η, defined by

$$r = a(N/\omega)\sqrt{\xi^2+1}\sqrt{1-\eta^2}, \quad z = ac\xi\eta, \quad c = N/\sqrt{\omega^2 - N^2}. \qquad \text{(A.2)}$$

We have

$$(r/a)^2 + (z/a)^2 = (N/\omega)^2(\xi^2+1) + \{c^2\xi^2 - (N/\omega)^2(\xi^2+1)\}\eta^2$$

and this equals 1 when $\xi = 1/c$. From Eq. (A.2),

$$\frac{\omega^2 r^2}{(aN)^2} + \frac{z^2}{(ac)^2} - 1 = \xi^2 - \eta^2 = \xi^2 - \frac{z^2}{(ac)^2\xi^2} = \frac{z^2}{(ac)^2\eta^2} - \eta^2.$$

Differentiating with respect to r gives

$$\frac{\omega^2 r}{(aN)^2} = \frac{1}{\xi}\frac{\partial\xi}{\partial r}\left(\xi^2 + \frac{z^2}{(ac)^2\xi^2}\right) = -\frac{1}{\eta}\frac{\partial\eta}{\partial r}\left(\frac{z^2}{(ac)^2\eta^2} + \eta^2\right).$$

Hence,

$$\frac{\partial\xi}{\partial r} = \frac{\omega^2 r\xi}{(aN)^2(\xi^2+\eta^2)}, \quad \frac{\partial\eta}{\partial r} = -\frac{\omega^2 r\eta}{(aN)^2(\xi^2+\eta^2)}.$$

Similarly, differentiating with respect to z gives

$$\frac{\partial\xi}{\partial z} = \frac{z(\xi^2+1)}{(ac)^2\xi(\xi^2+\eta^2)}, \quad \frac{\partial\eta}{\partial z} = \frac{z(1-\eta^2)}{(ac)^2\eta(\xi^2+\eta^2)}.$$

Then, using $N^2 = c^2(\omega^2 - N^2)$,

$$\begin{aligned}
\frac{\partial p}{\partial r} &= \frac{\omega^2 r}{(aN)^2(\xi^2+\eta^2)}\left(\xi\frac{\partial p}{\partial\xi} - \eta\frac{\partial p}{\partial\eta}\right), \\
\frac{\partial p}{\partial z} &= \frac{(\omega^2 - N^2)z}{(aN)^2(\xi^2+\eta^2)}\left(\frac{\xi^2+1}{\xi}\frac{\partial p}{\partial\xi} + \frac{1-\eta^2}{\eta}\frac{\partial p}{\partial\eta}\right).
\end{aligned}$$

These are the same as [28, Eq. (8.8)] except that the quantity $(aN)^2(\xi^2+\eta^2)$ is replaced by $R^2\sqrt{\omega^2 - \Sigma_+^2}\sqrt{\omega^2 - \Sigma_-^2}$, with $\Sigma_\pm$ defined by [28, Eq. (8.6)].

For the boundary condition, Eq. (A.1), we need the combination

$$(\omega^2 - N^2)r\frac{\partial p}{\partial r} + \omega^2 z\frac{\partial p}{\partial z} = \frac{\omega^2(\omega^2 - N^2)}{(aN)^2(\xi^2+\eta^2)}\left(A\frac{\partial p}{\partial\xi} + B\frac{\partial p}{\partial\eta}\right),$$

where

$$\begin{aligned}
A &= r^2\xi + (z^2/\xi)(\xi^2+1) = a^2\xi(\xi^2+1)\{(N/\omega)^2(1-\eta^2) + c^2\eta^2\} \\
B &= -r^2\eta + (z^2/\eta)(1-\eta^2) = a^2\eta(1-\eta^2)\{c^2\xi^2 - (N/\omega)^2(\xi^2+1)\}.
\end{aligned}$$

Now, on the sphere, $\xi = 1/c$ and $(N/\omega)^2(\xi^2+1) = 1$ so that $B = 0$. Also,

$$(N/\omega)^2(1-\eta^2) + c^2\eta^2 = (Nc/\omega)^2(c^{-2} + \eta^2).$$

Thus, on $\xi = 1/c$,

$$(\omega^2 - N^2)r\frac{\partial p}{\partial r} + \omega^2 z\frac{\partial p}{\partial z} = \frac{\omega^2}{c}\frac{\partial p}{\partial\xi}$$

and so Eq. (A.1) becomes

$$\frac{\partial p}{\partial\xi} = \frac{\mathrm{i}ac}{\omega}(\omega^2 - N^2)f(\eta) \quad \text{on } \xi = 1/c. \tag{A.3}$$

