Axisymmetric magnetic vortices with swirl

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Abstract

New solutions for compact vortex rings and vortices in the presence of swirl and toroidal magnetic field are presented. Steady solutions are found for thin-core vortex rings, generalizing the classical results of Kelvin and Hicks. An exact family of steady spherical solutions is found in the case when the underlying governing equation, originally due to Hicks, is linear. Proper matching of the pressure is critical in obtaining these solutions, for which another 19th century equation proves key. Contour dynamics simulations show the existence of counter-propagating dipolar structures when both swirl and magnetic fields are present.

Keywords: Vortex rings, magnetohydrodynamics, swirling flows

1. Introduction

The history of vortex structures with swirl is long, dating back to Hicks in 1899 [1]. Quite a few results seem to have been rediscovered a number of times. For example, the vorticity equation due to Hicks has been linked to the names of Bragg and Hawthorne [2] and of Long and Squire (who mentions Fraenkel) [3, 4, 5]. In plasma physics, an isomorphic equation relating magnetic field, current density and pressure is called the Grad–Shafranov equation [6, 7]. Solutions of the Grad–Shafranov equation do not translate directly across to the fluids problem, since the boundary conditions are different.

The number of explicit solutions for vortices with swirl is small. The only explicit solution is that that found by Hicks [1], which is also presented in [8]. A number of algorithms to calculate vortices and proofs of the existence of the vortices have been presented [e.g. 9, 10], but they have not usually been constructive. Recent non-axisymmetric simulations of vortex rings with swirl are presented in Cheng *et al.* [11].

Understanding the effect of swirl on vortex rings is of interest for a number of reasons. Virk *et al.* [12] argue that most vortical structures in transitional and turbulent flows have a swirl component. Vortical floss with swirl are also relevant in the study of vortex breakdown [13, § 4.4]. Finally, obtaining exact solutions to the Euler equations is always of use for validation of numerical simulations.

Hattori & Moffatt (HM henceforth) [14] describe axisymmetric magnetic eddies, that is structures with a localized toroidal magnetic field that moves under the effect of the vortex sheet generated by the discontinuity in the magnetic field. The eddies have no distributed vorticity. These are contour dynamics (CD) calculations in which the magnetic field has a special spatial structure that automatically satisfies the governing equations. A class of steady solutions related to Hill's Spherical Vortex (HSV) was also found, again with a vortex sheet on the boundary. The goal of this work was to understand the evolution of inviscid flows with magnetic field in situations where the topological constraint enforced by conservation of helicity was not present since the helicity vanishes for this flow. In fact the magnetic helicity and cross-helicity both vanish for this flow.

The CD approach for vortex rings was pioneered by Pozrikidis [15] and Shariff *et al.* [16]. A clear presentation is given in Riley [17]. The extension to swirl has been derived by Llewellyn Smith & Morin in an unpublished

☆This research was supported by Collaborative Research Project 2010, Institute of Fluid Science, Tohoku University, Project Code J10022.
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manuscript and has many similarities to the situation with magnetic field considered by HM. (Steady CD solutions for the Grad–Shafranov equation are given in [18].)

In this paper we consider the general case of axisymmetric vortical structures with vorticity, swirl and magnetic field. We present the governing equations and then investigate a number of solutions using different methods. In § 2 we derive results for thin-core vortex rings with swirl and magnetic field. In § 3 we examine a class of steady spherical solutions and show that only a limited class of solutions exist. In § 4 we show results of unsteady CD calculations with swirl and magnetic field. We find that when both are present, the initial condition evolves into two oppositely-moving vortex dipoles that move at nearly constant speed. In § 5 we calculate the motion of these dipoles. Finally in § 6 we conclude.

We limit ourselves to inviscid axisymmetric flows with toroidal magnetic field. We use cylindrical polar coordinates (r, θ, z) with corresponding velocity components (u, v, w). The magnetic field B is purely azimuthal (toroidal field). Then the governing equations can be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}(rv) = 0, \qquad \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\omega}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial z}(v^2 - B^2), \qquad \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{B}{r}\right) = 0, \qquad \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{1}$$

where ω is the azimuthal component of vorticity. The first equation corresponds to the conservation of angular momentum. One can show from these equations that the hydrodynamic impulse (cf. §3.9 of [13]) is conserved in the presence of magnetic field.

