Scattering of acoustic waves by a superfluid vortex

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Abstract. The scattering of plane acoustic waves by a vortex in a two-dimensional superfluid is examined for small Mach number M. The solution is developed from a systematic expansion of the governing equations in three separate regions: an inner vortical region which scales as the healing length, an interaction region governed by irrotational hydrodynamics, and an outer wave region. The solutions in the different regions are matched together. The leading-order scattered field occurs at $\mathrm{O}(M^2\delta)$ in the wave region, where δ is the small non-dimensional amplitude of the incoming acoustic wave. The far-field behaviour of the wave-region solution shows that a different form of the expansion is required in the forward scatter direction: this corresponds to the expression previously derived for the acoustic Magnus force.

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1. Introduction

The advent of high-temperature superconductivity has led to a renewal of interest in acoustic scattering due to vortices in a superfluid, starting with Ao and Thouless (1993) and continuing with Sonin (1997), Stone (2000) and other papers. Similarly, the proposal to measure vorticity in classical fluids using acoustic methods (Lund and Rojas 1990) has led to the development of useful experimental techniques (Labbé and Pinton 1998, Oljaca et al. 1998), and also led to renewed interest in acoustic scattering due to vortices in a classical fluid.

The scattering of sound by vortices in classical fluids and superfluids has a long history, dating back to Obukhov (1948) and Pitaevskii (1959) respectively. The original source of interest in the classical fluids literature came primarily from attempts to understand the scattering of sound by turbulence (Kraichnan 1953, Lighthill 1953, Batchelor 1956). Subsequent work concentrated generally on the scattering of sound by simple vortical structures. The superfluid literature has concentrated mostly on the momentum transfer between the acoustic field and the vortex (for a review of this and many other topics to do with vortices in nonlinear fields, see Pismen 1999).

The classical fluids literature considered the cases of short and long waves, measured relative to the size of the vortex, which have been called the WKB and Born limits.

The latter case was investigated, among others, by Müller and Matschat (1959), Fetter (1964) and Fabrikant (1983). These authors all found that the scattered field had infinite scattering far from the vortex in the forward direction, i.e. the direction in which he plane was originally propagating.

Colonius, Lele and Moin (1994) carried out direct numerical simulations of the Navier–Stokes equations to compute the scattering. The results of their computational aeroacoustic calculation agreed well with the predictions of the Born limit, except in the forward scatter direction, where the scattering amplitude was finite. (Nore et al. 1994 performed numerical simulations for superfluids, but not specifically of scattering.) Independently, Sakov (1993) had found a uniformly valid solution which showed there was a transition region in the forward scatter direction.

All of the above work used what might be called the Lighthill analogy, where the equations of motion are transformed into a forced wave equation, and the forcing term is replaced by the interaction term between wave and vortex. The validity of this approach is not immediately apparent, since it is not based on a systematic treatment of the equations of motion. Lighthill theory for aerodynamic sound generation had been shown to be consistent using matched asymptotic expansions (MAEs) by Crow (1970). Ford and Llewellyn Smith (1999; hereafter FLS) solved the problem of two-dimensional acoustic scattering of a plane wave by an arbitrary axisymmetric vortex with non-zero circulation using MAEs and recovered previous results, thereby putting them on a rigorous footing. Llewellyn Smith and Ford (2001a,b) carried out the same program for acoustic scattering by three-dimensional vortical flows. The results are different in the sense that there is no apparent singularity to resolve, but there are also similarities because matching is also needed to obtain the far-field scattered field.

Parallel lines of investigation in the superfluid literature have used similar Lighthill-analogy techniques, and obtained the same predictions for the scattering amplitude. The aim of this work is to compute scattering of a plane acoustic wave due to a superfluid vortex from a systematic expansion of the superfluid equations. We shall use the Gross–Pitaevskii equation to model the superfluid.

Following FLS (and originally Crow 1970), the procedure used here starts from an asymptotic expansion in terms of the Mach number, M, which is as usual the ratio of the characteristic velocity of the vortex to the sound speed. The superfluid vortex has a natural scale given by the healing length, and there is a wave region with length scale given by the wavelength of incoming waves, where the dominant dynamics are those of linear acoustics. However, unlike the MAE approach to dissipative dynamics of Neu (1990) and Pismen and Rodriguez (1990), a third region is required: this is an interaction region, where the dominant dynamics are those of irrotational hydrodynamics and where the vortex and wave interact. The length scale of this region is fixed by relating the Mach number of the vortex to the Strouhal number of the incoming wave compared to the vortex. The Strouhal number is essentially the nondimensional frequency of the incoming wave.