A.1 Pulsating sphere

In this case, $f = U_0$, a constant, so the right-hand side of Eq. (A.3) is constant and the appropriate solution of the partial differential equation for p has the form

$$p = \mathcal{A}_0 Q_0(\mathrm{i}\xi) = \frac{1}{2}\mathcal{A}_0 \log\left(\frac{\mathrm{i}\xi + 1}{\mathrm{i}\xi - 1}\right),$$

where Q_n is a Legendre function. Application of the boundary condition, Eq. (A.3), gives $\mathcal{A}_0 = a(\omega/N)U_0\sqrt{\omega^2 - N^2}$, in agreement with [28, Eq. (8.12)]. For large ξ, $p \sim -\mathrm{i}\mathcal{A}_0/\xi$, in agreement with [28, Eq. (8.17)].

After analytic continuation, we obtain

$$\mathcal{A}_0 = \mathrm{i}a(\omega/N)U_0\sqrt{N^2 - \omega^2} = \mathrm{i}aU_0 N \cos\theta_\mathrm{c} \sin\theta_\mathrm{c}.$$

In Region III, $z_0 \sim r_0 \cos\theta_\mathrm{c}$ and, after analytic continuation, [28] gives

$$\xi^2 \sim \cos\theta_\mathrm{c} \sin\theta_\mathrm{c}(r_0/a)\,\mathrm{e}^{\mathrm{i}\psi_0} \quad \text{with} \quad \psi_0 = \arccos(\sigma_0/a). \tag{A.4}$$

Hence,

$$p \sim aU_0 N\sqrt{(a/r_0)\cos\theta_\mathrm{c}\sin\theta_\mathrm{c}}\,\mathrm{e}^{-\mathrm{i}\psi_0/2} = -\frac{aU_0 N}{2\sqrt{r_0}}\sqrt{\sin 2\theta_\mathrm{c}}\,\mathcal{F}_0(\sigma_0). \tag{A.5}$$

A.2 Vertical oscillations of a rigid sphere

For this case, $f = W_0 n_3 = W_0 z/a = W_0\eta$ on $\xi = 1/c$. Then, the appropriate solution for p is

$$p = \mathcal{B}_0 \eta Q_1(\mathrm{i}\xi) = \mathcal{B}_0\eta\left[\frac{\mathrm{i}\xi}{2}\log\left(\frac{\mathrm{i}\xi + 1}{\mathrm{i}\xi - 1}\right) - 1\right]$$

(see [1, Section 2.1]), with $\mathcal{B}_0$ chosen to satisfy Eq. (A.3):

$$\mathcal{B}_0[Q_1(\mathrm{i}/c) + N^2/\omega^2] = \mathrm{i}(a/\omega)(\omega^2 - N^2)W_0. \tag{A.6}$$

On the sphere,

$$p = \mathcal{P}\eta \quad \text{with} \quad \mathcal{P} = \mathcal{B}_0 Q_1(\mathrm{i}/c). \tag{A.7}$$

In the far field, ξ is large, so that $p \sim -\mathcal{B}_0 z_0/(3ac\xi^3)$. Thus, using Eq. (A.4),

$$p \sim p_\infty (a/r_0)^{1/2}\,\mathrm{e}^{-3\mathrm{i}\psi_0/2} \quad \text{with} \quad p_\infty = -\tfrac{1}{3}\mathrm{i}\mathcal{B}_0[\cos\theta_\mathrm{c}\sin\theta_\mathrm{c}]^{-1/2}. \tag{A.8}$$

For this case, $f = U_0 + W_0 z/a$. By linearity, $p = \mathcal{A}_0 Q_0(\mathrm{i}\xi) + \mathcal{B}_0 \eta Q_1(\mathrm{i}\xi)$, with $\mathcal{A}_0$ and $\mathcal{B}_0$ given above. The combined far-field solution is given by Eq. (8.1) with

$$F(\sigma_0) = aA\mathcal{F}_0(\sigma_0) + B\mathcal{F}_0^3(\sigma_0),$$

where $A = -U_0(N/2)\sqrt{\sin 2\theta_c}$ is a real constant (see Eq. (A.5)), $B = -p_\infty/(2a\sqrt{2})$ is a complex constant and p_∞ is given by Eq. (A.8). Differentiating,

$$F'(\sigma_0) = (aA + 3B\mathcal{F}_0^2)(\mathrm{i}/2)(a^2 - \sigma_0^2)^{-1/2}\mathcal{F}_0(\sigma_0).$$