The velocity field is

$$\boldsymbol{u} = v\boldsymbol{e}_\theta + \boldsymbol{u}_P, \tag{2}$$

where $\boldsymbol{u}_P$ is the poloidal velocity and ψ is the Stokes streamfunction. Hence the radial and axial velocity components are, respectively,

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad w = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{3}$$

The vorticity is

$$\boldsymbol{\omega} = \omega\boldsymbol{e}_\theta + \boldsymbol{\omega}_P = -\frac{1}{r}E\psi\boldsymbol{e}_\theta + \frac{1}{r}\boldsymbol{\nabla}(rv) \times \boldsymbol{e}_\theta, \tag{4}$$

with the operator E given by

$$E = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{5}$$

This inviscid system conserves total energy, helicity and cross helicity, given in general by

$$E_T = E_H + E_M = \frac{1}{2}\int (|\boldsymbol{u}|^2 + |\boldsymbol{B}|^2)\,\mathrm{d}V, \qquad H_M = \int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V, \qquad H_C = \int \boldsymbol{u}\cdot\boldsymbol{B}\,\mathrm{d}V \tag{6}$$

respectively, where $\boldsymbol{A}$ is the magnetic potential. In our case, $H_M = 0$, while the energy and cross helicity can be written as

$$E_T = \frac{1}{2}\int (v^2 + |\boldsymbol{u}_P|^2 + B^2)\,\mathrm{d}V, \qquad H_C = \int vB\,\mathrm{d}V. \tag{7}$$

The cross helicity is zero (as in HM) unless both swirl and toroidal field are non-zero.

2. Thin-core magnetic vortices with swirl

We first ask if the analog of vortex rings exist for our situation, namely vortices with non-vanishing vorticity, swirl and magnetic field inside an annular region. It is straightforward to follow the analysis in § 10.3 of [13]. Lamb's transformation still works and the only change is that (in our notation) v^2 is replaced by $v^2 - B^2$. Hence for velocity and toroidal magnetic fields inside the vortex that depend only on the radial coordinate defined with respect to the center of the vortex, the thin-core vortex ring formula becomes

$$U = \frac{\Gamma}{4\pi R}\left(\log\frac{8R}{a} - \frac{1}{2} + 2\frac{\pi^2 a^2 \overline{v_\theta^2}}{\Gamma^2} - 4\frac{\pi^2 a^2(\overline{v_0^2} - \overline{B^2})}{\Gamma^2}\right), \tag{8}$$

where R is the ring radius, a the core radius, Γ the circulation, v_0 the swirl velocity, B the toroidal magnetic field and v_θ the angular velocity of the flow in the ring. Overbars denote the average over the cross-section of the rings, e.g.

$$\overline{w_0^2} = \frac{1}{\pi a^2}\int_0^a r w_0^2 \,\mathrm{d}r. \tag{9}$$

We have

$$v_\theta = \frac{\Gamma_0(r)}{2\pi r}, \qquad \Gamma_0(r) = 2\pi \int_0^r s\omega_0(s)\,\mathrm{d}s, \tag{10}$$

so that $\Gamma_0(r)$ is the circulation within radius r, given that $\omega_0(r)$ is the vorticity in the ring. Note that one needs to distinguish between the total circulation of the ring, Γ, and the circulation inside the ring, Γ_I, that is, without the vortex sheet on the boundary. For a uniform ring with a vortex sheet on the boundary, the result is

$$U = \frac{\Gamma}{4\pi R}\left(\log\frac{8R}{a} - \frac{1}{2} + \frac{\Gamma_I^2}{4\Gamma^2} - 4\frac{\pi^2 a^2(\overline{v_0^2} - \overline{B^2})}{\Gamma^2}\right). \tag{11}$$

This recovers the results of Kelvin ($\Gamma_I = \Gamma$) and of Hicks ($\Gamma_I = 0$). This result only makes sense for vortices with non-vanishing circulation. We will have reason later to consider vortices with zero circulation but non-zero dipole moment. The magnetic contribution to the motion of a thin vortex ring was previously derived independently [19].

