How do singularities move in potential flow?

Stefan G. Llewellyn Smith[a]

[a]*Department of Mechanical and Aerospace Engineering, Jacobs School of Engineering, UCSD, 9500 Gilman Drive, La Jolla CA 92093-0411, USA*

Abstract

The equations of motion of point vortices embedded in incompressible flow go back to Kirchhoff. They are a paradigm of reduction of an infinite-dimensional dynamical system, namely the incompressible Euler equation, to a finite-dimensional system, and have been called a "classical applied mathematical playground". The equation of motion for a point vortex can be viewed as the statement that the translational velocity of the point vortex is obtained by removing the leading-order singularity due to the point vortex when computing its velocity. We review the arguments used to obtain this result and discuss their history and limitations. We give a formulation that can be extended to study the motion of higher singularities (e.g. dipoles). Extensions to more complex physical situations are also discussed.

Keywords: Point vortex; Dynamical system; Euler equation; Irrotational flow.

1. Introduction

Vorticity has been a fundamental concept in fluid mechanics since its introduction by Helmholtz in 1858 [1]. Helmholtz's paper was translated by Tait in 1867 [2], sparking a wave of work by the Scottish school, including Kelvin and Thomson, and others, who for a time sought a theory of "vortex atoms" to explain the structure of matter.

Understanding elementary vortex structures has been a focus of extensive research. Given the complexity of the problem, simplified situations have been much considered. Two-dimensional flows are a good approximation for flows that do not vary much in the third dimensional, or that are constrained by effects such as stratification and rotation to move along near-horizontal surfaces. The next obvious approximation is that of using singular vorticity distributions: this holds the promise of being able to replace partial different equations by a system of ordinary differential equations. Point vortices are the natural candidate for constructing such a system. In many cases the scale of the vortices is much smaller than the other scales in the system, so replacing the vortices by elementary structures with no intrinsic scale is a natural modelling step.

Point vortices have been called a "classical applied mathematical playground" [3]. Applications include chaotic advection [4]; integrable systems [5–7]; control of fluid flows [8, 9]; biological locomotion and models of vortex shedding and wakes [10–15] and geophysical applications [16, 17]. Related problems arise in superfluids [18] and in dislocation theory [19], but we limit ourselves here to potential flow.

The complex potential for a point vortex at $z_n = x_n + \mathrm{i}y_n$ with circulation Γ_n is

$$\phi_n = \frac{\Gamma_n}{2\pi\mathrm{i}} \log(z - z_n). \tag{1}$$

Then the equation of motion, or Point Vortex Equation (PVE), is simply

$$\dot{z}_n = \overline{\tilde{w}_n}. \tag{2}$$

The tilde indicates the desingularized complex velocity at z_n. i.e. the limit as $z \to z_n$ of

$$\tilde{w}_n = \lim_{z\to z_n} \left[w - \frac{\Gamma_n}{2\pi\mathrm{i}} \frac{1}{z - z_n} \right], \tag{3}$$

where w is the full velocity field which may include contributions from other point vortices and from a smooth irrotational flow (e.g. due to boundaries). We stress that (2) is nothing to do with a sum of other point vortex velocity fields: it is the statement that the velocity of the point vortex is obtained by removing the leading-order singularity due to the point vortex when computing its velocity.

An extension to PVE is the Brown–Michael equation (BME) that has been proposed to govern the motion of a point vortex shed from a sharp corner. In potential flow, the velocity field near a non-reentrant corner is singular. This singularity can be related to the conformal mapping of a plane in which the contour is smooth to the physical plane. A vortex can be associated with each corner and its circulation set so as to make the velocity at the edge finite. The resulting circulation varies in time and one needs an equation for the motion of the vortex. The result, as obtained by a number of researchers in the 1950s (not just Brown and Michael), is not PVE but rather BME.

One can ask how other singularities might move, for examples point sinks or sources, or dipoles. A more general equation, or possibly set of equations, is needed, which we may call the point singularity equation or PSE.

If one considers the PVE separately from their vast history and popularity, one can ask how they are justified and what they mean. The answer to this question is not as obvious as it may appear at first sight. To gain some insight into the nature of the

Email address: sgls@ucsd.edu (Stefan G. Llewellyn Smith)

problem, we start in § 2 with a historical review of the justifications given for these equations (PVE, BME, PSE). We consider in § 3 an argument based on the conservation of momentum that gives PVE and BME. However, as we show in § 4, problems arise when moving to PSE. In § 5 we show how to resolve these problems. We give examples in § 6 and conclude in § 7, in which we also discuss possible extensions to more general situations.

2. Historical overview

We concentrate here strictly on how authors have justified or derived PVE, BME and PSE. Our review is biased in favour of the English language literature, and, once we are past the first few papers in the area, uses textbooks as indicators of the received wisdom on the subject. Four rough historical periods can be delineated. (An extensive bibliography of vortex dynamics is given by Meleshko and Aref [20].)

2.1. The pioneers: derivation (1858–1912)

Helmholtz's original work does not explicitly give the PVE. In fact there is no mention of point vortices at all. The behavior of parallel vortex lines (straight vortex filaments) is considered: "If there be two rectilinear vortex-filaments of indefinitely small section in an unlimited fluid, each will cause the other to move in a direction perpendicular to the line joining them. Thus the length of this joining line will not be altered. They will thus turn about their common centre of gravity at constant distances from it."

The PVE is given explicitly in Kirchhoff's 1876 Lecture notes [21]. The relevant Lesson is translated in Appendix A. This is essentially Helmholtz's argument turned into equations. Kirchhoff is talking about vortex filaments with infinitesimally small cross-section remaining at finite distances from each other, but allows the cross-section of the filaments to change.