Write $B = |B|\mathrm{e}^{\mathrm{i}\beta}$. Then, as $\mathcal{F}_0^2 = 2a\mathrm{e}^{-\mathrm{i}\psi_0}$, we find that

$$\operatorname{Im}(\overline{F}F') = \left[A^2 + 12|B|^2 + 8A|B|\cos(\beta - \psi_0)\right] a^2 \csc\psi_0.$$

As expected, this is positive when $B = 0$ (pulsations only) and when $A = 0$ (vertical oscillations only), but $\operatorname{Im}(\overline{F}F')$ can be negative in part of the wave beam. This is most easily seen by fixing B (fix W_0) and then varying A (vary U_0).

B Contribution from the small hemisphere when $P \in S$

Suppose that P at (x_0, y_0, z_0) is a point on S. Denote the unit normal vector at P (pointing out of S) by $\boldsymbol{n}_P$. Let H_ε denote the hemisphere of radius ε, centred at P, with $\boldsymbol{n}_P$ along its axis of symmetry, and with H_ε outside S. We are interested in integrating $(p\boldsymbol{v}^G - G\boldsymbol{v}^p)\cdot\boldsymbol{n}$ over H_ε, in the limit $\varepsilon \to 0$. The result is $p(P)$ multiplied by the value of the integral on the left-hand side of Eq. (4.5), with S_ε replaced by H_ε; denote this integral by I.

It is natural to introduce local spherical polar coordinates at P, with polar axis aligned with $\boldsymbol{n}_P$; this will lead to a formula for I as a repeated integral with constant limits of integration. However, as the integrand involves the global Cartesian coordinates, we require a coordinate rotation.

Let $\boldsymbol{x}' = (x', y', z')$ give local coordinates at P with z' in the direction of $\boldsymbol{n}_P$. Let $\boldsymbol{X} = (x - x_0, y - y_0, z - z_0)$. The integration point on H_ε is located using $\boldsymbol{x}'$ or (x, y, z), so that $|\boldsymbol{x}'| = |\boldsymbol{X}| = \varepsilon$. From [8, Section 4-4], we have $\boldsymbol{x}'^T = A\boldsymbol{X}^T$, where A is a 3×3 orthogonal matrix ($A^{-1} = A^T$) with entries given in terms of Euler angles. With $\boldsymbol{n}_P = (\sin\theta_P\cos\phi_P,\ \sin\theta_P\sin\phi_P,\ \cos\theta_P)$,

$$A = \begin{pmatrix} -\sin\phi_P & \cos\phi_P & 0 \\ -\cos\theta_P\cos\phi_P & -\cos\theta_P\sin\phi_P & \sin\theta_P \\ \sin\theta_P\cos\phi_P & \sin\theta_P\sin\phi_P & \cos\theta_P \end{pmatrix};$$

as a check, we note that $A\boldsymbol{n}_P^T = (0,0,1)^T$. Next, we locate the integration point using $\boldsymbol{x}' = \varepsilon(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and calculate $\boldsymbol{X}^T = A^T\boldsymbol{x}'^T$; this gives

$$\begin{aligned}
\varepsilon^{-1}(x-x_0) &= -(\sin\phi_P\cos\varphi + \cos\theta_P\cos\phi_P\sin\varphi)\sin\theta + \sin\theta_P\cos\phi_P\cos\theta, \\
\varepsilon^{-1}(y-y_0) &= (\cos\phi_P\cos\varphi - \cos\theta_P\sin\phi_P\sin\varphi)\sin\theta + \sin\theta_P\sin\phi_P\cos\theta, \\
\varepsilon^{-1}(z-z_0) &= \sin\theta_P\sin\theta\sin\varphi + \cos\theta_P\cos\theta;
\end{aligned}$$

as a check, $|\boldsymbol{X}| = \varepsilon$. Finally, we obtain

$$I = \frac{\mathrm{i}}{\omega}\int_0^{2\pi}\int_0^{\pi/2} \frac{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{\{\Lambda(\theta,\varphi;\boldsymbol{n}_P)\}^{3/2}} \tag{B.1}$$