3. Steady linear spherical solutions

We now move to solutions that are not thin. A family of steady CD rings was discovered by Norbury [20], who obtained their shapes numerically. The spherical end-member of this family is just HSV. HM found explicit spherical solutions in the case with magnetic field, as mentioned previously. For the case with swirl, spherical solutions were obtained by Hicks [1] and rediscovered by Moffatt [8]. We consider the generalized spherical case with swirl and magnetic field. To do this we use the Bragg–Hawthorne or Squire–Long equation, which was first derived by Hicks, but add magnetic field.

Exactly the same approach as in [13, § 3.13] leads to

$$E\psi = r^2\frac{\mathrm{d}H}{\mathrm{d}\Psi} - \frac{1}{2}\frac{\mathrm{d}C^2}{\mathrm{d}\Psi} - \frac{r^4}{2}\frac{\mathrm{d}D^2}{\mathrm{d}\Psi}, \tag{12}$$

where now $C(\psi) = rv$ and $D(\psi) = r^{-1}B$ are viewed as functions of the Stokes streamfunction ψ. The function H is the Bernoulli function, and in the case without magnetic field $\boldsymbol{\nabla} H = \omega \times \boldsymbol{u}$. The magnetic term is new. Note that the magnetic term in (12) is quite different from the form of the magnetic term in the Grad–Shafranov equation. The Hicks equation (12) gives the vorticity in terms of three independent functions of ψ, and in general leads to a nonlinear equation for ψ. For simplicity, we limit ourselves to choices of H, C and D that lead to a linear equation for ψ.

We consider spherical vortices with vorticity, swirl and magnetic field confined to the region $R < a$, where we define spherical polar coordinates using

$$r = R\sin\theta, \qquad z = R\cos\theta. \tag{13}$$

Outside, for $R > a$, we have $C = D = 0$ and $H = H_\infty$, a constant. Inside, the most general choice yielding a linear equation is

$$H = H_0 + H_1\psi + H_2\psi^2, \qquad C^2 = C_0^2 + C_1^2\psi + C_2^2\psi^2, \qquad D^2 = D_0^2 + D_1^2\psi + D_2^2\psi^2, \tag{14}$$

with 9 parameters. We immediately see that H_∞, H_0, C_0^2 and D_0^2 do not enter (12) directly. Such terms would correspond to a a vortex sheet on the boundary of the vortex. For small r, the C_0^2 and C_1^2 terms lead to unbounded vorticity near the axis, which we reject on physical grounds. Hence $C_0^2 = C_1^2 = 0$. In addition it is apparent that only the difference $H_\infty - H_0$ can be physically relevant, so we are down to 7 parameters. Substituting these forms into (12) gives

$$E\psi = R^2\sin^2\theta(H_1 + 2H_2\psi) - C_2^2\psi - R^4\sin^4\theta(D_1^2 + 2D_2^2\psi) \tag{15}$$

for $R < a$ and $E\psi = 0$ for $R > a$. Outside the vortex, the appropriate solution that vanishes on the boundary is the irrotational flow with velocity U at infinity past a sphere of radius a given by

$$\psi = -\tfrac{1}{2} U \sin^2\theta \left(R^2 - \frac{a^3}{R} \right). \tag{16}$$

Note that as $R \to a^+$, $\mathrm{d}\psi/\mathrm{d}R \to -(3/2)U \sin^2\theta$.

The operator E in spherical polar coordinates becomes

$$E = \frac{\partial^2}{\partial R^2} + \frac{\sin\theta}{R^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \right). \tag{17}$$

This operator has normal modes $z_n(\theta) = -\sin\theta(\mathrm{d}/\mathrm{d}\theta)P_{n-1}(\cos\theta)$ for $n \geq 2$, $z_1(\theta) = \cos\theta$ and $z_0 = 1$. We see that in (15) the terms in H_2 and D_2^2 couple all the modes together. The boundary conditions on ψ are that it be regular near the origin of space and vanish at $R = a$, the boundary of the vortex which is a streamline. This is a complete specification of the interior problem.