The amplitude of the incoming acoustic wave is taken to be small, in the sense that its characteristic velocity potential in the interaction region is $O(\delta)$ smaller that the potential due to the vortex, where $\delta \ll 1$. We are interested only in the linear scattering.

The plan of the paper is as follows. Section 2 outlines the equations and the physical flow considered, and sets up the asymptotic procedure. The leading-order vortical and wave flows are calculated. Sections 3, 4, and 5 solve the resulting equations at successive orders in δM^n. Section 6 examines the far-field behaviour of the solution, and Section 7 concludes. The calculations are similar to those of FLS, but the derivation and procedure are outlined in a way which should be intelligible on their own. The steps involved in the calculation of the far-field behaviour are not reproduced however. The results presented here are not new (see Pismen 1999 and Lund and Steinberg 1995), but the aim is to present results on a rigorous footing, and highlight the similarities and differences with the classical case.

2. Statement of the problem

2.1. Governing equations and flow configuration

We consider a superfluid governed by the Gross–Pitaevskii equation (Donnelly 1991)

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - \psi(E - V_0|\psi|^2), \tag{1}$$

appropriate to a weakly interacting Bose gas. This is a nonlinear Schrödinger equation for a the single-particle wavefunction $\psi(\boldsymbol{x}, t)$ governing an assembly of bosons of mass m, with V_0 the strength of the δ-function interaction potential between the bosons, and E the single-particle energy. The Madelung transformation $\psi = R\mathrm{e}^{\mathrm{i}S}$, with R and S real, leads to

$$\frac{\partial\rho_a}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_a\boldsymbol{u}) = 0, \tag{2}$$

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{\hbar^2}{2m^2}\frac{\nabla^2{\rho_a}^{1/2}}{{\rho_a}^{1/2}} - \frac{E}{m}\left(1 - \frac{V_0}{Em}\rho_a\right) = 0, \tag{3}$$

where $\rho_a = mR^2$ is the total density and $\phi = \hbar S/m$ is the velocity potential, to which the velocity $\boldsymbol{u}$ is related through $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. In the language of fluid mechanics, (2) is the continuity equation and (3) is Bernoulli's equation. The third term in (3) is the quantum pressure.

We now non-dimensionalize variables. The phase S is dimensionless, so the physical variable ϕ must scale like $\hbar/m$. In fluid terms, ϕ is a velocity potential and scales as $\phi \sim UL$, where U and L are the velocity and length scales appropriate to the interaction region where the acoustic flow will interact with the vortical flow. The time scale of the interaction region is taken to be $T = L/U$ so that advective effects will enter the problem. The natural scaling for density is $\rho_\infty \equiv Em/V_0$, while the speed of sound for the medium at infinity is given by $c_\infty^2 \equiv E/m$. The non-dimensional amplitude of the incoming wave field is taken to be $\delta \ll 1$, and the linear scattering problem requires the solution of the equations proportional to δ.

The Mach number appropriate to the interaction region is defined by $M \equiv U/c_\infty$. It may also be viewed as the ratio of vortical and interaction length scales, since $M^2 = 2a^2/L^2$, where the healing length, $a \equiv \hbar/\sqrt{2mE}$, is the length scale appropriate to the vortical region.

We may now determine the length scale of the interaction region by fixing the value of the Strouhal number $St \equiv \omega L/U$, which gives the ratio of the vortex turn-over time to the incoming acoustic wave time scale. The Mach number of the flow is then given by $M^2 = St^{-1}\hbar\omega/E$. We assume that this quantity is small. There are then three regions in the flow: a wave region with scale LM^{-1}, an interaction region with scale L and a vortical region with scale LM. Note that in the acoustic case, there are just two regions, called in FLS wave region and vortical region. The latter corresponds to both interaction and vortical regions here. In the wave and interaction regions, the density departs from its value at infinity by $\mathrm{O}(M^2)$, so we may write $\rho_a = \rho_\infty(1 + M^2\rho)$. We are thus led to three sets of equations in the three regions.

In the wave region, the appropriate physical coordinate is $\boldsymbol{X} \equiv M\boldsymbol{x}$, and the velocity potential and density (corresponding to ρ) are denoted by Φ and H respectively. The governing equations are then

$$\frac{\partial H}{\partial t} + \nabla^2\Phi + M^2\boldsymbol{\nabla}\cdot(H\boldsymbol{\nabla}\Phi) = 0, \tag{4}$$

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}M^2(\boldsymbol{\nabla}\Phi)^2 - M^2\frac{\nabla^2(1+M^2H)^{1/2}}{2(1+M^2H)^{1/2}} + H = 0, \tag{5}$$

where gradient operators act with respect to the wave variable $\boldsymbol{X}$.