where

$$\begin{aligned}
\Lambda = {} & [(\cos^2\theta_P + \Upsilon^{-1}\sin^2\theta_P)\sin^2\varphi + \cos^2\varphi]\sin^2\theta \\
& + (\sin^2\theta_P + \Upsilon^{-1}\cos^2\theta_P)\cos^2\theta + \frac{1}{2}(\Upsilon^{-1}-1)\sin 2\theta_P\sin 2\theta\sin\varphi;
\end{aligned}$$

as expected, $\Lambda = 1$ when $\Upsilon = 1$. Then, the quantity $\mathcal{A}(P)$ defined below Eq. (4.7) is given by $\mathcal{A} = (\omega^2 - N^2)^{1/2} I/(4\pi\mathrm{i})$.

C Details for the far field within the wave beams

C.1 Coordinate systems

The integration point, Q, is at (x, y, z). Following [29], introduce local Cartesian coordinates, (X_+, Y, Z_+) and (X_-, Y, Z_-), so that

$$x = X_\pm\cos\theta_\mathrm{c}\cos\phi_0 - Y\sin\phi_0 \pm Z_\pm\sin\theta_\mathrm{c}\cos\phi_0, \tag{C.1}$$
$$y = X_\pm\cos\theta_\mathrm{c}\sin\phi_0 + Y\cos\phi_0 \pm Z_\pm\sin\theta_\mathrm{c}\sin\phi_0, \tag{C.2}$$
$$z = \mp X_\pm\sin\theta_\mathrm{c} + Z_\pm\cos\theta_\mathrm{c}. \tag{C.3}$$

Note that as $x^2 + y^2 + z^2 = X_\pm^2 + Y^2 + Z_\pm^2$, $X_\pm$, Y and $Z_\pm$ are all bounded for $Q \in S$.

We locate points on S_a using Θ and Φ, assuming that P is in Region III; the reason for doing this is given at the end of Section 7.2. Our purpose here is to identify the (integration) region in the $\Theta\Phi$-plane.

Rather than relate Θ and Φ to x, y and z, we use X_+, Y and Z_+. To begin, Eq. (5.2) gives

$$x - x_0 = (z - z_0)\mathcal{T}\cos\Phi \quad \text{and} \quad y - y_0 = (z - z_0)\mathcal{T}\sin\Phi, \tag{C.4}$$

where $\mathcal{T} = \tan\Theta$. Also, Eqs. (7.1) and (C.3) give

$$z - z_0 = \tilde{Z}\cos\theta_{\rm c} - \tilde{X}\sin\theta_{\rm c} = \Lambda, \tag{C.5}$$

say, where $\tilde{X} = X_+ - \sigma_0$ and $\tilde{Z} = Z_+ - \zeta_0$. Then, as $x_0 = \varrho_0\cos\phi_0$ and $y_0 = \varrho_0\sin\phi_0$, Eqs. (C.1), (C.2) and (C.4) give

$$\begin{aligned} x - x_0 &= \tilde{X}\cos\theta_{\rm c}\cos\phi_0 - Y\sin\phi_0 + \tilde{Z}\sin\theta_{\rm c}\cos\phi_0 = \Lambda\mathcal{T}\cos\Phi, \\ y - y_0 &= \tilde{X}\cos\theta_{\rm c}\sin\phi_0 + Y\cos\phi_0 + \tilde{Z}\sin\theta_{\rm c}\sin\phi_0 = \Lambda\mathcal{T}\sin\Phi. \end{aligned}$$

These two equations give $Y = \Lambda\mathcal{T}s$ and $\tilde{Z}S + \tilde{X}C = \Lambda\mathcal{T}c$, where

$$C = \cos\theta_{\rm c}, \quad S = \sin\theta_{\rm c}, \quad c = \cos(\Phi - \phi_0), \quad s = \sin(\Phi - \phi_0).$$

Substituting for Λ from Eq. (C.5) and then solving for Y and $\tilde{X}$ in terms of $\tilde{Z}$ gives

$$\tilde{X} = \frac{(\mathcal{T}Cc - S)\tilde{Z}}{\mathcal{T}Sc + C}, \quad Y = \frac{\mathcal{T}s\tilde{Z}}{\mathcal{T}Sc + C}. \tag{C.6}$$