In 1881, Routh [22] uses conformal mappings to obtain the complex potential in domains for which the method of images fails. He never actually writes down what is now called the Routhian correction. Routh's prescription for obtaining the PVE is as follows: "the current function of P is obtained from that of Π by subtracting $(m/2)\log\mu$", i.e. he removes the singularity of the complex potential in the physical plane. This is a natural approach when one has the complex potential to hand, given the importance of isolated singularities in complex analysis and their usefulness for theorems such as the residue theorem. Routh is the first to use the tools of complex variable theory in the treatment of point vortices.

J. J. Thomson's 1883 work *A Treatise on the Motion of Vortex Rings* [23] is about vortex rings but first considers the stability of a polygonal array of point vortices. Thomson first shows (§ 39) that an almost circular column (i.e. a line vortex) has neutral modes: "We shall prove that if the cross sections of two such vortex columns are at any moment approximately circular they will always remain so". This result had been previously stated and shown by William Thomson (Lord Kelvin). J. J. Thomson later states (§ 42) "Our investigation of the motion of two infinite cylindrical vortices shews that to retain an approximately circular cross section the vortices must be at a distance from each other large compared with the diameter of the cross section of either". His vortex lines hence have a finite area and he requires them to be far enough apart to stop them from deforming each other's cores. Since then, numerical calculations have shown that if vortex patches are placed close enough together, vortex merger ensues [24]. Thomson then writes (§ 48) "We suppose the radius of a cross section of a vortex to be small compared with the distance between two vortices . . . The stream function due to a single vortex of strength m at a point whose distance from the vortex is $\rho = -(m/\pi)\log\rho$." This is implicitly the PVE. It is clear that that singular point vortex velocity field is being given physical credence because it has already been shown that vortices far enough away from each other remain circular to leading order.

Basset's 1888 *A Treatise on Hydrodynamics*[25] does not add anything new . He explicitly follows J. J. Thomson's account (§ 290): "Hence it follows that if any number of vortices of small cross sections are moving in the liquid, and the vortices never get very close to one another, we may neglect the effects produced by the deformations of their cross sections, which may therefore be regarded as approximately circular." Then (§ 292): "We shall pass on to consider the motion of a number of vortices of small and approximately circular cross sections" and the PVE is derived.

In his 1893 book *Théorie des tourbillons*, Chap. 6, § 65–68 [26], Poincaré treats point vortices. He takes narrow and straight vortex tubes with circulation $2\pi m_1, \ldots,$ and states that their strengths don't change. First he shows the center of vorticity of all the tubes stays fixed. Then he shows that the center of vorticity of a single tube is fixed (this is just the previous result actually). So, he says, to compute the motion of a single tube, we ignore its own velocity and take only that of the other tubes.

Zhukovskii wrote about point vortices in 1893 [27]. He writes down the streamfunction as an integral of vorticity with the Green's function, and then argues that the vorticity is concentrated into a small region. Hence it can be pulled out of the integral and the resulting simple integral gives the PVE when one subtracts out the leading-order singularity. This is an interesting approach, related to the idea of the far-field behavior of a concentrated vortex being essentially that of a point singularity, but nevertheless relies on the usual discarding of the singularity.

To end this era, we note that in Volume IV of the Collected Works of Kelvin, published in 1910 [28], there is no discussion of point vortices at all, despite a lot of discussion of vortex rings and filaments. Similarly the 1912 edition of Thomson and Tait's *Principles of Mechanics and Dynamics* (1st edition published in 1879 as *A Treatise on Natural Philosophy*) contains nothing on point vortices either [29].

We may therefore conclude that the derivations of PVE in this pioneering era are based on a verbal argument: the contribution of the self-induced velocity of the vortex is ignored. However, apart from Routh, all the authors talk about infinitesimal line vortices. It is clear that they are aware that the boundaries of the cross-sections of these vortices can be deformed, but the fact that the deformations take the form of neutral modes leads them to disregard these deformations if the other line vortices are far enough away. When Routh writes down the complex potential, the approach of removing the singularity directly from the potential becomes natural. From this point on, point vortices are usually viewed as singular structures rather than as having infinitesimal cross sections.

2.2. The classics: formalization (1912–1954)

A number of textbooks still used today originally date from this era (Lamb, Sommerfeld, Milne-Thomson). The complex variable formulation of irrotational flow is mature at this point, but we shall see that there is no change in presentation: the verbal argument of Helmholtz is still used.

Villat's 1930 work *Leçons sur la Théorie des Tourbillons* [30] uses complex potentials. In Chapter III he derives the velocity field outside a "tube infiniment delié", i.e. an infinitely thin tube, at a point A. "Quant au point A lui-même il reste évidemment immobile, par raison de symétrie, puisque le fluide est en repos a l'infini." Then in "Application au mouvement des tubes", we have "La vitesse est $(\mathrm{d}x_1/\mathrm{d}t, \mathrm{d}y_1/\mathrm{d}t)$, elle provient des vitesses provoquées par les tourbillons A_2, $\mathrm{A}_3, \ldots$, et de la vitesse eventuelle provoquee par le tube A_1 lui-même. Mais ce dernier, s'il etait seul, resterait immobile: il n'y a donc pas de vitesse de cette provenance, ..."

The final edition of Lamb's *Hydrodynamics* dates from 1932 [31] (the first edition dates from 1878). We find in § 155: "Since this centre remains at rest, the filament as a whole will be stationary. [...] The motion of each filament as a whole is entirely due to the other, and is therefore always perpendicular to AB."

Ewald, Pöschl and Prandtl's *The physics of solids and fluids, with recent developments* [32] discusses vortex filaments, not point vortices: "If there are several vortex filaments, the separate velocity fields are superposed and each vortex filament takes part in the motion which the others produce at the place where it is situated.