In the interaction region, the equations become

$$M^2\frac{\partial \rho}{\partial t} + \nabla^2\phi + M^2\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\phi) = 0, \tag{6}$$

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{\nabla^2(1+M^2\rho)^{1/2}}{2(1+M^2\rho)^{1/2}} + \rho = 0. \tag{7}$$

In this region, once the potential ϕ at a certain order has been calculated, the density ρ may be read off directly from (7). This is different from the classical case where the pressure and density must be solved after computing the velocity, which includes a rotational component.

Finally, there is a vortical region in which the quantum pressure is a leading-order quantity. The physical length scale is then the healing length, a, and the appropriate physical coordinate is $\boldsymbol{\xi} \equiv \sqrt{2}M^{-1}\boldsymbol{x}$. The equations will be written in terms of the velocity potential φ and the nondimensional total density squared, R, so that now $\rho_a = \rho_\infty R^2$. This leads to

$$M^2\frac{\partial R}{\partial t} + 2\boldsymbol{\nabla}R\cdot\boldsymbol{\nabla}\varphi + R\nabla^2\varphi = 0, \tag{8}$$

$$M^2\frac{\partial \varphi}{\partial t} + (\boldsymbol{\nabla}\varphi)^2 - \frac{\nabla^2 R}{R} + R^2 - 1 = 0, \tag{9}$$

where the gradients act with respect to the vortical variable $\boldsymbol{\xi}$. The time-derivative terms in the vortical region are small in light of the scalings considered here.

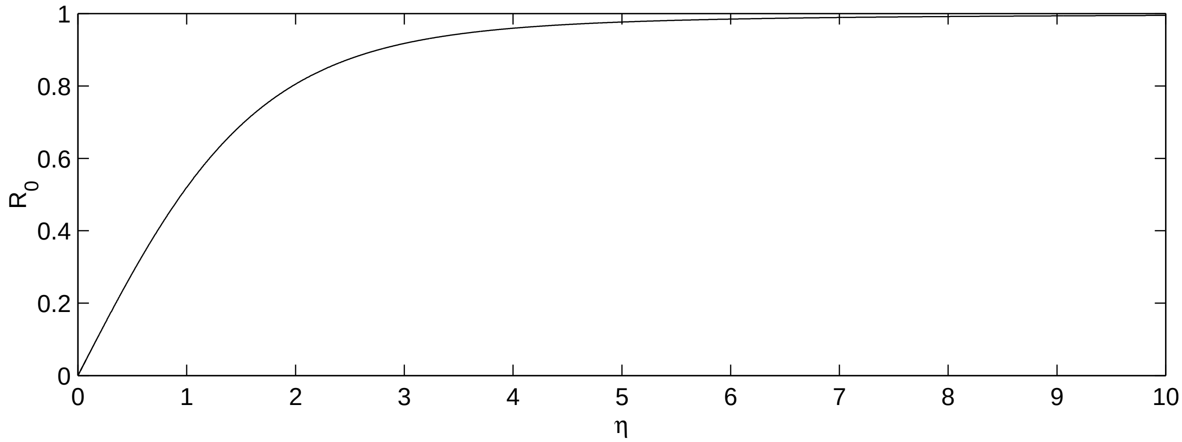

Figure 1. The GP vortex $R_0(\eta)$.

2.2. Leading-order solution in the vortical region

The expansion for the solution in the vortical region takes the form

$$\varphi = \varphi_0 + \delta(\varphi_{01} + M\varphi_{11} + M^2\varphi_{21}) + \mathrm{O}(M^2, \delta M^3, \delta^2), \tag{10}$$

$$R = R_0 + \delta(R_{01} + M\rho_{11} + M^2\rho_{21}) + \mathrm{O}(M^2, M^3\delta, \delta^2). \tag{11}$$

The leading-order solution in the vortical region is the GP vortex given by $\varphi_0 = \theta$, $R_0(\eta)$, where η and θ are polar coordinates in the vortical region. The function $R_0(\eta)$ satisfies

$$R_0'' + \frac{1}{\eta}R_0' - \frac{R_0}{\eta^2} + R_0(1 - R_0^2) = 0. \tag{12}$$

The function $R_0(\eta)$, shown in figure 1, is monotonically increasing and satisfies $R_0 = 0.583\eta + \mathrm{O}(\eta^3)$ for small η and $R_0 = 1 - \frac{1}{2}\eta^{-2} + \mathrm{O}(\eta^{-4})$ for large η.

