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Scattering of acoustic waves by a vortex

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We investigate the scattering of a plane acoustic wave by an axisymmetric vortex in two dimensions. We consider vortices with localized vorticity, arbitrary circulation and small Mach number. The wavelength of the acoustic waves is assumed to be much longer than the scale of the vortex. This enables us to define two asymptotic regions: an inner, vortical region, and an outer, wave region. The solution is then developed in the two regions using matched asymptotic expansions, with the Mach number as the expansion parameter. The leading-order scattered wave field consists of two components. One component arises from the interaction in the vortical region, and takes the form of a dipolar wave. The other component arises from the interaction in the wave region. For an incident wave with wavenumber k propagating in the positive X-direction, a steepest descents analysis shows that, in the far-field limit, the leading-order scattered field takes the form $\mathrm{i}(\pi-\theta)\mathrm{e}^{\mathrm{i}kX}+\frac{1}{2}\cos\theta\cot(\frac{1}{2}\theta)(2\pi/kR)^{1/2}\,\mathrm{e}^{\mathrm{i}(kR-\pi/4)}$, where θ is the usual polar angle. This expression is not valid in a parabolic region centred on the positive X-axis, where $kR\theta^2 = O(1)$. A different asymptotic solution is appropriate in this region. The two solutions match onto each other to give a leading-order scattering amplitude that is finite and single-valued everywhere, and that vanishes along the X-axis. The next term in the expansion in Mach number has a non-zero far-field response along the X-axis.

1. Introduction

When sound propagates through a flow with vorticity, the sound field may be significantly modified by its interaction with the vortical flow. Examples are the propagation of sound through turbulence (Kraichnan 1953; Lighthill 1953; Batchelor 1956), and the propagation of sound through the wakes of jet engines (Ferziger 1974). Lund & Rojas (1989) have proposed using ultrasound to probe turbulence in laboratory experiments (see also Labbé & Pinton 1998; Oljaca *et al.* 1998).

A first step is to understand the following fundamental problem: when a plane sound wave is incident on an axisymmetric vortex in two dimensions, what is the resulting scattered sound field?

There are two limits which afford both conceptual and analytical simplification. One is the limit in which the acoustic waves have small wavelength compared with the scale of the vortex; we refer to this as the WKB limit. The other is the opposite limit, in which the acoustic waves have long wavelength compared with the scale of the vortex; we refer to this as the Born limit.

In the WKB limit, acoustic waves travel many wavelengths as they propagate through the vortex, and their ray paths are deflected by the vortical flow. When the Mach number of the flow in the vortex is small, this deflection causes a caustic to form along a straight line which extends from the vortex in the direction opposite to the direction from which the plane wave is incident (Georges 1972). We refer to this direction as the 'forward scatter direction'. The WKB limit was first investigated by Lindsay (1948) for rays originating from a single point (see also Salant 1969; Broadbent 1977). It was argued by Georges (1972) that the WKB limit is appropriate for sound waves in the atmosphere. However, it is inappropriate for sound waves propagating though vortices in laboratory apparatus, such as that used by Oljaca *et al.* (1998), or for the propagation of sound generated by turbulence in the wakes of jet aircraft, where the vortices are small compared with the typical wavelength of sound waves (Lighthill 1952). For these problems, it is necessary to consider the opposite, Born, limit.

In the Born limit, the nature of the interaction is quite different. The large-scale acoustic wave induces pressure and velocity perturbations in the vortex, which in turn induce a contribution to the scattered wave field. In addition, when the vortex has non-zero circulation, the velocity due to the vortex has a long-range nature, with azimuthal velocity inversely proportional to distance from the vortex. The interaction between this long-range azimuthal velocity and the incident plane wave induces a second contribution to the scattered wave field. Both contributions must be included in any consistent theory of scattering.

The Born limit was first investigated by Müller & Matschat (1959) and Pitaevskii (1959). Müller & Matschat (1959) considered a distributed vortex with velocity discontinuities, while Pitaevskii (1959), and subsequently Fetter (1964) and Ferziger (1974), considered the case of a point vortex. O'Shea (1975) considered an incoming wave generated by a point source. In the 'far-field' limit, i.e. the limit of large distance from the vortex, these authors predicted that the scattered field has infinite amplitude in the forward scatter direction. However, the validity of the point vortex model is questionable for compressible flows, since it implies that a vacuum exists in a small but finite core at the centre of the vortex (see, for example, Barsony-Nagy, Er-El & Yungster 1987). Using Lighthill's acoustic analogy, Fabrikant (1983) investigated the scattered field produced by a distributed vortex and again found an infinite scattered wave amplitude in the forward scatter direction in the far-field limit.

The problem of scattering of sound by a distributed vortex was investigated both numerically and analytically by Colonius, Lele & Moin (1994). They integrated the nonlinear equations of motion for a two-dimensional compressible gas numerically, and compared the resulting solutions with analytical solutions in both the WKB and Born limits. Their numerical solutions in the case of non-vanishing circulation indicated no tendency for the amplitude of the scattered wave field in the forward scatter direction to become infinite.

Sakov (1993) returned to the problem of scattering of sound by a point vortex in the Born limit. The solution in a region about the axis in the forward scatter direction in the far field takes a different form from elsewhere, but no singularity develops. The dominant far-field contribution to the solution has angular dependence $(\pi - \theta)$, with a smooth transition about the direction $\theta = 0$ which has the form of a Fresnel function. However, no detailed analysis of the solution was undertaken, and the issue of the validity of the point vortex model combined with Lighthill's method remains.

In this paper, we consider scattering of sound by a distributed vortex in the Born limit. We suppose that the incident plane wave is of small amplitude, so that all terms quadratic in the wave amplitude may be neglected. We suppose further that the

Mach number of the flow in the vortex is small – an assumption which is necessarily violated by point vortices. These assumptions enable us to identify two asymptotic regions: an inner, vortical region, in which the flow consists of the initial vortex, with perturbations induced by the acoustic wave, and an outer, wave region, in which the flow consists of the long-range azimuthal velocity of the vortex plus the incident and scattered acoustic waves. We develop matched asymptotic expansions for the flow in both regions. The analysis is similar in spirit to that undertaken by Crow (1970) for the problem of aerodynamic sound generation (see also Batchelor 1956). The principal difference is that in our case sound propagates towards the vortical region as a plane wave, whereas in Crow (1970) all the sound is generated spontaneously by the motion in the vortical region. It is important to undertake this matched asymptotic analysis because, as pointed out by Crow (1970), there is no guarantee that the acoustic analogy formulation of Lighthill (1952), as used previously for this problem, represents a consistent asymptotic analysis of the equations of motion. Furthermore, our matched asymptotic analysis shows in detail how the interaction between the incident acoustic wave and the vortex occurs. Such insight cannot be obtained from a straightforward application of Lighthill's acoustic analogy.

In § 2 we develop the equations and asymptotic scalings in the two asymptotic regions. In § 3 we show how the incident wave induces a time-dependent flow in the vortical region, which can in turn induce a contribution to the scattered wave field in the wave region. In § 4 we derive the leading-order scattered flow in the wave region, and we consider its far-field behaviour in § 5. The leading-order scattered wave field vanishes along the forward scatter direction, and in § 6 we show that a scattered wave field with non-zero amplitude along the forward scatter direction arises at the next order. In § 7 we discuss briefly how our analysis is modified when the circulation of the vortex vanishes, and offer some concluding remarks.

2. Statement of the problem

2.1. *Governing equations and flow configuration*

We consider flow in a two-dimensional homentropic ideal gas. The corresponding equations of motion are

$$\rho_a \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p_a, \tag{2.1a}$$

$$\frac{\mathrm{D}\rho_a}{\mathrm{D}t} + \rho_a \nabla \cdot \boldsymbol{u} = 0, \tag{2.1b}$$

$$\frac{p_a}{p_0} = \left(\frac{\rho_a}{\rho_0}\right)^{\gamma}. \tag{2.1c}$$

Here, $\boldsymbol{u}$ is the two-dimensional velocity, p_a is the absolute pressure, ρ_a is the absolute density, and ∇ is the two-dimensional gradient operator. The relation (2.1*c*) is the equation of state for a homentropic ideal gas, and $\gamma > 1$ is the constant ratio of specific heats. The constants p_0 and ρ_0 are reference values of the pressure and density respectively. We shall take p_0 and ρ_0 to be the uniform values that the pressure and density take when the fluid is at rest. The equations (2.1) also describe motion in a shallow water layer, with ρ_a now standing for the depth of the fluid; this corresponds to $\gamma = 2$ (see for example Stoker 1957).

In this paper, attention will be restricted to flows in which vorticity is concentrated in some region of size L, which we refer to as the 'vortical region'. In this region, the

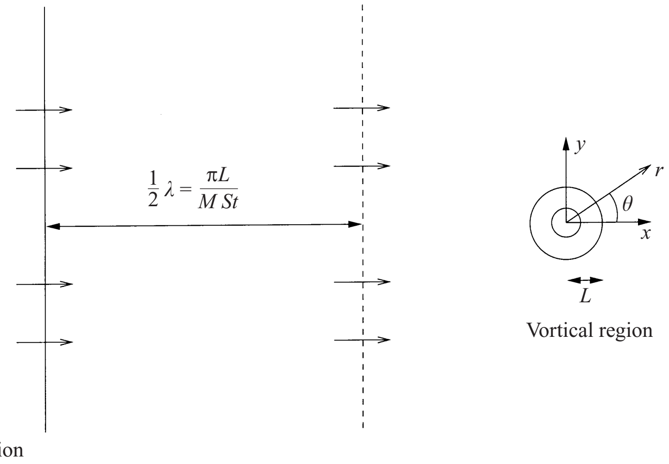

FIGURE 1. A schematic picture showing the vortical region and the surrounding wave region. The vortical region is characterized by the scale L, and the wave region by the longer scale LM^{-1}. The concentric circles in the vortical region represent contours of vorticity in the undisturbed vortex. Outside this region, the vorticity is exponentially small. The lines in the wave region represent the wave crests of the incident plane wave, which propagates in the direction of positive x. Solid and dashed lines represent negative and positive pressure perturbations respectively.

flow is dominated by the vortex which, in the absence of any incident acoustic wave, is assumed to be axisymmetric. The Mach number M of the vortex is then defined to be $M \equiv U/c_0 \ll 1$, where U is the typical magnitude of the azimuthal velocity in the vortex, and $c_0 \equiv (\gamma p_0/\rho_0)^{1/2}$ is the speed of linear sound waves in (2.1). The velocity scale U and length scale L define a time scale $\tau_v \equiv L/U$ appropriate to flow in the vortical region.