It is interesting to compare a classic hydraulics textbook, which one might expect to have a practical bent. Rouse's 1938 book *Fluid mechanics for hydraulic engineers* [33] has the novelty of mentioning possible free surface effects. The derivation of the PVE is "The existence of two or more neighboring filaments thus results in a relative movement of each filament in accord with the velocity fields of the others." He is hence considering explicitly line vortices (straight filaments).

Sommerfeld, in his Lectures on Physics (1950, vol. 2, IV.21.2) [34] just states that vortices move with induced velocity: "The velocity v_1 imparted to the filament F_1 by F_2 is equal and parallel to the velocity v_2 imparted to F_2 by F_1: $v_1 = v_2 = \mu/(2\pi c) = v$."

Milne-Thomson's *Theoretical Hydrodynamics* had its first edition in 1938 (making it a successor to Lamb) and its final edition in 1968 [35]. We read in § 13.22: "We have seen (13.10) that a circular vortex alone in the fluid possesses no tendency to set itself in motion and the same therefore applies to a vortex filament. If therefore there are several vortex filaments, the motion of the filament at the point P is the same as the motion which would be produced at P by the remaining vortices if the vortex at P did not exist." Milne-Thomson specifically writes down the equation by subtracting off the singular $\log(z - z_0)$ term in the potential as in (2).

2.3. The golden age: expansion (1952–1984)

The post-Second World War era of increasing research in aerodynamics and funding of fluid dynamics led, as part of research into supersonic flow past delta wings, to BME. BME was developed by Brown, Michael, Edwards, Cheng and Rott in the period 1952–1956. These authors initially found equations describing steady vortex sheets shed off delta wings, viewing the sheets as point vortices in cross-section, and later considered moving vortices, such as those shed by shocks passing over wedges. This development is rather interesting, but not central to the topic of this historical review, so it is outlined in Appendix B. However, treatments of the PVE in the textbooks and monographs of the time shows no real change from before, with one exception (Friedrichs).

A notable outlier is Truesdell in his 1954 book *The Kinematics of Vorticity* [36]. Ever the individualist, Truesdell talks only about kinematic properties of vortices. Consequently, vortex lines are mentioned, but there are no dynamics and no point vortices. One might wonder whether the omission of point vortices meant that Truesdell viewed them as dynamical entities, i.e. entities for which forces are important. It is more likely that his emphasis on three dimensions, as noted in McVittie's 1955 book review [37]: "A curious feature of Truesdell's treatment of vorticity is his extreme insistence on the importance of three spatial dimensions", excluded them from his consideration.

Feynman's *Lectures on Physics* (Vol. 2. 40.4–5) [38] discuss circulation and Helmholtz's laws of vorticity. Point vortices do not appear. Feynman's description of inviscid fluid mechanics as "dry fluid mechanics" is worth bearing in mind should one be tempted to take inviscid point vortices too seriously.

A number of important Russian books were translated during this period. One is Kochin, Kibel' and Roze's book *Theoretical Hydromechanics*, a 1964 English translation of the 1955 Russian original [39]. § 5.13 treats the case of one vortex: "As a consequence of the symmetry of the fluid motion around a point vortex, it is obvious that the vortex will remain fixed". In § 5.14 (two vortices), we find: "We will study the motion of the vortices in the fluid. The vortex at the point z_1 moves only under the influence of the other vortex, since an individual vortex does not move (a vortex does not act on itself); ..." Another interesting Russian book by Sedov, *Two-dimensional problems in hydrodynamics and aerodynamics* (1965 English translation of 1950 Russian original) [40] does not discuss vorticity at all. Sedov's 1971 (1968 in Russian) *A course in continuum mechanics. Vol 3: Fluids, gases and the generation of thrust* [41] does consider point vortices. We read in § 10.8.13: "In order to determine the velocity of a particle at the location z_{0s} of the vortex, one must employ the sum (10.8.22) after omitting from it the term $\Gamma_s/(2\pi\mathrm{i})(z - z_{0s})^{-1}$ corresponding to the point z_{0s}". Finally the influential Landau and Lifshitz (first English translation in 1959, second edition in 1987 with Pitaevskii) [42] rather surprisingly does not mention point vortices at all.

The first textbook to take a different approach is Friedrichs' 1966 *Special Topics in Fluid Dynamics* [43]. In it, he computes the force exerted by the fluid on a vortex filament (point vortex) and argues that if the vortex is free (as opposed to bound), this force must vanish. The idea of the force acting on a vortex filament was presumably inspired by the BME work mentioned above and will reoccur in later books. It is also very common in the superfluid literature.

Two well-known English textbooks are Batchelor and Tritton [44, 45]. There are no point vortices in the latter, which has a strong physical emphasis. In the former, we find in § 7.3: "Motion of a group of point vortices. The above integral invariants take a simpler form when the vorticity is concentrated at a number of points. [...] The strengths of these vortices remain constant [...] the velocity of movement of the vortex of strength κ_j is equal to the velocity of the fluid at the point (x_j, y_j) due to all the *other* vortices, since there is no self-induced movement of a point vortex."

We see essentially no change in the presentation of the PVE in the textbooks. However, in parallel, BME is developed and finds its ultimate form in 1956, but is only used in specialized contexts.

2.4. *The moderns: mathematics and dynamical systems (1984–present)*

There has been an explosion in the use of the PVE in recent years, driven by applications and by its role as a prototype dynamical system [46–48]. The development of vortex methods as a tool in computational fluid dynamics has been another source of interest in the dynamics of vortical structures. The starting date for this era can be loosely set as 1984 when Marchioro and Pulvirenti proved that systems of small vortex patches converge to vortex dynamics [49]. Given the vast amount of modern material, we limit ourselves to textbooks or articles that explicitly discuss or derive the PVE or generalizations.