Thus,

$$(\mathcal{T}Sc + C)^2(X_+^2 + Y^2) = [(\mathcal{T}Sc + C)\sigma_0 + (\mathcal{T}Cc - S)\tilde{Z}]^2 + [\mathcal{T}s\tilde{Z}]^2. \tag{C.7}$$

Now, the equation defining S_a is $x^2 + y^2 + z^2 = X_+^2 + Y^2 + Z_+^2 = a^2$, so $X_+^2 + Y^2 = a^2 - (\tilde{Z} + \zeta_0)^2$. Hence, eliminating $X_+^2 + Y^2$, using Eq. (C.7), gives a quadratic equation for $\tilde{Z}$,

$$\begin{aligned} 0 &= [(\mathcal{T}Cc - S)\tilde{Z} + (\mathcal{T}Sc + C)\sigma_0]^2 + [\mathcal{T}s\tilde{Z}]^2 + (\mathcal{T}Sc + C)^2[(\tilde{Z} + \zeta_0)^2 - a^2] \\ &= \tilde{Z}^2\{(\mathcal{T}Cc - S)^2 + \mathcal{T}^2s^2 + (\mathcal{T}Sc + C)^2\} + 2\tilde{Z}(\mathcal{T}Sc + C)Q_1 + (\mathcal{T}Sc + C)^2Q_2, \end{aligned}$$

where $Q_1 = (\mathcal{T}Cc - S)\sigma_0 + (\mathcal{T}Sc + C)\zeta_0$ and $Q_2 = \sigma_0^2 + \zeta_0^2 - a^2$. As the coefficient multiplying $\tilde{Z}^2$ simplifies to $\sec^2\Theta$, the solution for $\tilde{Z}(\Theta, \Phi)$ is

$$\tilde{Z} = (\mathcal{T}Sc + C)\cos^2\Theta\left\{-Q_1 \pm (Q_1^2 - Q_2\sec^2\Theta)^{1/2}\right\} = (C\cos\Theta + cS\sin\Theta)\mathcal{Q}_\pm, \tag{C.8}$$

say, where

$$\mathcal{Q}_\pm(\Theta,\Phi) = -Q_3 \pm \sqrt{\Delta}, \quad \Delta(\Theta,\Phi) = Q_3^2 - Q_2, \tag{C.9}$$
$$Q_3 = Q_1 \cos\Theta = (cC\sin\Theta - S\cos\Theta)\sigma_0 + (cS\sin\Theta + C\cos\Theta)\zeta_0. \tag{C.10}$$

Then, Eq. (C.6) gives

$$\tilde{X}(\Theta,\Phi) = (cC\sin\Theta - S\cos\Theta)\mathcal{Q}_\pm \quad \text{and} \quad Y(\Theta,\Phi) = s\mathcal{Q}_\pm \sin\Theta. \tag{C.11}$$

Evidently, we require that $\Delta(\Theta,\Phi) \geq 0$. When $\Theta = \theta_\mathrm{c}$ and $\Phi = \phi_0$, we have $c = 1$, $Q_3 = \zeta_0$ and $\Delta(\theta_\mathrm{c},\phi_0) = a^2 - \sigma_0^2 \geq 0$, because $|\sigma_0| < a$ when P is in Region III. Moreover, when P is in the far field, the region of the $\Theta\Phi$-plane in which $\Delta \geq 0$ is small, so we can approximate. Thus, for $\Theta \simeq \theta_\mathrm{c}$ and $\Phi \simeq \phi_0$, write $\vartheta = \Theta - \theta_\mathrm{c}$ and $\varphi = \Phi - \phi_0$. Then

$$\begin{aligned} Q_3 &= \zeta_0 \cos\vartheta + \sigma_0 \sin\vartheta + (c-1)(C\sigma_0 + S\zeta_0)\sin\Theta \\ &\simeq \zeta_0 + \sigma_0\vartheta - \tfrac{1}{2}\zeta_0\vartheta^2 - \tfrac{1}{2}(C\sigma_0 + S\zeta_0)S\varphi^2, \end{aligned}$$

correct to second order in ϑ and φ. Then, from Eq. (C.9),

$$\Delta \simeq a^2 - \sigma_0^2 + 2\zeta_0\sigma_0\vartheta + (\sigma_0^2 - \zeta_0^2)\vartheta^2 - (C\sigma_0 + S\zeta_0)S\zeta_0\varphi^2. \tag{C.12}$$