The incident acoustic wave is assumed to be a monochromatic plane wave, with radian frequency ω, wavenumber $k = \omega/c_0$, and wavelength $\lambda = 2\pi/k$. In this paper we are interested in the interaction between the wave and the vortex in the Born limit, in which the wavelength of waves of frequency ω is much larger than the length scale L of the vortical region, i.e. $\lambda \gg L$. The frequency ω defines a second time scale, $\tau_w \equiv 2\pi/\omega$, equal to the period of the wave. Defining the Strouhal number $St \equiv \omega L/U$, the ratio $\tau_v/\tau_w = St/2\pi$. Since we have $M \ll 1$ by assumption, it is natural to take $\tau_v/\tau_w = O(1)$ and the condition $\lambda \gg L$ may be written equivalently as $(\tau_v/\tau_w)M \ll 1$, i.e. the dimensionless wavenumber is $O(1)$. The limits of large and small Strouhal number can be recovered from our analysis by taking the appropriate limits of large and small ω respectively. The wavelength of the wave is related to the vortical length scale L by $\lambda = (\tau_v/\tau_w)^{-1}LM^{-1}$. The long length scale LM^{-1} characterizes a second region, which we refer to as the 'wave region', appropriate to the wavelength of the incident plane wave. The two regions are shown in figure 1.

The governing parameters of the problem are as follows. The incoming wave has frequency ω, wavenumber $k = \omega/c_0$ and pressure amplitude P^i. The vortex is entirely described by its vorticity $(U/L)\zeta_0(r/L)$, where $r \equiv |\boldsymbol{x}|$ is the radial coordinate and where $\zeta_0(r/L)$ is non-dimensional. We shall require that $\zeta_0(r/L)$ decay faster than any power of r/L for large r/L. We will also use the non-dimensional parameters M defined above and δ, the non-dimensional wave amplitude, to be defined below.

We now develop non-dimensional equations suitable for the flow in the two regions. It is convenient to replace the absolute pressure p_a and absolute density ρ_a by scaled perturbations away from their uniform reference values. We introduce the non-dimensional variables p and ρ, defined such that

$$p_a = p_0(1 + \gamma M^2 p), \quad \rho_a = \rho_0(1 + M^2\rho). \tag{2.2}$$

With the velocity $\boldsymbol{u}$ scaled on U, the spatial coordinate $\boldsymbol{x}$ scaled on L, and the time t scaled on L/U, and with the non-dimensional pressure and density variables p and ρ as defined in (2.2), the dimensionless equations appropriate to the flow in the vortical region are

$$(1 + M^2\rho)\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p, \tag{2.3a}$$

$$M^2\left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{u}\right) + \nabla\cdot\boldsymbol{u} = 0, \tag{2.3b}$$

$$\gamma M^2 p = (1 + M^2\rho)^\gamma - 1. \tag{2.3c}$$

Thus, in the vortical region, the leading-order dynamics ($M = 0$ in (2.3*a*) and (2.3*b*)) is simply the two-dimensional incompressible Euler equations.

To obtain equations for the surrounding wave region, we rescale the equations (2.3) using the long length scale LM^{-1}. We therefore introduce the wave-region spatial variable $\boldsymbol{X} = M\boldsymbol{x}$. Non-dimensional fields in the wave region are represented by capital letters (except for the density ρ, which we denote in the wave region by H). As we shall see, the velocity field is one order in M smaller in the wave region, so instead of $\boldsymbol{U}$ we write $M\boldsymbol{U}$. With these scalings, the non-dimensional equations (2.3) become

$$(1 + M^2 H)\left(\frac{\partial \boldsymbol{U}}{\partial t} + M^2\boldsymbol{U}\cdot\nabla\boldsymbol{U}\right) = -\nabla P, \tag{2.4a}$$

$$\frac{\partial H}{\partial t} + \nabla\cdot\boldsymbol{U} + M^2\nabla\cdot(\boldsymbol{U}H) = 0, \tag{2.4b}$$

$$\gamma M^2 P = (1 + M^2 H)^\gamma - 1, \tag{2.4c}$$

where the gradient operator acting on a wave-region quantity corresponds to differentiation with respect to $\boldsymbol{X}$. In the wave region, the nonlinear terms are of small order in M, and the leading-order dynamics of (2.4) admits propagating acoustic waves as solutions. Equations (2.3) and (2.4) are the non-dimensional equations obtained by Crow (1970).

We denote the non-dimensional amplitude of the incident acoustic wave by $\delta \equiv P^i/(\rho_0 c_0^2)$. Since linear waves are supported at leading order in (2.4), it appears that we may take $\delta = O(1)$. This implies that the pressure perturbations associated with the incident acoustic wave are the same magnitude as the pressure perturbations associated with the axisymmetric vortex in the vortical region. Taking $\delta = O(1)$, however, leads to significant complications in the analysis. This is because the nonlinear terms in (2.4*a*) and (2.4*b*) imply that nonlinear wave effects occur at $O(M^2\delta^2)$. The purpose of this paper is to investigate the interaction of the acoustic wave with the vortex. Therefore, although it is important to retain the nonlinear terms in (2.4*a*) and (2.4*b*) to account for the interaction of the incident acoustic wave with the long-range velocity field due to the vortex, we neglect any nonlinearity that arises from interaction of the acoustic wave with itself. Consequently we take δ sufficiently small that terms $O(\delta^2)$ can always be neglected when we expand in M. It turns out that $\delta \ll M^4$ is sufficient for the present purposes. However, we shall retain δ and M as independent asymptotic parameters throughout our analysis. The flow in the two regions is now expressed as an asymptotic expansion in M and δ, and matched where the regions overlap.

2.2. *Leading-order solution in the vortical region*

The expansion for the flow in the vortical region takes the form

$$\boldsymbol{u} = \boldsymbol{u}_0 + \delta(\boldsymbol{u}_{01} + M\boldsymbol{u}_{11} + M^2\boldsymbol{u}_{21} + M^3 \ln M\boldsymbol{u}_{311} + M^3\boldsymbol{u}_{31}) + O(\delta M^4 \ln M, \delta^2), \tag{2.5a}$$

$$p = p_0 + \delta(p_{01} + Mp_{11} + M^2p_{21} + M^3 \ln Mp_{311} + M^3p_{31}) + O(M^2, \delta M^4 \ln M, \delta^2), \tag{2.5b}$$

$$\rho = \rho_0 + \delta(\rho_{01} + M\rho_{11} + M^2\rho_{21} + M^3 \ln M\rho_{311} + M^3\rho_{31}) + O(M^2, \delta M^4 \ln M, \delta^2). \tag{2.5c}$$

Our normalization condition on $\boldsymbol{u}$ is that the vorticity, and hence the velocity, vanishes at $O(M^2)$ and higher orders in M^2. We could calculate the $O(M^2)$ and higher-order δ-independent contributions to pressure and density, but we will not need them so they have been suppressed in the expansion.

We specify the leading-order vorticity, $\zeta_0 \equiv \boldsymbol{k} \cdot (\nabla \times \boldsymbol{u}_0)$, where $\boldsymbol{k}$ is a unit vector normal to the plane. This defines the basic vortex, which depends only on r. Equation (2.3*b*) implies $\nabla \cdot \boldsymbol{u}_0 = 0$, and so we may define a streamfunction ψ_0 such that $\boldsymbol{u}_0 = \boldsymbol{k} \times \nabla\psi_0$. The rest of (2.3) and the definition of vorticity then lead to

$$\nabla^2\psi_0 = \zeta_0, \quad \frac{\mathrm{d}p_0}{\mathrm{d}r} = r\Omega^2(r), \quad p_0 = \rho_0, \tag{2.6}$$

where

$$\Omega(r) = \frac{1}{r}\frac{\mathrm{d}\psi_0}{\mathrm{d}r} \tag{2.7}$$

is the angular velocity of the vortex. Solving (2.6) subject to the boundary condition $p_0 \to 0$ as $r \to \infty$, we obtain

$$\psi_0(r) = \ln r \int_0^r s\zeta_0(s)\,\mathrm{d}s + \int_r^\infty s \ln s\, \zeta_0(s)\,\mathrm{d}s, \tag{2.8a}$$

$$p_0 = \rho_0 = -\int_r^\infty s\Omega^2(s)\,\mathrm{d}s. \tag{2.8b}$$

In the limit $r \to \infty$, we have

$$\psi_0 = \frac{\Gamma}{2\pi} \ln r + O(r^{-\infty}), \tag{2.9}$$

where $O(r^{-\infty})$ denotes terms that decay faster than any power of r (Llewellyn Smith 1995), and the circulation Γ is given by

$$\Gamma = 2\pi \int_0^\infty s\zeta_0(s)\,\mathrm{d}s. \tag{2.10}$$

In addition, as $r \to \infty$, we have

$$\Omega = \frac{\Gamma}{2\pi r^2} + O(r^{-\infty}), \quad p_0 = -\frac{\Gamma^2}{8\pi^2 r^2} + O(r^{-\infty}). \tag{2.11}$$

It is convenient for later use to replace the momentum equation (2.3*a*) in the vortical region by the vorticity equation and the divergence equation. The vorticity equation is

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (\boldsymbol{u}\zeta) = 0. \tag{2.12}$$

Taking the divergence of (2.3*a*) we obtain

$$\nabla \cdot \left((1 + M^2\rho)\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \right) + \nabla^2 p = 0. \tag{2.13}$$

2.3. *Leading-order solution in the wave region*

The expansion for the flow in the wave region takes the form

$$\boldsymbol{U} = \boldsymbol{U}_0 + \delta(\boldsymbol{U}_{01} + M^2\boldsymbol{U}_{21} + M^4 \ln M \boldsymbol{U}_{411}) + O(\delta M^4, \delta^2), \tag{2.14a}$$

$$P = \delta(P_{01} + M^2 P_{21} + M^4 \ln M P_{411}) + O(M^2, M^4\delta, \delta^2), \tag{2.14b}$$

$$H = \delta(H_{01} + M^2 H_{21} + M^4 \ln M H_{411}) + O(M^2, M^4\delta, \delta^2). \tag{2.14c}$$

Once again, we neglect pure powers of M^2 in the expansion since they do not affect the scattering problem. Strictly speaking, the truncation of (2.14) is not asymptotically consistent, since we have separated terms with the same algebraic power of M, namely $M^4 \ln M\, \delta$ and $M^4\delta$. However, since we will be interested only in the form of the $M^4 \ln M\, \delta$ term, and will not be summing the asymptotic series to this order, we may safely ignore the $M^4\delta$ term.