In § 6.5 of Lighthill's 1986 *An informal introduction to theoretical fluid mechanics* [50], after a comment on vortex sheets, we find: "Similarly, a line vortex of given strength thickness may be modelled as a line vortex with identical strength K but with zero thickness, i.e. as a line where the velocity tangential to a small circle of radius r around it behaves asymptotically like eqn (155) as $r \to 0$. These can be useful idealizations provided that we do not forget that in either case the nonzero thickness of the real structure, which must be present if only because of viscous diffusion." Then in § 6.4: "Line vortices do, of course, move with the fluid. Figure 37, illustrating the two velocity fields which together make up the flow field due to a line vortex near a plane boundary, shows that the velocity field (a) does not move the line vortex at all. However, the velocity field does move it, parallel to the plane, ..." This is a pictorial justification. Lighthill is clear however that point vortices are "useful idealizations."

As an aside, Krasny [51] uses a combined vortex sheet and vortex-dipole sheet model for the numerical simulation of a wake. The vortex-dipole distribution $\mathbf{D}$ evolves according to

$$\mathbf{D}_t = -\nabla\mathbf{u}^T \cdot \mathbf{D} \tag{4}$$

which is the evolution governing the gradient of vorticity. This is a Lagrangian equation. It is not quite the same as PVE or PSE because a vortex(-dipole) sheet is considered rather than a point vortex(-dipole). The result (4) will be useful later.

Ting and Klein's 1991 book *Viscous Vortical Flows* (updated in 2007 with Knio) [52, 53] presents work that goes back to the 1960s. The three-dimensional case is the real motivation, but we find a Matched Asymptotic Expansion (MAE) calculation for the Rankine vortex in a uniform stream in § 2.1.1.2. The result is the PVE equations in the far field (i.e. on scales far larger than the vortex) and neutral modes on the edge of the vortex.

In his 1992 textbook [54], Saffman (§ 2.3) gives a momentum flux argument using velocity potential: "it is appropriate to give a direct argument based on momentum conservation". Subsequently he presents "[...] an alternative argument based on vortex force".

Meleshko and Konstantinov's 1993 book *Dynamics of Vortex Structures* [55] says that Helmholtz's law that vorticity is frozen into fluid lines justifies the PVE.

Other textbooks from the 1990s approach the PVE in a number of ways. Chorin and Marsden's 1993 *A Mathematical Introduction to Fluid Mechanics* [56] states on pp. 61–62: "As the fluid moves according to Euler's equations, the circulations Γ_j associated with each vortex will remain constant" and later "Each one ought to move as if convected by the net velocity field of the other vortices". Chorin's 1994 *Vorticity and Turbulence* [57] is interesting in that it combines the smoothed kernel argument related to numerical vortex methods with the Victorian argument about being able to neglect the deformations of small patches, We find in § 1.3 the following: "Consider the motion of the centers of each of the functions of small support, neglecting the deformation of that support by the flow; their velocities are ... (The exclusion of $i = j$ is convenient and at this stage obviously harmless.)" The may no be not quite so harmless once one replaces functions with small support by delta functions. Faber's 1995 *Fluid Dynamics for Physicists* [58] is unusual for a fluid mechanics book in that it is written from a physicist's perspective. It contains a long discussion of vortex filaments, unsurprisingly, since they are so important in superfluid helium. In particular § 4.11–4.14 has an extensive discussion of forces on vortex lines. The vortex lines are viewed as physical entities that exert forces on each other, which is ultimately what makes the vortices move.

Panel methods in aerodynamics are based upon single- and double-layer potentials for Laplace's equations, and their numerical solution uses discrete representations of the potentials. A single-layer potential can hence be viewed as a collection of point vortices along a boundary. A clear book on the topic is Katz and Plotkin's *Low-Speed Aerodynamics* [59]. The velocity fields due to vortex filaments and line vortices are worked out, but the vortices are not considered as moving entities.

Newton's 2001 book *The N-Vortex Problem – Analytical Techniques* [60] is entirely about point vortices. In § 1.1.4, Newton refers to the question of whether "point vortices faithfully track the centers of vorticity of smoothed out vorticity distributions (such as vortex patches) for sufficiently short times". However, when deriving the PVE, one finds "since each point vortex moves with the local velocity of the fluid", i.e. back to the old argument.

Majda and Bertozzi's 2002 book *Vorticity and Incompressible Flow* [61] is mathematical in flavor. The derivation of the PVE in § 7.3 is rather physical however: "Ignoring the fact that the velocity of a point vortex is infinite at its center, intuitively as in the exact radial eddies described in Example 2.1, Chap. 2, we find that a point vortex induces no motion at its center. [...] It is worth noting here that such dynamic equations as those of Eqs. (7.90) have a self-consistent derivation through formal asymptotic expansions as the high Reynolds number limit of suitable solutions of the 2D Navier-Stokes equations. The interested reader can consult the book by Ting and Klein (1991) for a detailed discussion." Later in 7.5.3: "In Chap. 2 of their book, Ting and Klein (1991) present a detailed formal asymptotic derivation of the point-vortex equations of Section 7.3 from solutions of the Navier-Stokes equations. Can this formal work be combined with estimates for the 2D Navier-Stokes equations to rigorously justify this approximation?"

In their 2006 monograph, Wu, Ma and Zhou [62] state "For a free point-vortex system, which experiences no external force and in which each vortex moves under the induction of others", so there is a passing mention of forces, but no explanation of what happens to the self-induction. However, they later write "Equation (8.103) can also be derived by using asymptotic expansion as the high-Reynolds-number limit of the two-dimensional Navier–Stokes equation (see Ting and Klein 1991)", so a reference to the MAE approach is given.