2.3. Leading-order solution in the intermediate region

The expansion for the solution in the intermediate region takes the form

$$\phi = \phi_0 + \delta(\phi_{01} + M\phi_{11} + M^2\phi_{21}) + \mathrm{O}(\delta M^3, \delta^2), \tag{13}$$

$$\rho = \rho_0 + \delta(\rho_{01} + M\rho_{11} + M^2\rho_{21}) + \mathrm{O}(M^2, M^3\delta, \delta^2). \tag{14}$$

The leading-order solution is given by the steady axisymmetric solution to

$$\nabla^2\phi_0 = 0, \qquad \tfrac{1}{2}(\boldsymbol{\nabla}\phi_0)^2 + \rho_0 = 0, \tag{15}$$

with purely azimuthal flow. This leads to $\phi_0 = A\theta$ and $\rho_0 = -\frac{1}{2}A^2/r^2$. Matching to the vortical region gives $A = 1$. This is just the velocity field due to a point vortex with circulation 2π, which is a solution of the irrotational flow equation for ϕ_0. This solution could have been obtained as the large-η limit of (12). Note that the leading-order part of R_0 in the vortical region matches onto the constant background density 1, and it is the correction term of order η^{-2} that matches to ϕ_0 in the intermediate region.

For large r, the velocity potential remains an $\mathrm{O}(1)$ term, but the density decays like r^{-2}, and will hence match to an $\mathrm{O}(M^2)$ term in the wave region. However, this term is not linear in δ, and does not appear in the linear scattering problem.

2.4. Leading-order solution in the wave region

The expansion for the flow in the wave region takes the form

$$\Phi = \Phi_0 + \delta(\Phi_{01} + M^2\Phi_{21}) + \mathrm{O}(\delta M^4, \delta^2), \tag{16}$$

$$H = \delta(H_{01} + M^2 H_{21}) + \mathrm{O}(M^2, M^4\delta, \delta^2). \tag{17}$$

The velocity potential $\Phi_0 = \theta$ matches onto the velocity potential ϕ_0 in the intermediate region. This is again the potential due to a point vortex at the origin in an irrotational fluid.

2.5. The solution in the wave region at $\mathrm{O}(\delta)$

The incoming acoustic wave satisfies

$$\frac{\partial H_{01}}{\partial t} + \nabla^2\Phi_{01} = 0, \qquad \frac{\partial \Phi_{01}}{\partial t} + H_{01} = 0. \tag{18}$$

We take the solution

$$H_{01} = \mathrm{e}^{\mathrm{i}(kX-\omega t)}, \qquad \Phi_{01} = -\frac{\mathrm{i}}{k}\mathrm{e}^{\mathrm{i}(kX-\omega t)}, \tag{19}$$

to aid comparison with the classical case (see FLS). The real part of all fields will be understood in what follows. The dispersion relation is taken here to be $\omega = k$, so that the wave is coming from $X = -\infty$. The $\mathrm{O}(\delta)$ solutions in the intermediate and vortical regions will be computed as part of the full solution. For small values of X, the incoming wave takes the form

$$H_{01} = \mathrm{e}^{-\mathrm{i}\omega t}(1 + \mathrm{i}kX - \tfrac{1}{2}k^2X^2 + \cdots), \qquad \Phi_{01} = -\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}\omega t}(1 + \mathrm{i}kX - \tfrac{1}{2}k^2X^2 + \cdots). \tag{20}$$

3. The solution to $\mathrm{O}(\delta)$

3.1. The solution in the intermediate region at $\mathrm{O}(\delta)$

The solution in the intermediate region at $\mathrm{O}(M\delta)$ must match onto the solutions in the vortical and wave regions. The matching procedure could be carried out using van Dyke's rule or using an intermediate variable, but we shall be content merely to use informal limit arguments. As $r \to \infty$, we require

$$\phi_{01} \to -\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}\omega t} \tag{21}$$

to match onto the incoming wave, since the solution in the intermediate region at any order is determined by the velocity potential.

The governing equations at $\mathrm{O}(\delta)$ are

$$\nabla^2\phi_{01} = 0, \qquad \frac{\partial \phi_{01}}{\partial t} + \boldsymbol{\nabla}\phi_0 \cdot \boldsymbol{\nabla}\phi_{01} + \rho_{01} = 0. \tag{22}$$

The velocity potential at this order is a single-valued harmonic function and hence takes the form

$$\phi_{01} = \sum_{n=1}^{\infty}(a_n r^n + b_n r^{-n})\mathrm{e}^{\mathrm{i}n\theta} + a_0 + b_0 \log r. \tag{23}$$

Matching onto the wave region shows that the coefficients a_n must be zero for $n > 0$ since they would otherwise require terms of $\mathrm{O}(M^{-n}\delta)$ in the wave region, which do not exist. The coefficient b_0 is also excluded because it has no counterpart in the wave region at $\mathrm{O}(\delta)$. Matching onto the vortical region shows that the remaining b_n must all be zero, since there are no terms of $\mathrm{O}(M^{-n}\delta)$ in the vortical region. We are hence left with

$$\phi_{01} = -\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}\omega t}, \qquad \rho_{01} = \mathrm{e}^{-\mathrm{i}\omega t}, \tag{24}$$

after the matching condition has been applied and ρ_{01} has been obtained from (22)..