At this point we need to consider the behavior of the pressure on the boundary of the vortex. We can add magnetic field to the appendix by Hill to [1] and obtain a relation valid over all space:

$$0 = -\boldsymbol{\nabla}(p + \tfrac{1}{2}|\boldsymbol{u}|^2 - H + B^2). \tag{18}$$

The relation (18) is a version of the steady Bernoulli equation for flow with vorticity, but with the Bernoulli constant being uniform over the whole flow rather than along streamlines. It is hence rather more powerful than the usual steady version. It does not appear to be well known, although a similar two-dimensional version for piecewise continuous vorticity can be found in Lamb's *Hydrodynamics*, § 159a [21].

The value of (18) comes when we consider the behavior of the pressure across the vortex boundary. We can also show (e.g. by evaluating the integral version of Newton's Second Law for a small pillbox spanning the boundary of the vortex) that the modified pressure $p_* \equiv p + \frac{1}{2}B^2$ must be continuous across the boundary. Now since (18) holds over all space, $p + \frac{1}{2}|\boldsymbol{u}|^2 - H + B^2$ is continuous over all space, and in particular across the boundary. Subtracting $p + \frac{1}{2}B^2$, we obtain the condition

$$\tfrac{1}{2}|\boldsymbol{u}|^2 - H + \tfrac{1}{2}B^2 \text{ is continuous across the boundary of the vortex.} \tag{19}$$

The advantage (19) is that we have removed the pressure dependence. Alternatively, we can obtain (19) by integrating (12) across the boundary, using the fact that $\partial H/\partial r = rw\mathrm{d}H/\mathrm{d}\psi$ and so on.

Let us assume now that we have solved for the solution in the interior and try to match it to the exterior solution (16). We have

$$\psi = \sum_n z_n(\theta) f_n(R) \tag{20}$$

and by construction $\psi = 0$ on $R = a$, so that $f_n(a) = 0$. The R-derivative of ψ outside the vortex from (16) corresponds purely to mode 2, while the R-derivative of the interior ψ depends on the form of the solution of (15). Unless it is pure mode 2, which is not possible if the modes are coupled by the presence of H_2 and D_2^2, there will be a discontinuity in tangential velocity, i.e. a vortex sheet, on the boundary. 7 However, we also need to satisfy (19) across the boundary. This gives

$$\frac{9}{4}U^2 \sin^2\theta - H_\infty = \frac{1}{2a^2 \sin^2\theta} \left(\sum z_n(\theta) f_n'(a) \right)^2 - H_0 + \frac{1}{2} D_0 a^2 \sin^2\theta. \tag{21}$$

Hence any streamfunction in the interior that contains modes other than 0 and 2 will not satisfy this relation and there will then be no solution to the problem. This is the case when H_2 and D_2 are non-zero as these coefficients lead to a solution with all modes, and also when D_1 is non-zero, since it forces a mode-4 solution in the interior.

A solution will exist when the interior or the vortex sheet is pure mode 2. If H_1 is non-zero, this corresponds to HSV and the Hicks–Moffatt vortex for C_2 zero and non-zero respectively. If H_1 vanishes, there can be a solution if D_0 is non-zero, corresponding to the magnetic eddies of HM, which have a vortex sheet on the boundary.

Hence the solution to this problem depends on the quantities $H_0 - H_\infty$, H_1, C_2 and D_0. We write $\alpha = C_2$, $A = H_1$ and $\kappa = D_0$ to make contact with previous results. The solutions then satisfy

$$\frac{\partial^2\psi}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) + \alpha^2\psi = AR^2\sin^2\theta. \tag{22}$$

The solution inside the vortex is the same as for the Hicks–Moffatt vortex:

$$\psi = R^2\sin^2\theta[-A/C_2^2 + c(a/R)^{3/2}J_{3/2}(\alpha R)]. \tag{23}$$

The constant c comes from requiring the edge of the vortex to be a streamline, so that

$$c = \frac{A}{\alpha^2 J_{3/2}(\alpha a)}. \tag{24}$$

The pressure matching condition on the boundary gives

$$\frac{9}{8}U^2\sin^2\theta - H_\infty = \frac{1}{2}\sin^2\theta\left(\frac{AaJ_{5/2}(\alpha a)}{\alpha J_{3/2}(\alpha a)}\right)^2 - H_0 + \frac{1}{2}\kappa^2 a^2\sin^2\theta. \tag{25}$$