Alekseenko, Kuibin and Okulov in 2007 [63] equate the motion of a point vortex with the fluid velocity at the vortex, and find this velocity as follows; "The fluid velocity at the point coinciding with the vortex position $\mathbf{z}_{0p}$ is determined by the following rule: in sum (2.25) we eliminate the term including $\mathbf{z}_{0p}$ which is responsible for singularity". They also mention the vortex force argument: "The idea of the vortex force can also be applied to the determination of velocity of the straight vortex filament subject to the external force $\mathbf{F}$. Indeed, in the coordinate system moving together with the vortex, the vortex force is $\rho(\mathbf{u}-\mathbf{u}_V)\times\boldsymbol{\Gamma}$, where $\mathbf{u}$ is the flow velocity, $\mathbf{u}_V$ is the vortex velocity. The equilibrium condition requires that $\mathbf{F}+(\mathbf{u}-\mathbf{u}_V)\times\boldsymbol{\Gamma} = \mathbf{0}$. If the vortex filament is oriented along basis vector $\mathbf{k}$, we obtain $\mathbf{u} = \mathbf{u} + \mathbf{k} \times F/\boldsymbol{\Gamma}$".

From this list we see that in the modern era almost all authors continue to use the original Helmholtz argument. The force balance argument is followed up in the BME literature but not in textbooks, except in Saffman and Alekseenko. The Ting approach, which is really an MAE formalization of the original physical argument, is mentioned by Majda and Bertozzi and by Wu.

3. Conservation of momentum

This is the argument outlined in Saffman's 1992 book [54]. It has been used to derive BME [13], but we limit ourselves to PVE. It provides a mathematical formalization of the physical arguments used in the original derivations.[1]

For a general contour C, enclosing only fluid, that moves and deforms with a position-dependent velocity $\mathbf{u}_c$, Newton's second law for the fluid inside C is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \rho\mathbf{u}\,\mathrm{d}S = -\int_C p\mathbf{n}\,\mathrm{d}l - \int_C \rho\mathbf{u}[(\mathbf{u}-\mathbf{u}_c)\cdot\mathbf{n}]\,\mathrm{d}l, \tag{5}$$

where the left-hand side is the rate of change of the momentum inside the contour C and the terms on the right-hand side are respectively the force applied by the outside fluid on the contour and the flux of momentum through C. Assuming that the flow is irrotational and the density constant, one can write the complex potential and velocity as $F = \phi + \mathrm{i}\psi$ and $w = u - \mathrm{i}v$ respectively, and obtain from Bernoulli's equation

$$p = p_0(t) - \tfrac{1}{2}\rho(F_t + \bar{F}_t + w\bar{w}). \tag{6}$$

Then (5) can be written as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_C \rho\bar{w}\,\mathrm{d}S = \dot{M} = -\frac{\mathrm{i}\rho}{2}\int_C \left(F_t + \bar{F}_t\right)\mathrm{d}z \\ \overline{+\frac{\mathrm{i}\rho}{2}\int_C w(w-w_c)\,\mathrm{d}z} - \frac{\mathrm{i}\rho}{2}w_c\int_C \bar{w}\,\mathrm{d}z.\end{aligned} \tag{7}$$

Now shrink the contour down to a circle centered at the vortex position with radius ϵ. The complex potential and velocity can be decomposed as

$$F = \frac{\Gamma_n}{2\pi\mathrm{i}}\log(z-z_n) + \tilde{F}_n(z), \tag{8}$$

$$w = \frac{\Gamma_n}{2\pi\mathrm{i}(z-z_n)} + \tilde{w}_n(z) \tag{9}$$

with $\tilde{F}_n$ and $\tilde{w}_n$ single-valued and analytic on and inside C except at the vortex position z_n. As $\epsilon \to 0$, the velocity of the contour becomes uniform with $w_c = \dot{\bar{z}}_n$. Using these results, the integrals in (7) can be evaluated and we obtain

$$\dot{M} \to \mathrm{i}\rho\Gamma_n(\dot{z}_n - \bar{\tilde{w}}_c). \tag{10}$$

Near the vortex, the flow is purely azimuthal, so the linear momentum goes to zero. More precisely,

$$M = \int_C \rho\bar{w}\,\mathrm{d}S \sim \rho\int_0^{\epsilon}\int_0^{2\pi}\frac{\Gamma_n e^{\mathrm{i}\theta}}{2\pi\mathrm{i}r}r\,\mathrm{d}r\,\mathrm{d}\theta \to 0. \tag{11}$$

Hence $\dot{M} = 0$ (Saffman implicitly uses this result) and we obtain PVE. We have satisfied Newton's Second law in an integral sense for the fluid around the vortex.

[1] Graham [64] carries out a similar procedure in reverse for BME, computing the force on a solid from the form of the complex potential at infinity, using BME to obtain the result. However his argument cannot really be reversed to obtain BME from Newton's Second Law. In particular only one contour is used, which cannot lead to separate equations for each vortex.

This argument does not actually require the flow outside the vortex to be irrotational. The non-singular term retained from the rest of the velocity field is constant, which is always an irrotational flow whatever the nature of the $O(\epsilon)$ terms. Similarly the pressure could be obtained by integrating the leading-order (differential) momentum equation, which would be equivalent to the local form of the irrotational Bernoulli equation. Hence the irrotational form can be viewed as a convenient way to carry out the calculation. The same process applied to angular momentum carries through for PVE.

The same approach also gives BME:

$$\dot{z}_n + (z_n - z_{n,0})\frac{\dot{\Gamma}_n}{\Gamma_n} = \overline{\tilde{w}_n}. \tag{12}$$

It is necessary in this case to integrate through the branch cut associated with the complex logarithm in the potential. There is then an unbalanced torque when angular momentum is considered, and angular momentum is not conserved for BME.