We are interested in locating the curve defined by $\Delta(\Theta,\Phi) = 0$, given approximately by

$$(C\sigma_0 + S\zeta_0)S\zeta_0\varphi^2 + \mathcal{Z}_0\vartheta^2 - 2\zeta_0\sigma_0\vartheta = a^2 - \sigma_0^2, \tag{C.13}$$

where $\mathcal{Z}_0 = \zeta_0^2 - \sigma_0^2$ is taken to be positive. Completing the square shows that Eq. (C.13) defines an ellipse, $\varphi^2/\mathcal{A}^2 + (\vartheta - \vartheta_0)^2/\mathcal{B}^2 = 1$, where $\vartheta_0 = \zeta_0\sigma_0/\mathcal{Z}_0$,

$$\mathcal{A}^2 = \frac{a^2 - \sigma_0^2 + \mathcal{Z}_0\vartheta_0^2}{(C\sigma_0 + S\zeta_0)S\zeta_0} \quad \text{and} \quad \mathcal{B}^2 = \vartheta_0^2 + \frac{a^2 - \sigma_0^2}{\mathcal{Z}_0}. \tag{C.14}$$

Denote the interior of this ellipse by $\mathcal{E}$. In the $\Theta\Phi$-plane, $\mathcal{E}$ is aligned with the coordinate axes, and its centre is as $(\Theta,\Phi) = (\theta_\mathrm{c} + \vartheta_0, \phi_0)$. The endpoints of the major axis are at $(\Theta,\Phi) = (\theta_\mathrm{c} + \vartheta_0, \phi_0 \pm \mathcal{A})$. The endpoints of the minor axis are at $(\Theta,\Phi) = (\theta_\mathrm{c} + \vartheta_0 \pm \mathcal{B}, \phi_0)$. The area of $\mathcal{E}$ is approximately $\pi a^2/(r_0^2 \sin\theta_\mathrm{c})$ for large r_0. As $\mathcal{B} > \vartheta_0$, $(\Theta,\Phi) = (\theta_\mathrm{c},\phi_0)$ is always in $\mathcal{E}$. We conclude that $\Delta \geq 0$ in $\mathcal{E}$. Explicitly, $\mathcal{E}$ is defined by

$$\Phi_-(\Theta) < \Phi < \Phi_+(\Theta), \quad \Theta_- < \Theta < \Theta_+, \tag{C.15}$$

where

$$\Phi_\pm(\Theta) = \phi_0 \pm (\mathcal{A}/\mathcal{B})\sqrt{(\Theta_+ - \Theta)(\Theta - \Theta_-)}, \quad \Theta_\pm = \theta_\mathrm{c} + \vartheta_0 \pm \mathcal{B}. \tag{C.16}$$

Then, from Eq. (C.12), we obtain

$$\Delta(\Theta,\Phi) = (C\sigma_0 + S\zeta_0)S\zeta_0\,[\Phi_+(\Theta) - \Phi][\Phi - \Phi_-(\Theta)]\,. \tag{C.17}$$

In the far field, where ζ_0 is large, we have $\mathcal{Z}_0 \simeq \zeta_0^2$, $\vartheta_0 \simeq \sigma_0/\zeta_0$, $\mathcal{A} \simeq a/(\zeta_0 S)$ and $\mathcal{B} \simeq a/\zeta_0 \simeq S\mathcal{A}$. Thus, $\mathcal{E}$ shrinks as ζ_0 increases, and its centre moves towards (θ_c, ϕ_0). Also, $\zeta_0 \sim r_0$ and

$$\Theta_\pm \simeq \theta_c \pm (a \pm \sigma_0)/r_0, \tag{C.18}$$

$$[\Phi] = \Phi_+ - \Phi_- \simeq (2/S)\sqrt{(\Theta_+ - \Theta)(\Theta - \Theta_-)}. \tag{C.19}$$

C.3 The Jacobian

In terms of Θ and Φ, we have $\mathrm{d}S = |(j_x, j_y, j_z)|\,\mathrm{d}\Theta\,\mathrm{d}\Phi$, where

$$(j_x, j_y, j_z) = \left(\frac{\partial X_+}{\partial\Theta}, \frac{\partial Y}{\partial\Theta}, \frac{\partial Z_+}{\partial\Theta}\right) \times \left(\frac{\partial X_+}{\partial\Phi}, \frac{\partial Y}{\partial\Phi}, \frac{\partial Z_+}{\partial\Phi}\right). \tag{C.20}$$