3.2. The solution in the vortical region at $\mathrm{O}(\delta)$

The form of (24) shows that there must be an $\mathrm{O}(\delta)$ component of the solution in the vortical region. At this order, the system (8–9) becomes

$$2R_0'\frac{\partial\varphi_{01}}{\partial\eta} + \frac{2}{\eta^2}\frac{\partial R_{01}}{\partial\theta} + R_0\nabla^2\varphi_{01} = 0, \tag{25}$$

$$\frac{2}{\eta^2}\frac{\partial\varphi_{01}}{\partial\theta} - \frac{1}{R_0}\nabla^2 R_{01} + \frac{\nabla^2 R_0}{R_0^2}R_{01} + 2R_0R_{01} = 0. \tag{26}$$

These equations form a linear fourth-order system with coefficients that depend only on η. Hence different azimuthal modes may be considered separately.

Matching onto (24) will give a condition for the axisymmetric component of R_{01} and φ_{01}. For azimuthal wavenumber 0, the velocity potential that satisfies (25) is $\varphi_{01}^{(0)} = A + B\int^{\eta}(uR_0^2)^{-1}\,\mathrm{d}u$. The integral has a logarithmic singularity at the origin, so we pick the solution with $B = 0$ that behaves acceptably there.

For large η, R_{01} behaves like $\mathrm{e}^{\pm\sqrt{2}\eta}$: this will either give an exponentially small solution that cannot be matched at any order, or an exponentially large solution, which is unphysical. The origin is a regular singular point for (25), and the resulting solution for R_{01} does in fact grows exponentially for large η. Its amplitude must hence be zero. The axisymmetric part of the solution at $\mathrm{O}(\delta)$ in the vortical region is hence

$$\varphi_{01}^{(0)} = -\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}\omega t}, \qquad R_{01}^{(0)} = 0, \tag{27}$$

matching the potential to ϕ_{01} in the intermediate region.

There may be other azimuthal modes present in the solution at $\mathrm{O}(\delta)$. As we shall see, azimuthal mode one is necessary to match onto terms of $\mathrm{O}(M\delta)$ in the intermediate region. This solution must be well-behaved at the origin and decay as $\eta \to \infty$. Such a solution exists and takes a very simple form: it is the Goldstone mode

$$\varphi_{01}^{(1)} = (\beta\cos\theta + \gamma\sin\theta)\eta^{-1}, \qquad R_{01}^{(1)} = (-\gamma\cos\theta + \beta\sin\theta)R_0', \tag{28}$$

where the functions β and γ depend on time. (It is convenient to consider the general case here and return to time-harmonic behaviour later.) The Goldstone mode (28) is different from the Goldstone mode of the radial Rayleigh equation in circular geometry (see FLS): the latter has a well-defined amplitude obtained by matching onto the wave field. Here, however, the relation between the coefficients β and γ is as yet undetermined.

4. The solution to $\mathrm{O}(M\delta)$

The matching condition for the $\mathrm{O}(M\delta)$ solution in the intermediate region is

$$\phi_{11} \to x\mathrm{e}^{-\mathrm{i}\omega t} \tag{29}$$

as $r \to \infty$. The governing equations at this order are again (22), but for the $\mathrm{O}(M\delta)$ variables. The solution takes the same general form (23). The requirements that the solution match onto the wave region, and not produce terms larger than $\mathrm{O}(1)$ in both vortical and wave regions leads to

$$\phi_{11} = x\mathrm{e}^{-\mathrm{i}\omega t} + (c\cos\theta + s\sin\theta)r^{-1}\mathrm{e}^{-\mathrm{i}\omega t} \tag{30}$$

after the matching is carried out, with c and s undetermined constants.

The $\mathrm{O}(r^{-1})$ terms will match onto terms in the wave field at $\mathrm{O}(M^2\delta)$ which is where the acoustic scattering takes place. The x-term in ϕ_{11} will match onto an $\mathrm{O}(M^2\delta)$ term in the vortical region, while the r^{-1} terms in ϕ_{11} will match onto $\mathrm{O}(\delta)$ terms in the vortical region, namely the solution given by (28).