In the wave region, the vorticity is small beyond all orders in M. This implies that, to all algebraic orders in M, the velocity field $\boldsymbol{U}$ may be written using a velocity potential Φ, defined by $\boldsymbol{U} = \nabla\Phi$. Substituting into (2.4*a*), using (2.4*c*), and integrating, we have

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}M^2 \nabla\Phi \cdot \nabla\Phi + \frac{1}{\gamma - 1} M^{-2}[(1 + \gamma M^2 P)^{(\gamma-1)/\gamma} - 1] = 0. \tag{2.15}$$

Equations (2.15), (2.4*b*) and (2.4*c*) are a complete set of equations for the flow in the wave region.

The flow in the wave region at $O(1)$ must match to the flow in the vortical region. Since the leading-order flow in the vortical region is independent of t, we seek a solution in the wave region that is also independent of t. Equations (2.15) and (2.4*c*) then imply that $P_0 = H_0 = 0$. This is reflected in the fact that the expansions (2.14*b*) and (2.14*c*) for P and H start at $O(M^2)$. Then (2.4*b*) implies

$$\nabla^2 \Phi_0 = 0. \tag{2.16}$$

The associated velocity, in the limit $R \to 0$, must match to the velocity in the vortical region in the limit $r \to \infty$, given in (2.11). The solution for Φ_0 is

$$\Phi_0 = \frac{\Gamma}{2\pi}\theta, \tag{2.17}$$

where θ is the polar angle. Although Φ_0 is multivalued, the corresponding angular velocity is not and clearly matches onto (2.11).

2.4. *The solution in the wave region at* $O(\delta)$

The amplitude of the incident acoustic wave δ is not related to M, and therefore the flow in the wave region at $O(\delta)$ must be a solution of the leading-order equations.

From (2.4) and (2.15), we have

$$\frac{\partial \Phi_{01}}{\partial t} + P_{01} = 0, \tag{2.18a}$$

$$\frac{\partial H_{01}}{\partial t} + \nabla^2 \Phi_{01} = 0, \tag{2.18b}$$

$$P_{01} = H_{01}. \tag{2.18c}$$

From (2.18), we obtain

$$\frac{\partial^2 \Phi_{01}}{\partial t^2} - \nabla^2 \Phi_{01} = 0. \tag{2.19}$$

This is simply the linear wave equation for sound waves propagating with a sound speed of unity. The same equation holds for P_{01}. In this problem, plane sound waves with a single frequency ω propagate towards the vortex from the negative X-direction. The solution for these waves is then

$$P_{01} = \mathrm{e}^{\mathrm{i}(kX-\omega t)}, \quad \Phi_{01} = -\frac{\mathrm{i}}{\omega}\mathrm{e}^{\mathrm{i}(kX-\omega t)}, \tag{2.20}$$

where k and ω satisfy the dispersion relation $\omega = k$ for waves propagating in the positive X-direction, and it is understood that in (2.20) and subsequent expressions the real part is taken. We shall henceforth take k and ω to be real and positive.

The wave equation (2.19) also supports outwardly propagating solutions, which correspond to a response at the same order as the incoming wave. In fact, there are no such terms, because they would have logarithmic or algebraic singularities for small R, and we shall see that there are no possible terms for them to match onto in the vortical region.

3. The solution at $O(M\delta)$

3.1. *The solution in the vortical region at $O(\delta)$*

At successive orders in M^2, the pressure and density in the vortical region are modified by compressibility, and the pressure and density in the wave region respond appropriately. The flow in the vortical region must also match onto the flow in the wave region at $O(\delta)$, and so we calculate the vortical flow at $O(\delta)$, $O(M\delta)$ and so forth.

Expanding the $O(\delta)$ solution in the wave region for small R gives

$$P_{01} = \mathrm{e}^{-\mathrm{i}\omega t}(1 + \mathrm{i}kX + O(R^2)), \tag{3.1a}$$

$$\boldsymbol{U}_{01} = \boldsymbol{i}\,\mathrm{e}^{-\mathrm{i}\omega t} + O(R), \tag{3.1b}$$

where $\boldsymbol{i}$ is the unit vector in the x-direction. An elementary application of asymptotic matching leads to the conditions

$$p_{01} \to \mathrm{e}^{-\mathrm{i}\omega t}, \quad p_{11} \to \mathrm{i}kx\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{3.2a}$$

$$\boldsymbol{u}_{01} \to \boldsymbol{0}, \quad \boldsymbol{u}_{11} \to \boldsymbol{i}\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{3.2b}$$

for the inner quantities for large r. We could use Van Dyke's rule or an intermediate variable approach to carry out the matching, but there are no complications in this problem and we can carry out the matching naively.

The governing equations in the vortical region at $O(\delta)$ are

$$\frac{\partial \zeta_{01}}{\partial t} + \nabla \cdot (\boldsymbol{u}_0 \zeta_{01} + \boldsymbol{u}_{01} \zeta_0) = 0, \tag{3.3a}$$

$$\nabla \cdot \boldsymbol{u}_{01} = 0, \tag{3.3b}$$

$$\nabla^2 p_{01} + \nabla \cdot (\boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_{01} + \boldsymbol{u}_{01} \cdot \nabla \boldsymbol{u}_0) = 0. \tag{3.3c}$$

Equation (3.3*b*) implies that we may write $\boldsymbol{u}_{01} = \boldsymbol{k} \times \nabla \psi_{01}$. Since ζ_0 is a function of r alone, and $\boldsymbol{u}_0$ is purely azimuthal, it is natural to express (3.3*a*) in plane polar coordinates:

$$\frac{\partial \zeta_{01}}{\partial t} + \Omega(r) \frac{\partial \zeta_{01}}{\partial \theta} - \frac{1}{r} \frac{\mathrm{d}\zeta_0}{\mathrm{d}r} \frac{\partial \psi_{01}}{\partial \theta} = 0, \tag{3.4}$$

where $\zeta_{01} = \nabla^2 \psi_{01}$. This is the linearized vorticity equation. We write the streamfunction as a sum over azimuthal modes: $\psi_{01} = \sum_{n=-\infty}^{\infty} g_n(r)\, \mathrm{e}^{\mathrm{i}(n\theta - \omega t)}$. The functions $g_n(r)$ are solutions to the radial Rayleigh equation,

$$g_n'' + \frac{1}{r} g_n' - \left(\frac{n^2}{r^2} - \frac{n\zeta_0'}{r(\omega - n\Omega)} \right) g_n = 0, \tag{3.5}$$

that are regular at the origin. Solutions for $|n| > 0$ may encounter critical layers where $\omega = n\Omega$, and in these cases equation (3.5) may be regularized by including a small amount of viscosity (Lin 1955; Reinschke, Möhring & Obermeier 1997) or by a nonlinear critical layer (see for example Stewartson 1978; Warn & Warn 1978).

For $n = 0$, we have $g_0(r) = \alpha_0$, and so $g_0(r)$ represents a constant contribution to the streamfunction, which has no dynamical significance. For $|n| > 0$, $g_n(r)$ is proportional to $r^{|n|}$ as $r \to 0$ and takes the form $\alpha_n(r^{|n|} + \beta_n r^{-|n|})$ as $r \to \infty$, since $\zeta_0(r)$ is localized. The constant α_n is arbitrary, since (3.5) is a linear equation, and must be determined by matching conditions in the limit $r \to \infty$. The constant β_n is determined by solving (3.5), and it depends on ω and $\Omega(r)$. For $|n| = 1$, it can be shown that β_n is finite for all values of ω and for all choices of $\Omega(r)$, and this solution encounters no critical-layer singularity at $\omega = n\Omega$ (cf. (3.10)). However, for $|n| > 1$, β_n may be unbounded for some discrete values of ω. These values of ω are the real eigenfrequencies of (3.5), with eigenfunctions $g_n(r)$ which are regular as $r \to 0$ and bounded as $r \to \infty$. Thus if the incident plane wave frequency ω is equal to an eigenfrequency of (3.5), resonance will occur. In that case, we would be unable to find a time-periodic solution to our problem. Instead, we would have to solve an initial-value problem, in which nonlinear effects would presumably arrest the resonant growth. To avoid this complication, we shall henceforth assume that the vortex has no eigenfunctions with eigenfrequency ω. However, as we shall see below, our analysis does not require the detailed structure of the $g_n(r)$ for $|n| > 1$.

We can now show that $\boldsymbol{u}_{01} = \boldsymbol{0}$. For $|n| > 0$, solutions of (3.5) take the form $r^{|n|}$ as $r \to \infty$. However, causal solutions for Φ in the wave region are Hankel functions of the first kind of order $|n|$. Thus Φ takes the form $R^{-|n|}$ as $R \to 0$, and the corresponding velocity cannot match to a velocity in the vortical region derived from solutions of (3.5). Hence we may exclude all solutions of (3.5) for $|n| > 0$ in the vortical region at $O(\delta)$.

For $n = 0$ the velocity is zero, and (2.3*a*) then implies that p_{01} is constant in space. The appropriate solution to (3.3) that satisfies the matching conditions (3.2) is

$$p_{01} = \rho_{01} = \mathrm{e}^{-\mathrm{i}\omega t}, \tag{3.6a}$$

$$\boldsymbol{u}_{01} = \boldsymbol{0}. \tag{3.6b}$$

This represents the complete solution of the flow in the vortical region at $O(\delta)$.