The fact that the BME model cannot conserve at the same time linear and angular momenta in an integral sense around the vortex and branch cut is no surprise. Introducing a point vortex in the flow provides three degrees of freedom for the system: two for vortex position and one for circulation. For BME the regularity condition fixes circulation, while conservation of momentum gives two equation for the components of position. Angular momentum is not in general conserved unless $\dot{\Gamma} = 0$.

The fact that $\dot{\Gamma}$ does not enter PVE does not mean that Γ is constant. This requires a separate argument (for BME $\dot{\Gamma}$ is given by considerations of regularity). For irrotational flow outside the vortex, integrating the vorticity equation around the vortex leads necessarily to the result that Γ is constant. Body forces will not affect BME or PVE provided that they are not as singular as r^{-2} near the vortex.

4. General singularities

One can naturally ask how other singular potentials would evolve, analogously to vortices. The first such attempt goes back to Fridman and Polubarinova in 1928 [65]. Two classes of further singular potential have been investigated in detail: points sinks and sources, and dipoles.

Point sinks or sources correspond to taking Γ imaginary; vortex-sinks or twisters have Γ complex [66–68]. The equation of motion in all cases was just obtained by using complex Γ in PVE, with no justification being advanced for this choice.

Newton [69] considers dipoles. By considering two point vortices that come closer together, he argues that the dipole strength will align itself with the flow, and writes down an *ad hoc* equation governing this alignment process. He writes down PVE for the position of the dipole.

In terms of general approaches to this problem, Fridman and Polubarinova use a different argument to find PSE. They compute what they call the linear and angular momenta of the fluid lying in an annulus $l_1 < r < l_2$ centered around the singularity and moving with a complex velocity expressed as the Laurent series

$$w = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{13}$$

The point vortex has a_{-1} purely complex. They argue that the linear and angular momentum are a_0 and Im $a_{-1}/(l_1^2 + l_2^2)$ respectively. They then ignore the latter term and argue that the point vortex moves with a_0, which is just $\tilde{w}$ as for PVE.

Saffman and Meiron [70] discuss generalizations of point vortices to three-dimensional "vortons" and conclude that the concept doesn't work. Their approach, which they claim works for point vortices, is based on weak solutions to the vorticity equation. Subsequent works [71, 72] argue that this approach relies implicitly on a certain special definition of regularization, essentially a choice of order of integration. Chefranov [73] argues that there is actually no problem for vortex dipoles both in two and three dimensions (there can be no point vortex equivalent in three dimensions because of the solenoidal nature of the vorticity field). His method discards the singularity in the energy and obtains the dynamical equations from the usual Hamiltonian equation. This method should also work for point vortices. It is, however, a formal procedure. Similar equations [74, 75] are produced for a dipoles, quadrupolar vortices and point vortices.

The PSE has been derived recently [76, 77] by writing the vorticity field as a series of delta functions, substituting into the vorticity equation, and equating degrees of singularity. This requires multiplying a delta function by another function that is singular where the argument of the delta function vanishes. This is not defined for standard distributions. This gives the form of the equations for point vortices and for dipoles, but does not really justify the procedure (cf. the comments of [49] for point vortices). For higher singularities, the resulting evolution equations are claimed to be inconsistent.

It is tempting to try the momentum conservation argument of § 3 to obtain PSE. This fails for a number of expected and unexpected reasons. For a twister, write $F = C_n/(2\pi)\log(z - z_n) + \tilde{F}_n(z)$. Then the right-hand side of Newton's Second Law gives

$$-\frac{\rho}{2}C_n\dot{z}_n + \frac{\rho}{2}[-2\overline{\tilde{w}_n} + \dot{z}_n]\overline{C_n}. \tag{14}$$

For real C_n (source or sink), we find $\tilde{w}_n = 0$, which is not an evolution equation.

For a dipole, with

$$w = \frac{D_n}{2\pi}\frac{1}{(z - z_n)^2} + \tilde{w}_n + \tilde{w}_n'(z - z_n) + \cdots, \tag{15}$$

the same approach gives

$$0 = -\frac{\rho}{2}\dot{D}_n - \rho\overline{D_n\tilde{w}_n'}. \tag{16}$$

This is an equation for the dipole strength, not for the position of the dipole. The factor of 2 is inconsistent with the known equation for the evolution of the vorticity gradient (4) [51, 76]. This is because the surface integral M has been interpreted in a certain sense by carrying out the azimuthal integral first to give 0, but it is not a regular integral and this interpretation leads to the wrong answer.

5. Generalized momentum argument

We can deal with these problems by using a generalized argument. We no longer write down Newton's Second Law in integral form; instead we multiply the Euler equation written in terms of $\mathbf{u}-\mathbf{u}_c$ by a test function T and carry out the same procedure. In vectorial form, this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \rho T(\mathbf{u}-\mathbf{u}_c)\,\mathrm{d}S = \int_S \left(\rho\frac{\mathrm{D}T}{\mathrm{D}t}(\mathbf{u}-\mathbf{u}_c) - \rho T\dot{\mathbf{u}}_c + p\boldsymbol{\nabla}T\right)\mathrm{d}S$$
$$-\int_C Tp\mathbf{n}\,\mathrm{d}l - \int_C \rho T(\mathbf{u}-\mathbf{u}_c)[(\mathbf{u}-\mathbf{u}_c)\cdot\mathbf{n}]\,\mathrm{d}l, \qquad (17)$$