Axisymmetric terms in ϕ_{11} are impossible: they would have to be either constant in space or logarithmic. The former cannot match onto any causal solution in the wave region, while the latter match onto the integral in the previous section which is not an acceptable solution. Hence there are no terms in the vortical region at $\mathrm{O}(M\delta)$.

5. The solution to $\mathrm{O}(M^2\delta)$

5.1. The solution in the vortical region at $\mathrm{O}(M^2\delta)$

The presence of the x-term in (30) shows that there is a non-zero azimuthal mode-one component to the solution at $\mathrm{O}(M^2\delta)$ in the vortical region. This form of this term in the limit of large η determines the coefficients β and γ of the Goldstone mode, as we shall see.

The equations in the vortical region at $\mathrm{O}(M^2\delta)$ differ from those at $\mathrm{O}(\delta)$ only by the presence of time-derivatives of the $\mathrm{O}(\delta)$ solution. The governing equations are

$$\frac{\partial R_{01}}{\partial t} + 2R_0'\frac{\partial \varphi_{21}}{\partial \eta} + \frac{2}{\eta^2}\frac{\partial R_{21}}{\partial \theta} + R_0\nabla^2\varphi_{21} = 0, \tag{31}$$

$$\frac{\partial \varphi_{01}}{\partial t} + \frac{2}{\eta^2}\frac{\partial \varphi_{21}}{\partial \theta} - \frac{1}{R_0}\nabla^2 R_{21} + \frac{\nabla^2 R_0}{R_0^2}R_{21} + 2R_0R_{21} = 0. \tag{32}$$

In general, these equations do not have a closed-form solution and must be solved numerically. However, we will be able to extract the information we need by considering (32) in the limit of large η.

The matching condition for the potential is $\varphi_{21} \to r\cos\theta\,\mathrm{e}^{-\mathrm{i}\omega t}$ as $\eta \to \infty$. Equating the various terms in (32) for azimuthal mode one shows that, for large η,

$$\varphi_{21}^{(1)} = \frac{\eta}{\sqrt{2}}\cos\theta\,\mathrm{e}^{-\mathrm{i}\omega t} + \mu\cos(\theta-\theta_0)\eta^{-1} + \mathrm{O}(\eta^{-3}), \tag{33}$$

$$R_{21}^{(1)} = (\sigma\cos\theta + \chi\sin\theta)\eta^{-1} + \mathrm{O}(\eta^{-3}), \tag{34}$$

where σ and χ are again functions of time. The second term in $\varphi_{21}^{(1)}$ is harmonic and plays no further part. Substituting in these values into the governing equations leads to four ordinary differential equations for γ, β, σ and χ, which may be solved to give

$$\gamma = \sqrt{2}\mathrm{e}^{-\mathrm{i}\omega t} + \gamma_0, \qquad \beta = \beta_0, \qquad \sigma = 0, \qquad \chi = 0, \tag{35}$$

where γ_0 and β_0 are constants. Since we are looking for a purely time-harmonic solution, we have $\gamma_0 = \beta_0 = 0$. Note that $R_{21}^{(1)} = \mathrm{O}(\eta^{-3})$ as expected, since ρ_{01} is axisymmetric.

We now finish the matching procedure to determine ϕ_{11} in the intermediate region, using (28) and (35). The $x\mathrm{e}^{-\mathrm{i}\omega t}$ term has already been matched onto the wave region and onto the vortical region. We now match the r^{-1} term to φ_{01} onto the vortical region and obtain $s = \mathrm{i}\omega^{-1}\mathrm{e}^{-\mathrm{i}\omega t}$.

5.2. The solution in the wave region at $\mathrm{O}(M^2\delta)$

At $\mathrm{O}(M^2\delta)$, the equations in the wave region are

$$\frac{\partial H_{21}}{\partial t} + \nabla^2\Phi_{21} + \boldsymbol{\nabla}\cdot(H_{01}\boldsymbol{\nabla}\Phi_0) = 0, \tag{36}$$

$$\frac{\partial \Phi_{21}}{\partial t} + \boldsymbol{\nabla}\Phi_0\cdot\boldsymbol{\nabla}\Phi_{01} + H_{21} = 0. \tag{37}$$

This leads to the forced wave equation

$$\frac{\partial^2\Phi_{21}}{\partial t^2} - \nabla^2\Phi_{21} = \boldsymbol{\nabla}\Phi_0\cdot\boldsymbol{\nabla}H_{01} - \boldsymbol{\nabla}\Phi_0\cdot\frac{\partial\boldsymbol{\nabla}\Phi_{01}}{\partial t} = -2\mathrm{i}k\frac{Y}{R^2}\mathrm{e}^{\mathrm{i}(kX-\omega t)}. \tag{38}$$