3.2. *The solution in the vortical region at $O(M\delta)$*

At $O(M\delta)$, the vorticity equation (2.12), continuity equation (2.3*b*), and pressure equation (2.13) are

$$\frac{\partial \zeta_{11}}{\partial t} + \nabla \cdot (\boldsymbol{u}_0 \zeta_{11} + \boldsymbol{u}_{11} \zeta_0) = 0, \tag{3.7a}$$

$$\nabla \cdot \boldsymbol{u}_{11} = 0, \tag{3.7b}$$

$$\nabla^2 p_{11} + \nabla \cdot (\boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_{11} + \boldsymbol{u}_{11} \cdot \nabla \boldsymbol{u}_0) = 0. \tag{3.7c}$$

Equations (3.7) for $\boldsymbol{u}_{11}$ and p_{11} are identical to (3.3) for $\boldsymbol{u}_{01}$ and p_{01}. Thus, we write $\boldsymbol{u}_{11} = \boldsymbol{k} \times \nabla \psi_{11}$. Then ψ_{11} is written as a sum over azimuthal modes, and the solution for each azimuthal mode is the solution of (3.5) that is regular as $r \to 0$. The matching condition (3.2*b*) implies

$$\psi_{11} \to -r \sin\theta \, \mathrm{e}^{-\mathrm{i}\omega t} \qquad \text{as } r \to \infty. \tag{3.8}$$

Azimuthal mode one is therefore non-zero, while the others are zero by the arguments given in § 3.1, and so we write

$$\psi_{11} = g_1(r)\, \mathrm{e}^{\mathrm{i}(\theta - \omega t)} + g_{-1}(r)\, \mathrm{e}^{\mathrm{i}(-\theta - \omega t)}. \tag{3.9}$$

The functions $g_1(r)$ and $g_{-1}(r)$ can be deduced from the results of Smith & Rosenbluth (1990) and Llewellyn Smith (1995), giving

$$\psi_{11} = K_1(\Omega - \omega) r\, \mathrm{e}^{\mathrm{i}(\theta - \omega t)} + K_{-1}(\Omega + \omega) r\, \mathrm{e}^{\mathrm{i}(-\theta - \omega t)}, \tag{3.10}$$

where the constants K_1 and K_{-1} must be determined by the asymptotic matching condition (3.8). This gives $K_1 = K_{-1} = -\mathrm{i}/2\omega$, and hence

$$\psi_{11} = -r \left(\sin\theta + \mathrm{i}\frac{\Omega}{\omega} \cos\theta \right) \mathrm{e}^{-\mathrm{i}\omega t}. \tag{3.11}$$

In the limit $r \to \infty$, (3.11) implies that

$$\psi_{11} = -r \sin\theta \, \mathrm{e}^{-\mathrm{i}\omega t} - \frac{\mathrm{i}\Gamma}{2\pi} (kr)^{-1} \cos\theta \, \mathrm{e}^{-\mathrm{i}\omega t} + O(r^{-\infty}). \tag{3.12}$$

When $r = O(M^{-1})$, the first term in the expression (3.12) matches to the incident acoustic wave at $O(\delta)$ by construction. The next term must match to a flow in the wave region at $O(M^2\delta)$. This supports the description of the scattering process given in § 1: the incident acoustic wave at $O(\delta)$ induces a flow in the vortical region at $O(M\delta)$, which in turn induces a scattered wave in the wave region at $O(M^2\delta)$. Note that this second term in (3.12) vanishes when the circulation vanishes, although the corresponding term in (3.11) is non-zero even in that case.

To complete the description of the flow in the vortical region at $O(M\delta)$, we obtain expressions for p_{11} and ρ_{11}. From (3.7*c*), and using (3.11), we have

$$\nabla^2 p_{11} = -\frac{\mathrm{i}}{\omega} [6\Omega\Omega' + 2r(\Omega'^2 + \Omega\Omega'')] \cos\theta \, \mathrm{e}^{-\mathrm{i}\omega t}. \tag{3.13}$$

The solution of (3.13) that satisfies the matching condition (3.2*a*) is

$$p_{11} = \rho_{11} = \mathrm{i} \left(1 - \frac{\Omega^2}{\omega^2} \right) kr \cos\theta \, \mathrm{e}^{-\mathrm{i}\omega t}. \tag{3.14}$$

In the limit $r \to \infty$, (3.14) implies that

$$p_{11} = \mathrm{i}kr\cos\theta\,\mathrm{e}^{-\mathrm{i}\omega t} - \frac{\mathrm{i}\Gamma^2}{4\pi^2 kr^3}\cos\theta\,\mathrm{e}^{-\mathrm{i}\omega t} + O(r^{-\infty}). \tag{3.15}$$

From (3.12) and (3.15) we see that $\boldsymbol{u}_{11}$ and p_{11} match to terms $O(M^2\delta)$ and higher in the wave region, and so there is no flow in the wave region at $O(M\delta)$, as indicated in (2.14).

4. The solution at $O(M^2\delta)$

4.1. *The solution in the wave region at $O(M^2\delta)$*

At $O(M^2\delta)$, the equations in the wave region are

$$\frac{\partial \Phi_{21}}{\partial t} + \boldsymbol{U}_0 \cdot \boldsymbol{U}_{01} + P_{21} = 0, \tag{4.1a}$$

$$\frac{\partial H_{21}}{\partial t} + \nabla^2\Phi_{21} + \nabla\cdot(\boldsymbol{U}_0 H_{01}) = 0, \tag{4.1b}$$

$$P_{21} = H_{21}. \tag{4.1c}$$

This leads to the forced wave equation

$$\frac{\partial^2\Phi_{21}}{\partial t^2} - \nabla^2\Phi_{21} = \boldsymbol{U}_0\cdot\nabla H_{01} - \boldsymbol{U}_0\cdot\frac{\partial \boldsymbol{U}_{01}}{\partial t} = -\frac{\mathrm{i}\Gamma k}{\pi}\frac{Y}{R^2}\mathrm{e}^{\mathrm{i}(kX-\omega t)}. \tag{4.2}$$

We may write the solution to this equation as

$$\Phi_{21} = -\frac{\mathrm{i}\Gamma}{4\pi}\varphi\,\mathrm{e}^{-\mathrm{i}\omega t} + \sum_{n=0}^{\infty} H_n^{(1)}(kR)\,[A_n\cos n\theta + B_n\sin n\theta]\;\mathrm{e}^{-\mathrm{i}\omega t}, \tag{4.3}$$

where φ satisfies

$$(-k^2 - \nabla^2)\varphi = 4k\,\mathrm{e}^{\mathrm{i}kX}\frac{\partial}{\partial Y}\ln R, \tag{4.4}$$

$H_n^{(1)}$ is a Hankel function of the first kind (Abramowitz & Stegun 1965), and the constants A_n and B_n must be determined by matching to the flow in the vortical region.

Using a Fourier transform in X, the solution to (4.4) may be written as

$$\varphi = \operatorname{sgn}(Y)\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}lX}}{l-k}[\mathrm{e}^{-|l-k||Y|} - \mathrm{e}^{-(l^2-k^2)^{1/2}|Y|}]\,\mathrm{d}l. \tag{4.5}$$

Equation (4.4) shows that φ is an odd function of Y, as may be seen from (4.5); we henceforth consider only $Y > 0$ in our analysis. It is also apparent that φ is a function of $k\boldsymbol{X}$ alone. For a causal solution, we require that waves propagate away from the X-axis for large $|Y|$, and this determines the analytic branch of $(l^2-k^2)^{1/2}$ in (4.5). We take branch cuts from $\pm k$ to $\pm(k+\mathrm{i}\infty)$, and take the branch of $(l^2-k^2)^{1/2}$ that, for real l, is real and positive when $l^2 > k^2$ and imaginary and negative when $l^2 < k^2$. Hence the integral in (4.5) exists as an integral along the real l-axis, with an integrable singularity at $l = k$.

To match to the flow in the vortical region, we require the asymptotic behaviour of (4.5) for small values of R. We divide the range of integration into three portions:

$$\varphi = \mathrm{e}^{\mathrm{i}kX}\left(\int_{-\infty}^{-1/\epsilon} + \int_{-1/\epsilon}^{1/\epsilon} + \int_{1/\epsilon}^{\infty}\right)\frac{\mathrm{e}^{\mathrm{i}uX}}{u}[\mathrm{e}^{-|u|Y} - \mathrm{e}^{-(u^2+2ku)^{1/2}Y}]\,\mathrm{d}u, \tag{4.6}$$

where ϵ is a small parameter, whose asymptotic order relative to R we are free to choose. In the inner integral we expand the exponentials in X and Y. In the outer integrals, we rescale the integration variable and expand. The conditions that we require on ϵ are $kR \ll k\epsilon \ll 1$. Then

$$\varphi = \mathrm{e}^{\mathrm{i}kX}\left\{-kY\int_{-\infty}^{-Y/\epsilon}\frac{\mathrm{e}^{v(\mathrm{i}X/Y+1)}}{v}\,\mathrm{d}v + kY\int_{Y/\epsilon}^{\infty}\frac{\mathrm{e}^{v(\mathrm{i}X/Y-1)}}{v}\,\mathrm{d}v \right.$$
$$\left. +Y\int_{-1/\epsilon}^{1/\epsilon}\frac{1}{u}[-|u| + (u^2+2ku)^{1/2}]\,\mathrm{d}u + \ldots\right\}$$
$$= 2kY\left(1-\gamma_E - \ln\left(\tfrac{1}{2}kR\right) - \tfrac{1}{2}\mathrm{i}\pi\right) + O(R^2\ln R), \tag{4.7}$$

where $\gamma_E = 0.5772\ldots$ is Euler's constant.

4.2. *The solution in the vortical region at $O(M^2\delta)$*

At $O(M^2\delta)$, (2.12) and (2.3b) are

$$\frac{\partial\zeta_{21}}{\partial t} + \nabla\cdot(\boldsymbol{u}_0\zeta_{21} + \boldsymbol{u}_{21}\zeta_0) = 0, \tag{4.8a}$$

$$\frac{\partial\rho_{01}}{\partial t} + \nabla\cdot\boldsymbol{u}_{21} = 0. \tag{4.8b}$$

To represent $\boldsymbol{u}_{21}$, we write $\boldsymbol{u}_{21} = \nabla\phi_{21} + \boldsymbol{k}\times\nabla\psi_{21}$. Equation (4.8$b$) then gives

$$\nabla^2\phi_{21} = \mathrm{i}\omega\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{4.9}$$

We cannot determine a unique solution for ϕ_{21}, since the asymptotic matching conditions must be applied to the velocity field $\boldsymbol{u}$, and not to the individual components ϕ and ψ. Therefore, we may take for ϕ_{21} any solution of (4.9). The matching conditions are then used to determine ψ_{21} uniquely. A convenient choice of solution to (4.9) is

$$\phi_{21} = \tfrac{1}{4}\mathrm{i}\omega r^2\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{4.10}$$

The vorticity equation (4.8a) is then

$$\frac{\partial\zeta_{21}}{\partial t} + \Omega\frac{\partial\zeta_{21}}{\partial\theta} - \frac{1}{r}\frac{\mathrm{d}\zeta_0}{\mathrm{d}r}\frac{\partial\psi_{21}}{\partial\theta} = -\frac{\mathrm{i}\omega}{2r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\zeta_0\right)\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{4.11}$$

where $\zeta_{21} = \nabla^2\psi_{21}$. By expanding $\boldsymbol{U}_{01}$ for small R up to $O(R)$, the matching condition on $\boldsymbol{u}_{21}$ is

$$\boldsymbol{u}_{21} \to \mathrm{i}kx\boldsymbol{i}\mathrm{e}^{-\mathrm{i}\omega t} = \tfrac{1}{2}\mathrm{i}k\,[r(1+\cos 2\theta)\boldsymbol{r} - r\sin 2\theta\,\boldsymbol{\theta}]\,\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{as } r\to\infty, \tag{4.12}$$

where $\boldsymbol{r}$ and $\boldsymbol{\theta}$ are unit vectors in the r- and θ-directions respectively. We can verify that $\nabla\phi_{21}$ matches to the θ-independent part of (4.12). The rest of (4.12) requires

$$\psi_{21} \to -\tfrac{1}{4}\mathrm{i}kr^2\sin 2\theta\,\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{as } r\to\infty. \tag{4.13}$$

Equation (4.11) is a forced linearized vorticity equation. We solve this equation by a decomposition into azimuthal modes. The forcing term on the right-hand side of (4.11) implies that, unlike (3.4), mode zero is non-trivial, while the solution at all other modes is given by solutions of the Rayleigh equation (3.5). The matching condition (4.13) implies that the amplitude of mode two must be non-zero. The solution to (4.11) is

$$\psi_{21} = f_{21}(r)\mathrm{e}^{-\mathrm{i}\omega t} + g_2(r)\,\mathrm{e}^{\mathrm{i}(2\theta-\omega t)} + g_{-2}(r)\,\mathrm{e}^{\mathrm{i}(-2\theta-\omega t)}, \tag{4.14}$$

where $f_{21}(r)$ satisfies

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f_{21}}{\mathrm{d}r}\right) = \frac{1}{2r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\zeta_0\right). \tag{4.15}$$

Equation (4.15) gives

$$f_{21}(r) = \frac{1}{2}\left(r^2\Omega(r) - \frac{\Gamma}{2\pi}\right). \tag{4.16}$$

Thus, $f_{21}(r) = O(r^{-\infty})$ as $r \to \infty$, consistent with the asymptotic matching condition (4.12).