In complex form and using Bernoulli's equation where appropriate, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \rho T(\bar{w}-\bar{w}_c)\,\mathrm{d}S = \dot{G} = -\frac{\mathrm{i}\rho}{2}\int_C T(F_t+\bar{F}_t)\,\mathrm{d}z$$
$$+\int_S\left(\rho\frac{\mathrm{D}T}{\mathrm{D}t}(\bar{w}-\bar{w}_c) - \rho T\dot{\bar{w}}_c + 2pT_{\bar{z}}\right)\mathrm{d}S$$
$$+\overline{\frac{\mathrm{i}\rho}{2}\int_C T(w-w_c)^2\,\mathrm{d}z} - \frac{\mathrm{i}\rho}{2}\int_C T(\bar{w}w_c + w\bar{w}_c - |w_c|^2)\,\mathrm{d}z. \quad (18)$$

For the monopole, we do not need T. We find the usual PVE whatever the phase of C_n. Integrating the vorticity equation around the singularity gives $\operatorname{Im} C_n = 0$, which hence is a consequence of the underlying dynamics. It is worth noting that the sink/source strength is not constrained by the dynamics and can be specified arbitrary; this freedom does not seem to have been exploited previously.

For the dipole, we take $T = 1$ and $T = (z-z_n)$. For $T = (z-z_n)$, the surface integrals do not contribute and we again recover $\dot{z}_n = \overline{\tilde{w}_n}$. For $T = 1$, the $pT_{\bar{z}}$ and $\bar{w}_c$ terms are small. The critical terms are the left-hand side of (18) and the $\mathrm{D}T/\mathrm{D}t$ term. This former is singular and we know that ignoring it leads to inconsistent results. However, the singular part of the integral is dynamically irrelevant. As discussed in [54], momentum is not defined in an infinite region since the integral is only conditionally convergent. The hydrodynamic impulse of a fluid is however well-defined and plays the part of momentum for unbounded fluid. We are faced here with a similar problem. We use the result 3.11.31 of [54]: $\int_S \mathbf{u}\,\mathrm{d}S = \frac{1}{2}\mathbf{D}$. This holds for a large circle, and the same result holds here, since the singularity in our small circle is the same. Hence the left-hand side of (18) becomes $\frac{1}{2}\rho\dot{D}_n$ in complex notation. The $\mathrm{D}T/\mathrm{D}t$ term in the integral is singular and must be ignored (again it vanishes if the azimuthal integral is carried out first). We obtain the expected equation for D_n, without the factor of 2 present in (16). The resulting PSE for the dipole is

$$\dot{z}_n = \overline{\tilde{w}_n}, \qquad \dot{\overline{D_n}} + D_n\tilde{w}_n' = 0. \qquad (19)$$

We define general singularities by the local behavior

$$w = \frac{A_n}{2\pi}(z-z_n)^{-l-1} + \tilde{w}_n(z) + \tilde{w'}_n(z)(z-z_n) + \cdots. \qquad (20)$$

Evaluating all moments for higher singularities will lead to an inconsistent set of equations [76]. If, as for BME, we take the view that these equations are nevertheless useful since they satisfy a subset of the moment integrals, we can proceed as follows. Taking $T = (z-z_n)^l$ leads to the usual result for $\dot{z}_n$. To find $\dot{A}_n$, we take $T = (z-z_n)^{l-1}$. The same issue as for the dipole arises, and we deal with the integrals in the same fashion. The only difference is a factor of l^{-1} in F. We find

$$\dot{z}_n = \overline{\tilde{w}_n}, \qquad \dot{\overline{A_n}} + \frac{2l}{l+1}A_n\tilde{w}_n' = 0. \qquad (21)$$

We see that the singularity strength evolves in time according to a very similar equation for all singularities. The irrotational approach used above for PVE works even when the point vortex is embedded in a rotational flow. For higher singularities, this is no longer likely to be true because it is in terms like $\tilde{w}'$ that the effects of background vorticity appear. This procedure gives a well-defined pair of equations. For hybrid singularities, i.e. ones in which there is more than one singular term in the potential, this approach will lead to PSE for the dominant singularity, but will not give evolution equations for the weaker ones.

The obtained PSE is different from the equations previous found: the singularity strengths of [65] do not evolve in time, the equation for the singularity strength of [69] is different, and in [76] it is claimed that only the dipole system is consistent. We do not expect to be able to satisfy all moments: we use 4 moments to obtain 2 complex equations.

6. Example

As a short example, we calculate the motion of a dipole with strength $D = D_r + \mathrm{i}D_i$ and position $z = x + \mathrm{i}y$ in the upper half-plane. We place an image dipole with strength $\bar{D}$ and position $\bar{z}$ to satisfy the no-normal flow condition along the x-axis. It can be shown that D_r is constant in time, while the other unknowns obey the system

$$\dot{x} = -\frac{D_r}{8\pi y^2}, \quad \dot{y} = -\frac{D_i}{8\pi y^2}, \quad \dot{D}_i = -\frac{D_r^2 + D_i^2}{8\pi y^3}. \qquad (22)$$

If $D_r = 0$, the dipole moves vertically, either away from the wall if $D_i(0) < 0$ or toward the wall if $D_i(0) > 0$ (the sign of D_i may look backward but the image dipole has opposite D_i and the physical dipole is moving in its field). The dipole's position is given by

$$y = \sqrt{y_0^2 - \frac{D_i(0)t}{4\pi y(0)}}, \qquad (23)$$

so the dipole reaches the wall at time $t = 4\pi y(0)^3/D_i(0)$ with infinite velocity.

If $D_r \neq 0$, the trajectory of the dipole is given by

$$y = \frac{|D_r|y_0}{\sqrt{D_r^2 + D_i(0)^2}}\cosh\left\{\cosh^{-1}\frac{\sqrt{D_r^2 + D_i(0)^2}}{|D_r|} + \frac{\sqrt{D_r^2 + D_i(0)^2}}{D_r y_0}(x - x_0)\right\}. \qquad (24)$$

For large times, the dipole moves away from the wall with decreasing velocity.