The difference $H_0 - H_\infty$ is independent of the other parameters. Note that on the vortex boundary, the swirl velocity vanishes. We find

$$\left(\frac{3U}{2}\right)^2 - \left(\frac{AaJ_{5/2}(\alpha a)}{\alpha J_{3/2}(\alpha a)}\right)^2 = \kappa^2 a^2. \tag{26}$$

Given α and κ, we have a two-parameter family of vortices that reduce to those of HM as $\alpha \to 0$ and to the Hicks–Moffatt family as $A \to 0$.

Solutions for the Grad–Shafranov equation have been sought for spherical geometries, but there is no far field to match onto, and hence general solutions are possible [22]. The fluids analog is the situation with a rigid boundary at the edge of the vortex, which can support pressure, as mentioned originally by Hicks.

4. Unsteady contour dynamics solutions

We now look for solutions with

$$ru = CH[f], \qquad B = \kappa rH[f], \qquad \omega = rAH[f] + r\Omega\delta(f)|\nabla f|, \tag{27}$$

where H and δ are the Heaviside and delta functions respectively, and $f(x, y, z, t)$ defines the boundary of the vortex with $f > 0$ inside the vortex. For these solutions, the volume V of the vortex is also conserved and we see that $H_C = uBV = \kappa CV$.

We follow the procedure of HM and obtain contour dynamics equations. The vortex sheet strength Ω evolves in time, but A is steady. It is more convenient to use γ, which is the circulation around a small area of the vortex boundary, given by $\gamma = \Omega LR$. The boundary of the vortex is $(r, z) = (R(s, t), Z(s, t))$ and $L^2(s, t) = (\partial R/\partial s)^2 + (\partial Z/\partial s)^2$, where s is a parameterization of the boundary. Equations (2.23) and (2.25–2.26) of HM become

$$\frac{\partial\gamma}{\partial t} = \left(\kappa^2 R - \frac{C^2}{R^3}\right)\frac{\partial R}{\partial s} \tag{28}$$

and

$$u_r(r, z) = u_r^{HM} + u_r^R, \qquad u_z(r, z) = u_z^{HM} + u_z^R. \tag{29}$$

The contributions to the poloidal velocity from the magnetic field and swirl are integrals of γ around the boundary of the vortex and are given in (2.32–2.33) of HM. The contribution to the poloidal velocity from the A term is given most simply by (2.11) and (2.14) of [17].

5. Dipolar solutions

Figures 1 and 2 show examples of numerical simulation using the contour dynamics method derived above. We consider the case $C = \kappa = 1$ and $A = 0$, so the vortex does not have a self-induced velocity along its axis of symmetry. Initially there is no poloidal flow: $\Omega = 0$. The boundary of the vortex is a torus of circular cross section, and the ratio of the core radius σ_0 to the torus radius R_0 is set to 0.2 and 0.5. Note that the magnetic and swirl effects in (28) balance at $r = 1$. Vorticity on the boundary is generated so that the boundary moves to $r = 1$. As time proceeds the torus splits into two tori connected by a thin cylindrical sheet. The tori move almost steadily in the opposite direction.

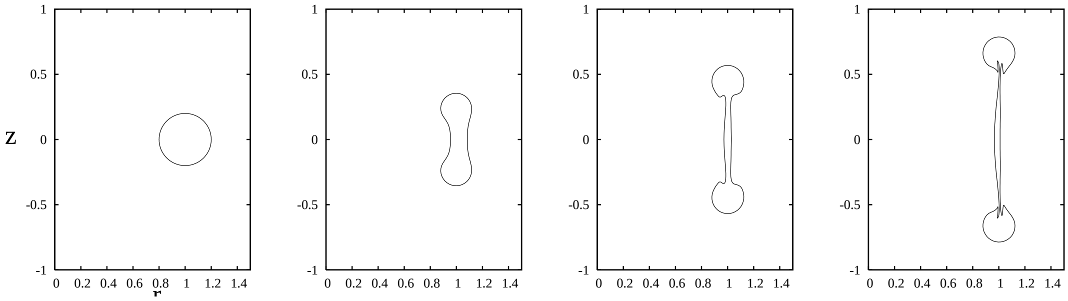

Figure 1: Contour dynamics simulation. $\sigma_0/R_0 = 0.2$. $Ct/R_0^2 = 0, 2, 4, 6$.