We may write the solution to this equation as

$$\Phi_{21} = -\tfrac{1}{2}\mathrm{i}\psi\,\mathrm{e}^{-\mathrm{i}\omega t} + \sum_{n=0}^{\infty} H_n^{(1)}(kR)\left[A_n\cos n\theta + B_n\sin n\theta\right]\mathrm{e}^{-\mathrm{i}\omega t}, \tag{39}$$

where ψ satisfies

$$(-k^2 - \nabla^2)\psi = 4k\,\mathrm{e}^{\mathrm{i}kX}\frac{\partial}{\partial Y}\ln R, \tag{40}$$

$H_n^{(1)}$ is a Hankel function of the first kind (Abramowitz and Stegun 1965), and the constants A_n and B_n must be determined by matching to the solution in the vortical region. We know that A_n and B_n must be zero for $n > 2$ to avoid solutions that cannot be matched to the vortical region.

From FLS, we may write ψ as

$$\psi = \operatorname{sgn} Y\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}lX}}{l-k}\left[\mathrm{e}^{-|l-k||Y|} - \mathrm{e}^{-(l^2-k^2)^{1/2}|Y|}\right]\mathrm{d}l. \tag{41}$$

The following result is required for the matching:

$$\psi = 2kY\left(1-\gamma-\ln\left(\tfrac{1}{2}kR\right)-\tfrac{1}{2}\mathrm{i}\pi\right)+\mathrm{O}(R^2\ln R), \tag{42}$$

where $\gamma = 0.5772\ldots$ is Euler's constant. Note that $\psi(k\boldsymbol{X})$ is an odd function of Y, as may be seen from (40).

5.3. The solution in the intermediate region at $\mathrm{O}(M^2\delta)$

At $\mathrm{O}(M^2\delta)$, the governing equation is

$$\nabla^2\phi_{21} = -\frac{\partial\rho_{01}}{\partial t} = \mathrm{i}\omega\mathrm{e}^{-\mathrm{i}\omega t}. \tag{43}$$

A convenient solution is

$$\phi_{21} = \tfrac{1}{2}\mathrm{i}\omega x^2\,\mathrm{e}^{-\mathrm{i}\omega t}+\sum_{n=1}^{\infty}(e_n r^n + f_n r^{-n})\mathrm{e}^{\mathrm{i}n\theta}+e_0+f_0\log r. \tag{44}$$

The matching condition as $r\to\infty$ is

$$\phi_{21}\to\tfrac{1}{2}\mathrm{i}kx^2\mathrm{e}^{-\mathrm{i}\omega t}. \tag{45}$$

Matching to the far field shows that $e_n = 0$, since these terms cannot match to any causal solutions in the wave region. Matching onto the vortical region also gives $f_n = 0$ for $n > 2$. However, the values of f_1 and f_2, which match onto the vortical region at $\mathrm{O}(M\delta)$ and $\mathrm{O}(\delta)$ respectively are not required since they only come into the matching to the wave region at orders higher than $\mathrm{O}(M^2\delta)$ As a result, we need not compute them.

5.4. Matching at $\mathrm{O}(M^2\delta)$

To complete the matching at $\mathrm{O}(M^2\delta)$, we need to compute A_n and B_n in (39). These coefficients must vanish for $n > 1$ for consistency with the vortical region.

There can be no A_0 or B_0 terms as was explained for the the $\mathrm{O}(M\delta)$ solution in the intermediate region. The ψ contribution to the $\mathrm{O}(M^2\delta)$ solution will match onto terms in the intermediate region at $\mathrm{O}(M^4\delta)$ (actually at $\mathrm{O}(M^4\log M\,\delta)$ to be precise). What remains is to compute A_1 and B_1. These terms match onto the intermediate region solution at $\mathrm{O}(M\delta)$ from (30). The matching gives $A_1 = 0$ and $B_1 = \frac{1}{2}\mathrm{i}ksF$, using the result $H_1^{(1)}(z)\sim -2\mathrm{i}(\pi z)^{-1}$ for small z.

Hence the solution in the wave region at $\mathrm{O}(M^2\delta)$ is

$$\Phi_{21} = -\tfrac{1}{2}\mathrm{i}\psi(k\boldsymbol{X})\mathrm{e}^{-\mathrm{i}\omega t}-\tfrac{1}{2}\pi H_1^{(1)}(kR)\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{46}$$

This may be obtained by substituting $\Gamma = 2\pi$ into (4.20) of FLS. The density is now obtained from (37) and takes the form

$$H_{21} = \tfrac{1}{2}\omega\psi(k\boldsymbol{X})\mathrm{e}^{-\mathrm{i}\omega t}-\tfrac{1}{2}\mathrm{i}\pi\omega H_1^{(1)}(kR)\sin\theta\,\mathrm{e}^{-\mathrm{i}\omega t}+\frac{Y}{R^2}\,\mathrm{e}^{\mathrm{i}(kX-\omega t)}. \tag{47}$$

This expression is identical to the pressure and density in the classical case. Both potential and density are odd functions of Y.