Direct calculation from Eqs. (C.8) and (C.11) gives

$$\begin{aligned}
\frac{\partial X_+}{\partial\Theta} &= (cC\sin\Theta - S\cos\Theta)\frac{\partial\mathcal{Q}_\pm}{\partial\Theta} + (cC\cos\Theta + S\sin\Theta)\mathcal{Q}_\pm,\\
\frac{\partial Y}{\partial\Theta} &= s\frac{\partial\mathcal{Q}_\pm}{\partial\Theta}\sin\Theta + s\mathcal{Q}_\pm\cos\Theta,\\
\frac{\partial Z_+}{\partial\Theta} &= (C\cos\Theta + cS\sin\Theta)\frac{\partial\mathcal{Q}_\pm}{\partial\Theta} + (-C\sin\Theta + cS\cos\Theta)\mathcal{Q}_\pm,\\
\frac{\partial X_+}{\partial\Phi} &= (cC\sin\Theta - S\cos\Theta)\frac{\partial\mathcal{Q}_\pm}{\partial\Phi} - sC\mathcal{Q}_\pm\sin\Theta,\\
\frac{\partial Y}{\partial\Phi} &= s\frac{\partial\mathcal{Q}_\pm}{\partial\Phi}\sin\Theta + c\mathcal{Q}_\pm\sin\Theta,\\
\frac{\partial Z_+}{\partial\Phi} &= (C\cos\Theta + cS\sin\Theta)\frac{\partial\mathcal{Q}_\pm}{\partial\Phi} - sS\mathcal{Q}_\pm\sin\Theta.
\end{aligned}$$

From these and Eq. (C.20), j_x, j_y and j_z can be calculated, whence

$$\frac{j_x^2 + j_y^2 + j_z^2}{\mathcal{Q}_\pm^2} = \mathcal{Q}_\pm^2\sin^2\Theta + \left(\frac{\partial\mathcal{Q}_\pm}{\partial\Theta}\right)^2\sin^2\Theta + \left(\frac{\partial\mathcal{Q}_\pm}{\partial\Phi}\right)^2.$$

We have $\mathcal{Q}_\pm = -Q_3 \pm \sqrt{\Delta}$ with $\Delta = Q_3^2 - Q_2$, and, from Eq. (C.10),

$$\frac{\partial Q_3}{\partial\Theta} = (cC\cos\Theta + S\sin\Theta)\sigma_0 + (cS\cos\Theta - C\sin\Theta)\zeta_0, \quad \frac{\partial Q_3}{\partial\Phi} = -s(C\sigma_0 + S\zeta_0)\sin\Theta.$$

Hence,

$$\frac{\partial\mathcal{Q}_\pm}{\partial\Theta} = \left(-1 \pm \frac{Q_3}{\sqrt{\Delta}}\right)\frac{\partial Q_3}{\partial\Theta} = \mp\frac{\mathcal{Q}_\pm}{\sqrt{\Delta}}\frac{\partial Q_3}{\partial\Theta},$$

with a similar formula for $\partial\mathcal{Q}_\pm/\partial\Phi$. Thus,

$$\begin{aligned} j_x^2 + j_y^2 + j_z^2 &= \frac{\mathcal{Q}_\pm^4}{\Delta}\left\{\Delta \sin^2\Theta + \left(\frac{\partial Q_3}{\partial\Theta}\right)^2 \sin^2\Theta + \left(\frac{\partial Q_3}{\partial\Phi}\right)^2\right\} \\ &= \frac{\mathcal{Q}_\pm^4}{\Delta}\left\{a^2 + Q_3^2 + \left(\frac{\partial Q_3}{\partial\Theta}\right)^2 - \sigma_0^2 - \zeta_0^2 + s^2(C\sigma_0 + S\zeta_0)^2\right\}\sin^2\Theta. \end{aligned}$$

The expression inside the braces simplifies to a^2, whence

$$|(j_x, j_y, j_z)| = \frac{\mathcal{Q}_\pm^2 a}{\sqrt{\Delta}} \sin\Theta. \tag{C.21}$$

Acknowledgement

The authors are grateful for thorough reviews of the paper, leading to significant corrections, clarifications and improvements.

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