The functions $g_2(r)$ and $g_{-2}(r)$ must in general be obtained numerically. However, they are not required for matching to the flow in the wave region at $O(M^2\delta)$. To see this, recall that $g_{\pm 2}(r) = \alpha_{\pm 2}(r^2 + \beta_{\pm 2}r^{-2}) + O(r^{-\infty})$ as $r \to \infty$, where the constants $\beta_{\pm 2}$ are determined by solving (3.5) for $n = \pm 2$. The coefficients $\alpha_{\pm 2}$ are determined from the asymptotic matching condition (4.13). The result is $\alpha_2 = -\alpha_{-2} = -k/8$, and so, for large r, we have

$$\psi_{21} = -\tfrac{1}{4}\mathrm{i}kr^2\sin 2\theta - \tfrac{1}{8}(\beta_2 - \beta_{-2})kr^{-2}\cos 2\theta - \tfrac{1}{8}(\beta_2 + \beta_{-2})\mathrm{i}kr^{-2}\sin 2\theta + O(r^{-\infty}). \tag{4.17}$$

The terms of form r^{-2} in (4.17) then match back onto the flow in the wave region at $O(M^4\delta)$. Therefore, to $O(M^2\delta)$, it is not necessary to solve for $g_2(r)$ and $g_{-2}(r)$.

4.3. *Matching at $O(M^2\delta)$*

The form of $\boldsymbol{u}_{11}$ as $r \to \infty$, derived from (3.12), implies that $\boldsymbol{U}_{21}$ must satisfy

$$\boldsymbol{U}_{21} \to -\frac{\mathrm{i}\Gamma k}{2\pi\omega^2}\boldsymbol{k} \times \nabla\left(R^{-1}\cos\theta\right)\,\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{as } R \to 0. \tag{4.18}$$

Equivalently,

$$\Phi_{21} \to \frac{\mathrm{i}\Gamma k}{2\pi\omega^2}R^{-1}\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{as } R \to 0. \tag{4.19}$$

Now, from (4.7), we have that $\varphi(kX)$ is finite as $R \to 0$. The condition (4.19) must therefore be satisfied by choosing B_1 in (4.3) to be non-zero (and all the other A_n and B_n to be zero). The result is

$$\Phi_{21} = -\frac{\mathrm{i}\Gamma}{4\pi}\varphi(kX)\mathrm{e}^{-\mathrm{i}\omega t} - \frac{\Gamma}{4}H_1^{(1)}(kR)\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{4.20}$$

This expression describes the leading-order scattered field in the wave region. The first component arises from the interaction of the incident plane wave with the long-range azimuthal velocity of the vortex in the wave region. The second component, which was obtained by Howe (1975), arises from the unsteady dynamics in the vortical region induced by the incident plane wave.

An expression for P_{21} can be obtained from (4.1*a*):

$$P_{21} = \frac{\Gamma\omega}{4\pi}\varphi(kX)\mathrm{e}^{-\mathrm{i}\omega t} - \frac{\mathrm{i}\Gamma\omega}{4}H_1^{(1)}(kR)\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t} + \frac{\Gamma Y}{2\pi R^2}\,\mathrm{e}^{\mathrm{i}(kX-\omega t)}. \tag{4.21}$$

It can be shown that P_{21}, unlike Φ_{21}, is $O(R\ln R)$ as $R \to 0$. The singular term in the expansion of $\mathrm{i}\omega\Phi_{21}$ for small R is exactly compensated by the last term in (4.21). This is essential, since we see from (3.15) that p_{11} has no terms of form $r^{-1}\sin\theta$ for large r, and hence P_{21} must have no terms of form $R^{-1}\sin\theta$ for small R. This provides a check on the asymptotic expansions.

A contour plot of the root-mean-square (r.m.s.) pressure amplitude at $O(M^2\delta)$, i.e. $P_{21}^{\mathrm{rms}} \equiv |P_{21}|/\sqrt{2}$, scaled by $\Gamma\omega/2\pi$, is shown in figure 2. The integral in (4.5) is

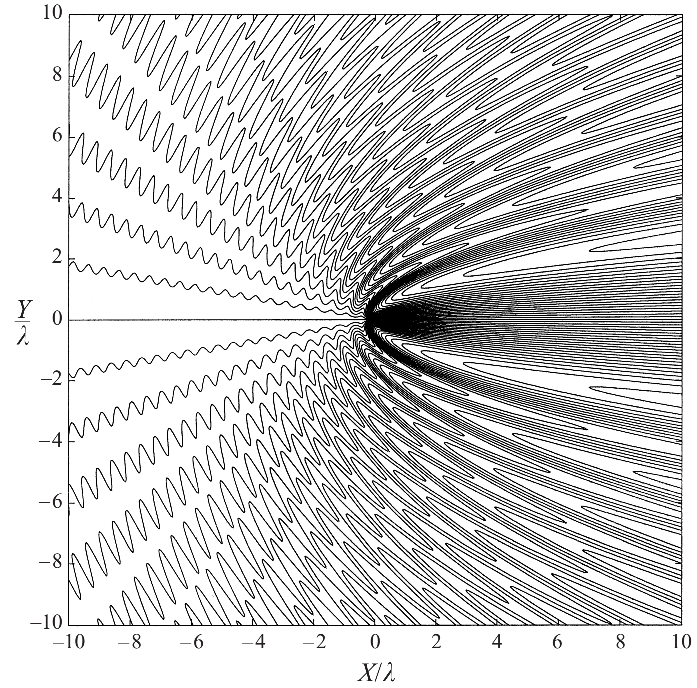

FIGURE 2. Contour plot of $P_{21}^{\mathrm{rms}}/(\Gamma\,\omega/2\pi)$ as a function of X/λ, where $\lambda = 2\pi/k$ is the wavelength of the incident wave. Contour interval: 0.25. Note that P_{21}^{rms} vanishes on the X-axis.

evaluated numerically, using the numerical integration procedure DQAWFE designed specifically to cope with trigonometric integrands and which uses an adaptive step size. The solutions presented here were obtained with an accuracy of six significant figures.

The r.m.s. pressure shown in figure 2 has small amplitude in the negative X-direction and vanishes along the X-axis. The r.m.s. pressure field is greatest in a parabolic region about the positive X-axis. The pattern is very similar to figure 2 of Colonius *et al.* (1994), which showed the r.m.s. pressure field in a nonlinear numerical simulation. The principal difference between our figure 2 and figure 2 of Colonius *et al.* (1994) is that our r.m.s. pressure is symmetric about $Y = 0$. The asymmetry obtained by Colonius *et al.* (1994) arises from higher-order effects which do not enter into our analysis at $O(M^2\delta)$.

5. Far-field analysis

As previously mentioned, Lund & Rojas (1989) have proposed using ultrasound to probe turbulence in laboratory experiments. In a typical experimental setting, the sensors, which detect the scattered sound field, will be placed several acoustic wavelengths from the vortex. It is therefore of interest to analyse P_{21} given by (4.21) in the far-field limit $R \to \infty$. We focus attention on the scattering cross-section, defined to be the r.m.s. scattered pressure as a function of θ, for large R.

In figure 3 we show the scattering cross-section at $R/\lambda = 1, 2, 3, 4$. For small values of R/λ the solution appears dipolar, as can be predicted from (4.21), using (4.7). As R/λ increases, an increasing number of local extrema appear. The angles $\theta = 0$ and

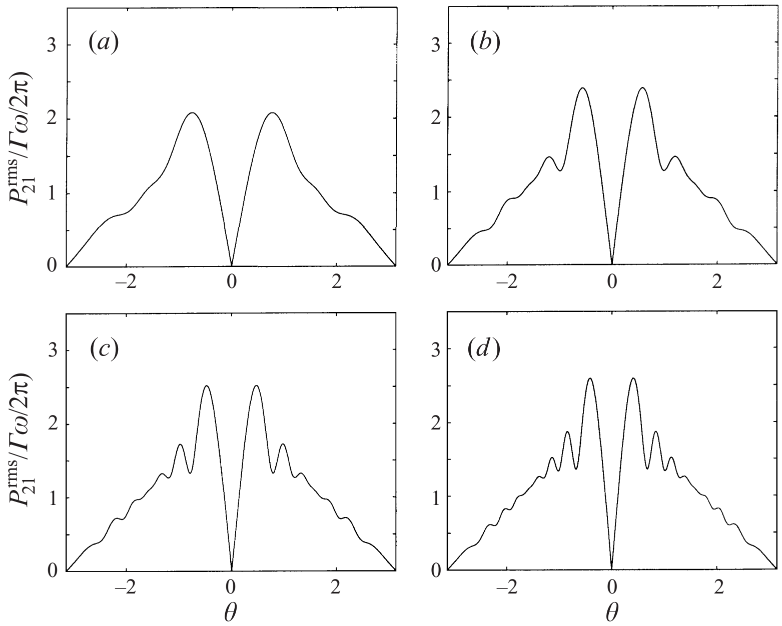

FIGURE 3. The scattering cross-section for (*a*) $R/\lambda = 1$, (*b*) 2, (*c*) 3, (*d*) 4.