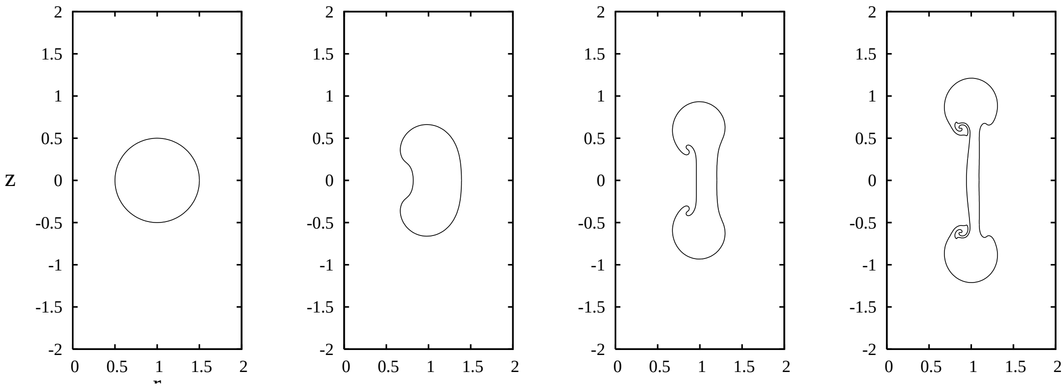

Figure 2: Contour dynamics simulation. $\sigma_0/R_0 = 0.5$. $Ct/R_0^2 = 0, 1, 2, 3$.

These numerical results suggest that there exists an exact solution for each localized vortex dipole. Using a perturbation expansion in which the expansion parameter $\epsilon = \sigma_0/R_0$ is assumed to be small, we obtain the following asymptotic solution:

$$u = w = 0, \qquad v = \frac{C}{r}, \qquad B = \kappa r, \qquad p_* = \epsilon^2 C^2 \left(\frac{1}{2R_0^2} - 2\frac{r^2}{R_0^4} \right) + O(\epsilon^3) \tag{30}$$

inside the core $r^2 + z^2 < \sigma_0^2$ and

$$u = \epsilon C R_0 \frac{2rz}{(r^2 + z^2)^2} + O(\epsilon^2), \tag{31}$$

$$w = \epsilon C \left(\frac{1}{R_0} + R_0 \frac{r^2 - z^2}{(r^2 + z^2)^2} \right) + O(\epsilon^2), \tag{32}$$

$$p_* = -\epsilon^2 C^2 \frac{r^2 - z^2 + \frac{1}{2}R_0^2}{(r^2 + z^2)^2} + O(\epsilon^3), \tag{33}$$

outside the core $r^2 + z^2 > \sigma_0^2$, where $v = B = 0$. This is a steady solution in the frame moving with speed $\epsilon C/R_0 = \sigma_0 C/R_0^2$ in the z-direction. Up to $O(\epsilon)$ the outer poloidal flow is same as the flow around a circular cylinder, where the role of the cylinder is replaced by a vortex sheet on the boundary and by a restoring force which is the sum of magnetic tension and centrifugal force. At higher orders the boundary will deform and no longer be circular.