6. Far-field analysis

Following FLS, we may compute the far-field behaviour of H_{21}. The asymptotics of the function ψ are slightly non-standard, but may be calculated by careful use of the method of steepest descents. The result comes out to be

$$\Phi_{21} = (\pi - \theta)\mathrm{e}^{\mathrm{i}kX} - \tfrac{1}{2}\mathrm{i}\cos\theta\cot\left(\tfrac{1}{2}\theta\right)\left(\frac{2\pi}{kR}\right)^{1/2}\mathrm{e}^{\mathrm{i}(kR-\pi/4)} + \mathrm{O}\left((kR)^{-3/2}\cot^3\left(\tfrac{1}{2}\theta\right)\right). \quad (48)$$

This expression is valid for θ in the range $0 < \pi < \theta$. The result for $-\pi < \theta < 0$ may be obtained from the fact that Φ_{21} is an odd function of Y. The second term in (48) is the one that gives the infinite scattering amplitude in the forward direction. Most papers starting from the Lighthill analogy for classical fluids have missed the first term, although not Sakov (1993). The expansion (48) breaks down close to the forward scatter direction. The expansion is non-uniform in space, and near the positive X-axis, a different expansion holds.

There is a distinguished scaling for $kR\theta^2 = \mathrm{O}(1)$, which leads us to define a new variable $\vartheta = \theta(kR/2)^{1/2}$. This corresponds to a parabolic region around the forward scatter direction. The solution in this area is called the acoustic Magnus force in Pismen (1999; see also Iordanskii 1966). In this region,

$$H_{21} = -\tfrac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}kR}F(\vartheta)\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{O}(kR^{-1/2}), \quad (49)$$

where the function F is defined by

$$F(\vartheta) = 4\pi^{1/2}\mathrm{e}^{\mathrm{i}(\pi/4-\vartheta^2)}\int_0^\eta \mathrm{e}^{\mathrm{i}u^2}\,\mathrm{d}u = 2^{-3/2}\pi\,\mathrm{e}^{\mathrm{i}(\pi/4-\vartheta^2)}(C+\mathrm{i}S)[(2/\pi)^{1/2}\vartheta]. \quad (50)$$

The functions C and S are Fresnel integrals (Abramowitz and Stegun 1965). The form (49) matches onto the expansion (48) for large ϑ.

7. Conclusions

The scattering due to a superfluid vortex turns out to be the same as that due to a classical vortex with circulation 2π, although the nature of the interaction region between wave and vortex is different. The asymptotic procedure used here forces the vortex to be 'oscillating' rather than 'pinned', which seems to be the situation that in any case is usually considered (cf. 4.61 of Pismen 1999). The scattered field has amplitude $\mathrm{O}(M^2\delta)$ in the far field. The acoustic drag and Magnus forces correspond naturally to the far-field expansion of the wave region scattered field in regions far from, and near to, the forward scatter direction, respectively.

The results of this work show that the classical and superfluid vortices scatter sound in a similar fashion, even though the superfluid vortex has a further level of complication in its dynamics. The Goldstone mode of the vortex plays a key role in the solution in both cases. For the superfluid vortex it is a steady mode to leading order.

As mentioned earlier, much of the recent interest in acoustical scattering by classical fluid vorticity has been aroused by the development of new non-intrusive experimental techniques to measure vorticity, as in Labbé and Pinton (1998), Oljaca et al. (1998) and Manneville et al. (1999). Lund and Steinberg (1995), following the approach of Lund and Rojas (1980) have suggested using second sound to detect and measure quantum vorticity, and their results are similar to those obtained here. Davidovitz and Steinberg (1997) have also advocated using the Aharonov–Bohm effect, which is implicit in (48), to measure quantized vorticity in a superfluid.

The problem of sound scattered by a more general vortical structure is interesting. The results of FLS and Llewellyn Smith and Ford (2001a) suggest that, in this MAE framework, the scattered field will just be determined by the total circulation of the vortices, provided they are separated by a distance of the order of the interaction length or less, with vortex-vortex interactions on the healing length scale being set aside.

Finally, a natural extension of this work is to consider how these results change for a two-fluid model. Given the results of this paper and of FLS, one might conjecture that the scattering amplitude would again be the same.

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