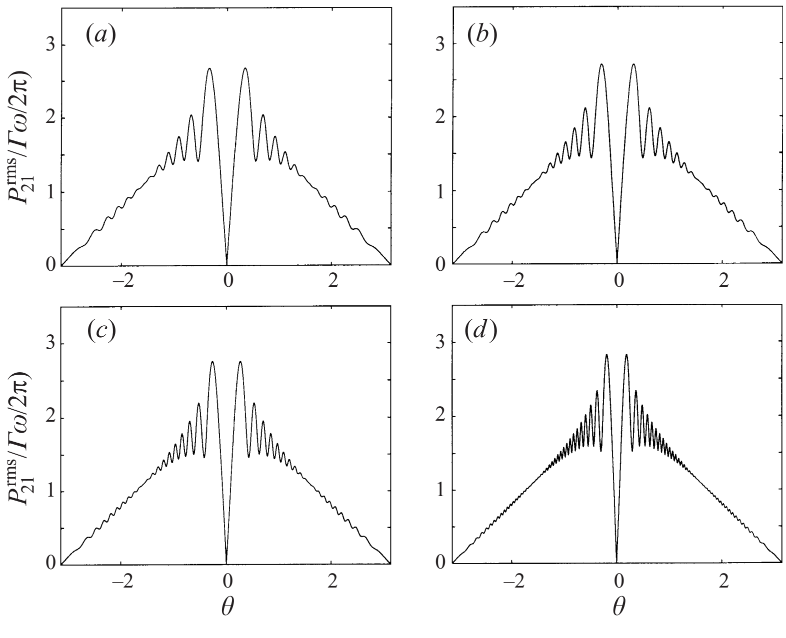

FIGURE 4. The scattering cross-section for (*a*) $R/\lambda = 6$, (*b*) 7.5, (*c*) 10, (*d*) 20.

$\theta = \pm\pi$ are global minima, where the amplitude of the scattered wave is zero. The global maxima are the two maxima on either side of the direction $\theta = 0$. As R/λ increases, the angle at which these global maxima occur moves progressively closer to $\theta = 0$, and the amplitude saturates. This interpretation is reinforced in figure 4, which shows scattering cross-sections for $R/\lambda = 6$, 7.5, 10, 20.

Figures 3 and 4 agree with the scattering cross-sections obtained by Colonius *et al.* (1994) from their analytical solution (figure 14 of their paper). However, Colonius *et al.* (1994) did not determine the asymptotic behaviour of their solution in the far-field limit. We will show that the maximum amplitude of P_{21}^{rms} approaches a constant value as $R \to \infty$, the angle at which this maximum occurs scales as $R^{-1/2}$, the local extrema become confined to a region about the positive X-axis, and their number increases without bound.

To obtain the pressure in the limit $R \to \infty$, we calculate the asymptotic behaviour of the integral (4.5) for large R. As before, we only consider $Y > 0$ since P_{21} is an odd function of Y. We note that, because the singularity at $l = k$ is integrable, the integral (4.5) is equal to the sum of two Cauchy principal part integrals I_1 and I_2, where

$$I_1 = \mathscr{P}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}lX}}{l-k}\mathrm{e}^{-|l-k|Y}\,\mathrm{d}l, \quad I_2 = -\mathscr{P}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}lX}}{l-k}\mathrm{e}^{-(l^2-k^2)^{1/2}Y}\,\mathrm{d}l. \tag{5.1}$$

The integral I_1 can be evaluated without approximation:

$$I_1 = \mathscr{P}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}lX}}{l-k}\mathrm{e}^{-|l-k|Y}\,\mathrm{d}l = \mathrm{e}^{\mathrm{i}kX}\ln\left(\frac{Y+\mathrm{i}X}{Y-\mathrm{i}X}\right) = \mathrm{i}\mathrm{e}^{\mathrm{i}kX}(\pi - 2\theta), \tag{5.2}$$

where here θ lies in the interval $(0,\pi)$. The expression (5.2) is non-zero at $Y = 0$, but we shall see that I_2 compensates for this.

The integral I_2 is the sum of the integral along two contours C_1 and C_2 in the complex l-plane, where C_1 runs along the real l-axis from $l = -\infty$ to $k - \epsilon$, and C_2 runs along the real l-axis from $l = k + \epsilon$ to ∞, and where the limit $\epsilon \to 0$ is taken. We can rewrite I_2 as the integral along a connected contour $C \equiv C_1 + C_3 + C_2$, minus the integral along C_3. The contour C_3 is a semicircle of radius ϵ below the real l-axis in the complex l-plane, centred at $l = k$, with counterclockwise direction of integration. Hence,

$$I_2 = -\int_C \frac{\mathrm{e}^{h(l)}}{l-k}\,\mathrm{d}l + \pi\mathrm{i}\mathrm{e}^{\mathrm{i}kX}, \tag{5.3}$$

where

$$h(l) = \mathrm{i}lX - (l^2-k^2)^{1/2}Y \tag{5.4}$$

is the argument of the exponential, and the second term in (5.3) arises from the integration along C_3. There is a single critical point of h in the complex l-plane at $l = k\cos\theta$. We deform the contour C onto C_s, the contour of stationary phase passing through the critical point. The leading-order contribution to the integral along C_s arises from the region $|l - k\cos\theta| = O(R^{-1/2}\sin\theta)$. When $kR\sin^2\theta \gg 1$, we may expand the function h about the critical point. Writing $l = k\cos\theta + u$, we have

$$h = kR - \frac{Ru^2}{2k\sin^2\theta} + O\left(\frac{Ru^3\cos\theta}{k^2\sin^4\theta}\right). \tag{5.5}$$

We retain k and θ in the order term to show the dependence of the asymptotic expansion on the Strouhal number and on the angle. Hence,

$$\varphi = I_1 + I_2 = 2\mathrm{i}(\pi-\theta)\mathrm{e}^{\mathrm{i}kX} + \cot\left(\tfrac{1}{2}\theta\right)\left(\frac{2\pi}{kR}\right)^{1/2}\mathrm{e}^{\mathrm{i}kR-\mathrm{i}\pi/4} + O((kR)^{-3/2}\cot^3\left(\tfrac{1}{2}\theta\right)). \tag{5.6}$$

Using (4.21), the expression for P_{21} in the limit $R \to \infty$ is

$$P_{21} = \frac{\Gamma\omega}{2\pi}\left[\mathrm{i}(\pi - \theta)\mathrm{e}^{\mathrm{i}kX} + \tfrac{1}{2}\cos\theta\cot\left(\tfrac{1}{2}\theta\right)\left(\frac{2\pi}{kR}\right)^{1/2}\mathrm{e}^{\mathrm{i}kR - \mathrm{i}\pi/4}\right]$$
$$+ O((kR)^{-1}\sin\theta, (kR)^{-3/2}\cot^3\left(\tfrac{1}{2}\theta\right)). \tag{5.7}$$

The first term was given by Sakov (1993). The second term in this expression is the one obtained by Pitaevskii (1959), Ferziger (1974), Fabrikant (1983) and Lund & Rojas (1989), and has the standard form for diffraction in two dimensions. It vanishes when $\theta = \pm\pi/2$, which accounts for the lack of oscillations at $\theta = \pm\pi/2$ in the scattering cross-sections shown in figures 3 and 4.

The combined contributions from I_1 and I_2 ensure that there is no discontinuity along $\theta = \pi$. Indeed, φ vanishes there, as can be seen from (4.5). However, (5.7) is infinite along $\theta = 0$. The expansion (5.6) is invalid when $kR\theta^2 = O(1)$ because then the critical point at $l = k\cos\theta$ approaches the singularity at $l = k$. We now explicitly consider small θ and write $l = k + \theta^2 v$. We find

$$h = \mathrm{i}kR(1 - \tfrac{1}{2}\theta^2) - kR\theta^2(v + \mathrm{i}(2v)^{1/2}) + O(kR^{-1/2}), \tag{5.8}$$

where we have used the fact $kR\theta^2 = O(1)$.

The integral I_2 is then

$$I_2 = \pi\mathrm{i}\mathrm{e}^{\mathrm{i}(kR-\eta^2)} - \mathrm{e}^{\mathrm{i}(kR-\eta^2)}\int_{C_v} \mathrm{e}^{2(\mathrm{i}v - (2v)^{1/2})\eta^2}\frac{\mathrm{d}v}{v} \tag{5.9}$$

where $\eta = \theta(kR/2)^{1/2}$ and is positive since we are considering only positive Y. The function $(2v)^{1/2}$ is real and positive when v is real and positive, and the branch cut runs from $v = 0$ to $\mathrm{i}\infty$. The contour C_v in the complex v-plane is the image of the contour C in the complex l-plane. After deforming C_v onto the branch cut in the v-plane, it is straightforward to show that

$$I = I_1 + I_2 = 4\mathrm{e}^{\mathrm{i}(kR-\eta^2)}\int_0^\infty \mathrm{e}^{-w^2}\sinh\left[2\mathrm{e}^{\mathrm{i}\pi/4}w\eta\right]\frac{\mathrm{d}w}{w} + O(R^{-1/2})$$
$$= \mathrm{e}^{\mathrm{i}kR}F(\eta) + O(kR^{-1/2}), \tag{5.10}$$

where

$$F(\eta) = 4\pi^{1/2}\mathrm{e}^{\mathrm{i}(\pi/4 - \eta^2)}\int_0^\eta \mathrm{e}^{\mathrm{i}u^2}\,\mathrm{d}u. \tag{5.11}$$

The integral in (5.11) is related to the Fresnel integrals† of Abramowitz & Stegun (1965) by $\int_0^\eta \mathrm{e}^{\mathrm{i}u^2}\,\mathrm{d}u = (\pi/2)^{1/2}(C + \mathrm{i}S)[(2/\pi)^{1/2}\eta]$. The corresponding asymptotic form for P_{21} is

$$P_{21} = \frac{\Gamma\omega}{4\pi}\mathrm{e}^{\mathrm{i}kR}F(\eta) + O(kR^{-1/2}). \tag{5.12}$$

For large η, the asymptotic behaviour of $F(\eta)$ is given by

$$F(\eta) = 2\pi\mathrm{i}\mathrm{e}^{-\mathrm{i}\eta^2} + 2\pi^{1/2}\mathrm{e}^{-\mathrm{i}\pi/4}\eta^{-1} + O(\eta^{-2}). \tag{5.13}$$

The expression (5.13) matches to (5.6) when the limit $\theta \to 0$ is taken in the latter. Thus, our far-field solution in the parabolic region $kR\theta^2 = O(1)$ matches smoothly to the far-field solution obtained by the steepest-descents analysis in the region $kR\theta^2 \gg 1$.

In figure 5 we show $|F(\eta)|$, together with its limiting form for large η from (5.13).

† We thank F. G. Leppington for deriving the connection between (5.10) and the Fresnel integrals.