6. Conclusions

We have investigated the problem of axisymmetric flow of a flow with swirl and a toroidal magnetic field. We have obtained the velocity of a thin-core vortex ring with swirl and magnetic field, generalizing the classical results of Kelvin and Hicks. We have also shown that the spherical vortices for which the Hicks equation is linear satisfy the relation (26) and move at a velocity U determined by the parameters a, α and κ. In the appropriate limits of no magnetic field and no swirl, they reduce to the Hicks–Moffatt vortices and the HM eddies. There may be other solutions which satisfy (15) but they will no longer be spheres and their shapes will have to be found numerically as in the Norbury family, although there may be special spheroidal solutions. Numerical simulations using a contour dynamics method designed for this problem show the creation of counter-propagating vortex dipoles. Approximate solutions for these dipoles are also found.

A number of generalizations are possible, including the role of axial strain and whether solutions inside a sphere exist that satisfy nonlinear extensions of (15). In addition, more investigation is needed of the contour dynamics of unsteady magnetic vortices. One interesting observation in this field is that the governing vorticity and pressure equations were both known to Hicks in 1899, and have been successively rediscovered over the years.

References

[1] Hicks WM. Researches in Vortex Motion. Part III: On Spiral or Gyrostatic Vortex Aggregates. Phil Trans R Soc Lond *A* 1899;192:33–99.
[2] Bragg SL, Hawthorne WR. Some exact solutions of the flow through annular cascade actuator discs. J Aero Sci 1950;17:243–9.
[3] Squire HB. Rotating fluids. In: Batchelor GK, Davies RM, editors. Surveys in mechanics; a collection of surveys of the present position of research in some branches of mechanics, written in commenmoration of the 70th birthday of Geoffrey Ingram Taylor. Cambridge: Cambridge University Press; 1956, p. .
[4] Long RR. A vortex in an infinite viscous fluid. J Fluid Mech 1961;11:611–24.
[5] Fraenkel LE. On the flow of rotating fluid past bodies in a pipe. Proc R Soc Lond *A* 1956;233:506–26.
[6] Grad H, Rubin H. Hydromagnetic equilibria and force-free fields. In: Proceedings of the 2nd United Nations Conference on the Peaceful Uses of Atomic Energy; vol. 31. Geneva: IAEA; 1958, p. 190–7.
[7] Shafranov VD. Plasma equilibrium in a magnetic field. Rev Plasma Phys 1966;2:103.
[8] Moffatt HK. The degree of knottedness of tangled vortex lines. J Fluid Mech 1969;35:117–29.
[9] Ambrosetti A, Struwe M. Existence of steady vortex rings in an ideal fluid. Arch Rational Mech Anal 1989;108:97–109.
[10] Turkington B. Vortex Rings with Swirl: Axisymmetric Solutions of the Euler Equations with Nonzero Helicity. SIAM J Math Anal 1989;20:57–73.
[11] Cheng M, Lou J, Lim TT. Vortex ring with swirl: A numerical study. Physics of Fluids 2010;22:097101.
[12] Virk D, Melander MV, Hussain F. Dynamics of a polarized vortex ring. J Fluid Mech 1994;260:23–55.
[13] Saffman PG. Vortex Dynamics. Cambridge: Cambridge University Press; 1992.
[14] Hattori Y, Moffatt HK. Evolution of toroidal magnetic eddies in an ideal fluid. J Fluid Mech 2006;558:253–79.
[15] Pozrikidis C. The nonlinear instability of Hill's vortex. J Fluid Mech 1986;168:337–67.
[16] Shariff K, Leonard A, Ferziger JH. Dynamics of a Class of Vortex Rings. Tech. Rep.; NASA; 1989.
[17] Riley N. The fascination of vortex rings. Appl Sci Res 1998;59:169–89.
[18] Gourdain P, Leboeuf J. Contour dynamics method for solving the Grad–Shafranov equation with applications to high beta equilibria. Phys Plasmas 2004;11:4372–81.
[19] Fukumoto Y, Moffatt HK, Ooshiro S. Motion of vortex rings with and without magnetic field. In: Proceedings of 22nd International Congress of Theoretical and Applied Mechanics. Adelaide; 2008,.
[20] Norbury J. A family of steady vortex rings. J Fluid Mech 1973;57:417–31.
[21] Lamb H. Hydrodynamics. Cambridge: Cambridge University Press; 6 ed.; 1932.
[22] Tsui KH. Toroidal equilibria in spherical coordinates. Phys Plasmas 2008;15:112506.