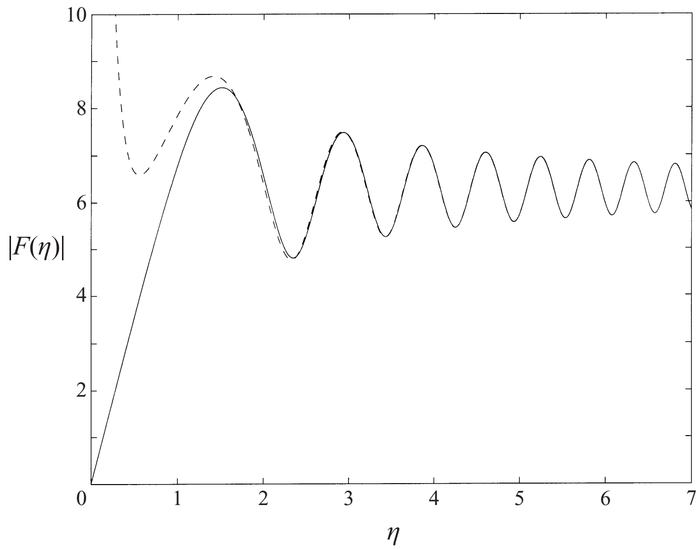

FIGURE 5. Amplitude of $F(\eta)$ (solid curve) and its asymptotic approximation (5.13) (dashed curve) for large η.

From this, we see that the maximum amplitude of $|F(\eta)|$ is $F_{\max} \approx 8.433$, and is achieved at $\eta = \eta_{\max} \approx 1.516$.

From (5.12), we see that the maximum value of $P_{21}^{\text{rms}}/(\Gamma\omega/2\pi)$ approaches the constant value $F_{\max}/2\sqrt{2} \approx 2.982$ as $R \to \infty$. The angle at which this occurs is given by $|\theta| = \eta_{\max}(kR/2)^{-1/2}$, as suggested by the results of Colonius *et al.* (1994) (figure 16 of their paper). Analysis of (5.13) shows that, for sufficiently large η, local extrema of $|F(\eta)|$ occur at $\eta^2 \approx \left(m - \frac{1}{4}\right)\pi$, for integer values of m. Hence the number of local extrema in the scattering cross-section increases without bound as $R \to \infty$, and all the extrema occur in the region $kR\theta^2 = O(1)$.

Figure 6(a) shows the scattering cross-section for $R/\lambda = 70$, together with P_{21}^{rms} evaluated from the asymptotic solution (5.7), valid for $kR\theta^2 \gg 1$. We can see that this asymptotic solution agrees well with the numerical solution, except in a small region about $\theta = 0$.

Figure 6(b) shows the scattering cross-section in a small region about $\theta = 0$ for the same value of R/λ, together with P_{21}^{rms} evaluated from the asymptotic solution (5.12), valid for $kR\theta^2 = O(1)$ and $kR \gg 1$, and from the asymptotic solution (5.7) as before. For the value of R/λ shown, the asymptotic solution (5.12) predicts the maximum amplitude of P_{21}^{rms}, and the spacing in η of the subsequent extrema. However, even at this large value of R, (5.12) predicts accurately the amplitudes of only the first few extrema.

6. The solution to $O(M^4 \ln M\,\delta)$

6.1. *The solution in the vortical region to $O(M^3\delta)$*

The scattered wave field at $O(M^2\delta)$ has zero amplitude at $Y = 0$. However, the numerical simulations of Colonius *et al.* (1994) indicate that the amplitude of the scattered wave along the X-axis is smaller than the amplitude of the scattered wave elsewhere, but nonetheless non-zero. In this section we develop the solution in the

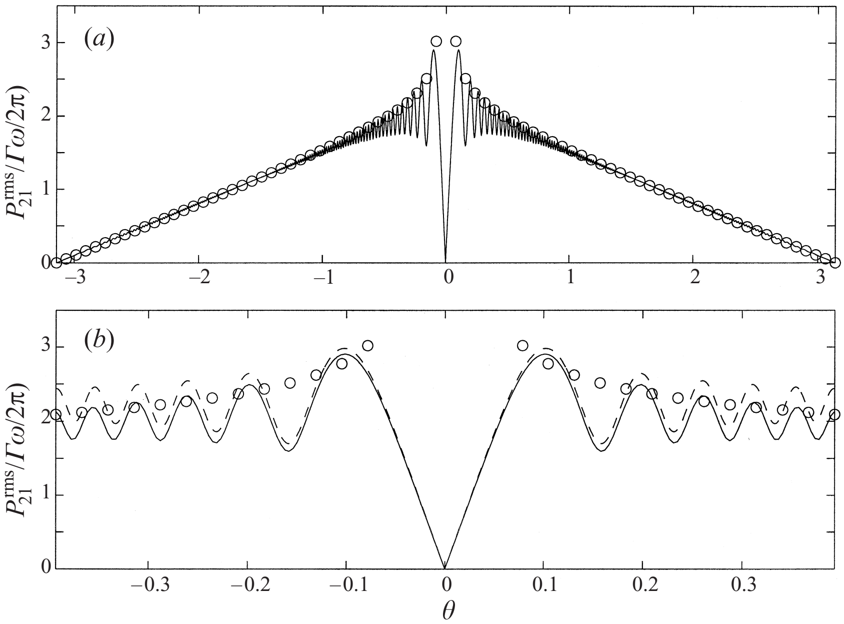

FIGURE 6. The scattering cross-section at $R/\lambda = 70$. (*a*) The full numerical solution (solid line) and the far-field asymptotic solution (5.7) (circles). (*b*) As (*a*), in the region $kR\theta^2 = O(1)$, with the scattering cross-section derived from the asymptotic solution (5.12) (dashed line).

vortical region to $O(M^3\delta)$, and show how this matches to a scattered wave in the wave region at $O(M^4 \ln M\,\delta)$ which has non-zero amplitude along the X-axis.

To determine the matching conditions to be satisfied by the flow in the vortical region to $O(M^3\delta)$, we expand the solution in the wave region for small R:

$$\begin{aligned}\Phi = &-\frac{\mathrm{i}\delta}{\omega}[1 + \mathrm{i}kR\cos\theta - \tfrac{1}{4}k^2R^2(1+\cos 2\theta) - \tfrac{1}{24}\mathrm{i}k^3R^3(\cos 3\theta + 3\cos\theta)]\,\mathrm{e}^{-\mathrm{i}\omega t}\\ &+\frac{\mathrm{i}M^2\delta\Gamma}{4\pi}\left(\frac{2}{kR} + kR[\ln(\tfrac{1}{2}kR) + \gamma_E - \tfrac{3}{2} + \tfrac{3}{2}\pi\mathrm{i}]\right)\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t}\\ &+O(M^4\delta, M^2\delta R^2\ln R).\end{aligned} \tag{6.1}$$

The form of (6.1) implies that there must be flow in the vortical region at $O(M^3 \ln M\,\delta)$ and $O(M^3\delta)$. In the limit $r \to \infty$, the velocity $\nabla\phi + \boldsymbol{k}\times\nabla\psi$ in the vortical region must be consistent with

$$\phi_{311} \to \frac{\mathrm{i}\Gamma}{4\pi}kr\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{6.2a}$$

$$\psi_{311} \to 0, \tag{6.2b}$$

$$\begin{aligned}\phi_{31} \to &-\frac{1}{24\omega}(kr)^3(\cos 3\theta + 3\cos\theta)\,\mathrm{e}^{-\mathrm{i}\omega t}\\ &+\frac{\mathrm{i}\Gamma}{4\pi}[\ln(\tfrac{1}{2}kr) + \gamma_E - \tfrac{3}{2} + \tfrac{3}{2}\pi\mathrm{i}]kr\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t},\end{aligned} \tag{6.2c}$$

$$\psi_{31} \to 0. \tag{6.2d}$$

We are free to add terms to the matching conditions on ψ_{311} and ψ_{31}, provided we subtract corresponding terms from the matching conditions on ϕ_{311} and ϕ_{31}, such that the matching conditions on the velocity remain unchanged.

In the vortical region, (2.12) and (2.3*b*) at $O(M^3 \ln M\,\delta)$ are

$$\frac{\partial \zeta_{311}}{\partial t} + \nabla \cdot (\boldsymbol{u}_0 \zeta_{311} + \boldsymbol{u}_{311}\zeta_0) = 0, \tag{6.3a}$$

$$\nabla \cdot \boldsymbol{u}_{311} = 0. \tag{6.3b}$$

Equations (6.3*a*) and (6.3*b*) for $\boldsymbol{u}_{311}$ are identical to (3.7*a*) and (3.7*b*) for $\boldsymbol{u}_{11}$. We write $\boldsymbol{u}_{311} = \boldsymbol{k} \times \nabla\psi_{311}$, with $\phi_{311} = 0$. The matching conditions (6.2*a*) and (6.2*b*) are then replaced by the equivalent condition

$$\psi_{311} \to \frac{\mathrm{i}\Gamma}{4\pi} kr \cos\theta \, \mathrm{e}^{-\mathrm{i}\omega t}. \tag{6.4}$$

The matching condition (6.4) for ψ_{311} differs from the matching condition (3.8) for ψ_{11} only by a constant factor and rotation though an angle of $\pi/2$ in θ, and (6.3) can be solved for ψ_{311} using a procedure identical to that used to determine ψ_{11}. The solution is

$$\psi_{311} = \frac{\Gamma}{4\pi} kr \left(\mathrm{i}\cos\theta + \frac{\Omega}{\omega}\sin\theta \right) \mathrm{e}^{-\mathrm{i}\omega t}. \tag{6.5}$$

In the limit $r \to \infty$, we have

$$\psi_{311} = \frac{\mathrm{i}\Gamma}{4\pi k} r\cos\theta\,\mathrm{e}^{-\mathrm{i}\omega t} - \frac{\Gamma^2}{8\pi^2 r}\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t} + O(r^{-\infty}). \tag{6.6}$$

At $O(M^3\delta)$, (2.3*b*) is

$$\nabla^2\phi_{31} = -\frac{\partial \rho_{11}}{\partial t} - \nabla \cdot (\boldsymbol{u}_0\rho_{11} + \boldsymbol{u}_{11}\rho_0) = -kr(\omega\cos\theta - \mathrm{i}\Omega\sin\theta)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{6.7}$$

The solution that satisfies (6.2*c*) is

$$\begin{aligned}\phi_{31} = {} & -\frac{1}{24\omega}(kr)^3(\cos 3\theta + 3\cos\theta)\mathrm{e}^{-\mathrm{i}\omega t} + \frac{\mathrm{i}\Gamma}{4\pi}[\ln(\tfrac{1}{2}kr) + \gamma_E - \tfrac{3}{2} + \tfrac{3}{2}\pi\mathrm{i}]kr\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t} \\ & -\tfrac{1}{2}\mathrm{i}k\sin\theta\left[\frac{1}{r}\int_0^r s^3\tilde{\Omega}(s)\,\mathrm{d}s + r\int_r^\infty s\tilde{\Omega}(s)\,\mathrm{d}s\right]\mathrm{e}^{-\mathrm{i}\omega t},\end{aligned} \tag{6.8}$$

where $\tilde{\Omega}(r) \equiv \Omega(r) - \Gamma/2\pi r^2$.

In the limit $r \to \infty$, (6.8) takes the form

$$\begin{aligned}\phi_{31} = {} & -\frac{1}{24\omega}(kr)^3(\cos 3\theta + 3\cos\theta)\mathrm{e}^{-\mathrm{i}\omega t} + \frac{\mathrm{i}\Gamma}{4\pi}[\ln(\tfrac{1}{2}kr) + \gamma_E - \tfrac{3}{2} + \tfrac{3}{2}\pi\mathrm{i}]kr\sin\theta\mathrm{e}^{-\mathrm{i}\omega t} \\ & -\frac{\mathrm{i}k}{2r}\sin\theta\int_0^\infty s^3\tilde{\Omega}(s)\,\mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\omega t} + O(r^{-\infty}).\end{aligned} \tag{6.9}$$

The integral in (6.9) is a ‘renormalized’ angular momentum.

The equation for ψ_{31} is then

$$\frac{\partial \zeta_{31}}{\partial t} + \Omega\frac{\partial \zeta_{31}}{\partial \theta} - \frac{1}{r}\frac{\partial \psi_{31}}{\partial \theta}\frac{\mathrm{d}\zeta_0}{\mathrm{d}r} = -\nabla \cdot (\zeta_0 \nabla\phi_{31}) \tag{6.10}$$

with $\zeta_{31} = \nabla^2\psi_{31}$. Equation (6.10) is a forced linearized vorticity equation. The forcing term is non-zero for azimuthal modes one and three. This equation must be solved subject to the matching condition $\psi_{31} \to 0$ as $r \to \infty$. Because the forcing is localized, the solutions of (6.10) that satisfy the matching condition take the form $r^{-|n|}$ for large r. This means that

$$\psi_{31} = (D_1 + D_{-1})r^{-1}\cos\theta\, \mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{i}(D_1 - D_{-1})r^{-1}\sin\theta\, \mathrm{e}^{-\mathrm{i}\omega t} + O(r^{-3}), \tag{6.11}$$

where D_1 and D_{-1} are constants which are determined by solving (6.10). As we shall see, however, the values of these constants are not required to determine the flow in the wave region at $O(M^4 \ln M\,\delta)$. Note also that the form of all fields calculated so far in the vortical region shows that there is no flow in the wave region at $O(M^3\delta)$.

6.2. *The solution in the wave region at $O(M^4 \ln M\,\delta)$*

The expansion parameter in the wave region is M^2. Therefore, no terms at lower orders in the wave region can interact to force flow in the wave region at $O(M^4 \ln M\,\delta)$. The flow at this order therefore satisfies

$$\frac{\partial \Phi_{411}}{\partial t} + P_{411} = 0, \tag{6.12a}$$

$$\frac{\partial H_{411}}{\partial t} + \nabla^2\Phi_{411} = 0, \tag{6.12b}$$

$$P_{411} = H_{411}. \tag{6.12c}$$

The solution at this order arises entirely as a result of matching conditions with the vortical region, and must be causal. Therefore

$$\Phi_{411} = \sum_{n=0}^{\infty} H_n^{(1)}(kR)(A_n\cos n\theta + B_n \sin n\theta)\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{6.13}$$

where A_n and B_n are constants.

The asymptotic limits (6.6), (6.9), (6.11), and (4.17) imply

$$\Phi_{411} \to \frac{\Gamma^2}{8\pi^2}R^{-1}\cos\theta\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{6.14a}$$

$$\begin{aligned}\Phi_{41} \to &-\tfrac{1}{2}\mathrm{i}kR^{-1}\sin\theta\int_0^\infty s^3\tilde{\Omega}(s)\,\mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\omega t}\\ &+(D_1 + D_{-1})R^{-1}\cos\theta\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{i}(D_1 - D_{-1})R^{-1}\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t}\\ &-\tfrac{1}{8}(\beta_2 - \beta_{-2})kR^{-2}\cos 2\theta - \tfrac{1}{8}\mathrm{i}(\beta_2 + \beta_{-2})kR^{-2}\sin 2\theta. \end{aligned}\tag{6.14b}$$

Note that, although our interest here is in the constants A_n and B_n in the solution at $O(M^4 \ln M\,\delta)$, it is necessary to consider the matching conditions at both $O(M^4 \ln M\,\delta)$ and $O(M^4\delta)$.

For $n > 1$, all the constants A_n and B_n in (6.13) vanish, since there are no terms in the vortical region for them to match onto. The matching condition (6.14*a*) implies that $A_1 = \mathrm{i}\Gamma^2k^2/16\pi$ and $B_1 = 0$ in (6.13). The value of A_0 must be determined by matching to the azimuthal mode-zero flow at $O(M^4 \ln M\,\delta)$ in the vortical region. At this order, the flow in the vortical region is incompressible, and the mode-zero streamfunction satisfies the Rayleigh equation (3.5) with $n = 0$. Therefore, $A_0 = 0$ by the arguments given in § 3.1, and the solution in the wave region at $O(M^4 \ln M\,\delta)$ is

$$P_{411} = \mathrm{i}\omega\Phi_{411} = -\frac{\Gamma^2k^2}{16\pi}H_1^{(1)}(kR)\cos\theta\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{6.15}$$

This is non-zero along the axis $Y = 0$, and hence represents the leading-order scattered wave field along this axis.

7. Discussion

Our analysis shows that the problem of scattering of an acoustic plane wave by a vortex with arbitrary circulation in the Born limit is well posed. The solution (4.21) for the scattered pressure satisfies the causality conditions appropriate to this problem. This may be seen by adding a small imaginary part to k; then the scattered solution decays exponentially for large positive X. The integral representation (4.5) of the scattering which results from interaction in the wave region is readily obtained. To obtain the full scattered sound field, a second component must be added which arises from matching to the dynamics in the vortical region.

The steepest-descents contribution to the far-field behaviour, whose amplitude goes as $(kR)^{-1/2}$ for large R, is always smaller than the contribution $\mathrm{i}(\pi - \theta)\,\mathrm{e}^{\mathrm{i}kX}$ arising from other terms in (4.5), except in the parabolic region $kR\theta^2 = O(1)$. In this parabolic region a different expansion procedure is required. The far-field form of the solution remains bounded for all θ. Berthet & Lund (1995) had conjectured that the problem of scattering of a plane wave by a vortex with circulation might be ill-posed, but this interpretation appears incorrect.

The solution of Colonius *et al.* (1994) was obtained via the 'acoustic analogy' formulation of Lighthill (1952). Their starting point is the dimensional equation

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = \frac{\partial^2}{\partial x_i \partial x_j} T_{ij}, \tag{7.1}$$

where $T_{ij} = \rho u_i u_j + p_0[(1 + \rho'/\rho_0)^\gamma - \gamma\rho'/\rho_0 - 1]\delta_{ij}$ and $\rho' = \rho - \rho_0$. They then approximate (7.1) by taking $T_{ij} = \rho_0 u_i u_j$, which is valid in the limit of small Mach number, and also take the $O(\delta)$ contribution to this approximate T_{ij} that results from substituting into T_{ij} the velocity fields of the basic vortex and the incident plane wave.

To $O(M^2\delta)$, our matched asymptotic expansions yield the same equations in the wave region, because all the nonlinear terms (i.e. the terms on the right-hand-side of (4.2)) arise from the interaction of the basic vortex and the incident plane wave. However, in the vortical region, the acoustic analogy formulation is equivalent to replacing $\boldsymbol{u}_{11}$ in (3.7*c*) by $\boldsymbol{i}\mathrm{e}^{-\mathrm{i}\omega t}$. The result is that terms $O(\Omega^2/\omega^2)$ in the expression (3.14) for p_{11} are neglected. This does not affect the asymptotic matching conditions on P_{21}, because the terms $O(\Omega^2/\omega^2)$ in (3.14) match onto terms $O(M^4\delta)$ in the wave region. Since our solution and the solution of Colonius *et al.* (1994) both satisfy causality conditions, it appears that they must be identical to $O(M^2\delta)$. However, there is no *a priori* reason to neglect terms $O(\Omega^2/\omega^2)$ in (3.14), except in the limit of large Strouhal number, in which $\Omega^2/\omega^2 \ll 1$ by assumption.

If the circulation Γ of the vortex vanishes, all the scattered fields that we have computed vanish. Moreover, the scattered waves in the wave region satisfy the unforced wave equation (2.19) to all orders in M. In that case, the leading-order scattered waves arise at $O(M^4\delta)$, and must satisfy the matching condition (6.14*b*). All terms in (6.14*b*) are in general non-zero when the circulation vanishes, but the constants D_1, D_{-1}, β_2, β_{-2} must be determined numerically. The resulting scattered wave field consists of a dipole and a quadrupole, as found by Colonius *et al.* (1994) using the acoustic analogy formulation. Integral representations can be derived for D_1 and D_{-1}, but β_2 and β_{-2} are obtained by solving (3.5) for $n = \pm 2$, which can be done analytically only in special cases. However, vortices with vanishing circulation are likely to be susceptible to inviscid shear-flow instability, because their vorticity must be a non-monotonic function of radius (Dritschel 1988). Vortices with monotonic vorticity may still be susceptible to acoustic destabilization (Broadbent & Moore 1979), but the growth rates of such instabilities are small ($O(M^4)$) at small Mach number. The three-dimensional problem, in which necessarily $\oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = 0$, leads to faster spatial decay of the velocity field due to the vortex. The leading-order scattered wave in the three-dimensional satisfies an unforced wave equation at leading order (cf. (4.2)). The leading-order scattered wave is a dipolar wave, and arises entirely from interaction between the vortex and the incident wave in the vortical region (Kambe & Mya Oo 1981).

The matching condition (6.14*b*) shows that the scattered wave field at $O(M^4\delta)$ contains information about the internal structure of the vortex, while the scattered wave field at $O(M^2\delta)$ and $O(M^4 \ln M\,\delta)$ depends only on the circulation of the vortex. If ultrasound is to be used to probe vortices, as proposed by Lund & Rojas (1989), it may be desirable to use the scattered wave fields at $O(M^4\delta)$ in an inverse problem to determine information about the internal structure of the vortices.

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