7. Conclusion and future work

We have shown how to derive the PVE, BME and PSE using generalized momentum arguments. The Euler equation is satisfied pointwise everywhere outside the singularity, and moments of it are satisfied in an integral sense around a contour arbitrarily close to the moving singularity. The singularity moves with the flow, but its strength evolves for dipoles and higher singularities. The evolution equation for the strength requires certain choices in regularizing singular integrals. For the dipole we are guided by previous results. It is disappointing that two different regularizations are needed, and the general PSE result should possibly be viewed with some suspicion. It does not satisfy all moments of the Euler equation (this is also true for BME). To a certain extent, the utility of such singularities as dynamical entities lies in how well and how simply they describe interesting physical phenomena. Mathematically they provide a new class of dynamical systems that may be of some intrinsic interest. The physical underpinning for the treatment of the singular integrals in PSE would benefit from further explanation.

A historical overview of the PVE shows that the earliest workers knew that line vortices with circular cross-section supported neutral modes. Hence parallel line vortices that were sufficiently far from each other could be treated as dynamical objects, neglecting their internal core structure. The later complex variable formalism removed the singularity, but did not address the internal structure of the vortices. The conservation of momentum argument that appears with BME provides a justification for treating higher singularities.

The matched asymptotic expansion approach [52] can be viewed as a mathematical reformulation of the original argument. However it does not appear to work for higher singularities.

Additional physical effects have been added to point vortices, including the influence of viscosity [78, 79] and mass, using "mass vortices" (with infinite density) [80, 81]. Any effect that can be described simply as an extra term in the incompressible Euler equation falls into the current framework. Any body force that is not singular does not modify PSE. Hence *ad hoc* approaches such as the beta-plane point vortices [82] (with no associated vorticity field) are inconsistent with momentum conservation.

The effect of compressibility is particularly interesting. Point vortices in a compressible flow have an obvious problem: close to the center of the vortex, the velocity becomes supersonic. Barsony-Nagy *et al.* constructed steady point-vortex like solutions with hollow internal structure for small Mach number using the Imai–Lamla version of the Rayleigh–Janzen expansion [83]. A number of considerations lead to a standard problem in complex variable theory, one of these being that the force on the vortex (obtained by the appropriate generalization of Blasius theorem) vanish. This leads to the obvious equation of the corresponding generalization to the unsteady case. It is not clear that the internal structure that is used is appropriate and more work is required in this interesting area.

Partial support from NSF grant CTS-01133978 is acknowledged, as well as a UCSD Senate Travel Grant. The author is grateful to N. Rott and to the late H. K. Cheng for conversations on their contribution to the original development of the BM equation, and to A. Borisov for providing a copy of Fridman and Polubarinova. Conversations with Sébastien Michelin, Darren Crowdy and David Hill were extremely helpful.

Appendix A. Extract from Kirchhoff's Lesson Twenty

Kirchhoff's remarkable 1876 work *Lectures on Mathematical Physics* [21] does not appear ever to have been translated into English. We hence provide our translation of the relevant section, using the original notation (italics in the original) and formatting.

> [Vortex filaments. Straight and parallel vortex filaments. Motion of several such threads of infinitely small cross-section. Straight filaments that fill a cylinder of elliptical cross-section. Circular vortex filaments with a common axis. Motion of a vortex ring and of two vortex rings of infinitely small cross-section.]
>
> § *1*.. [. . .]
>
> § *2*.. [. . .]
>
> § *3*.. We now want to apply the results of the previous paragraphs to the case of a *single* filament or a number of vortex filaments of infinitely small cross-section. We assume next that only *one* filament exists and set
>
> $$\int \zeta \,\mathrm{d}f = m; \tag{A.1}$$
>
> hence we take m to be finite; ζ must hence be infinitely large. We do not set ζ to be finite in what follows, but ζ must not change its sign; the center of gravity of the vortex filament then always lies inside or infinitely close to its cross-section. For all points that lie at a finite distance from the vortex filament, the equations, according to (K11)[2], are
>
> $$u = \frac{\mathrm{d}W}{\mathrm{d}y}, \quad v = -\frac{\mathrm{d}W}{\mathrm{d}x}, \qquad W = -\frac{1}{\pi} m \log\rho, \tag{A.2}$$
>
> where the origin of ρ is any point of the cross-section of the filament. Infinitely close and inside the filament, W, u, v are in general infinitely large and their values depend on its cross-section and the values that ζ takes for the individual particles; according to the results of the end of § 2,

[2]Derived in § 2 of the Lesson.

we have for the center of gravity of the vortex filament $u = v = 0$. To this extent we can say that the vortex filament stays in place, although in general its cross-section changes and its center of gravity occupies different locations in the fluid at different times; each fluid element at a finite distance from the filament describes a circle with uniform velocity

$$\frac{m}{\pi}\rho. \tag{A.3}$$

Now let there be other such vortex filaments, as previously we had considered only one; let m_1, m_2, ... be the values of the integrals given by m in (A.1) for these filaments; let x_1, y_1, x_2, y_2, ... be the coordinates of their centers of gravity at time t, and let ρ_1, ρ_2, ... be the distances of the centers from the point (x, y); then for all the points that lie at a finite distance from the vortex filaments,

$$u = \frac{\mathrm{d}W}{\mathrm{d}y}, \quad v = -\frac{\mathrm{d}W}{\mathrm{d}x},$$
$$W = -\frac{1}{\pi}\sum m_1 \log\rho_1, \tag{A.4}$$

where the sum is to be carried out over the index. The centers of gravity of the vortices move; the parts of the velocity u and v at the center of a vortex from *that* vortex vanish however; it is hence assumed, when we refer to u_1 and v_1 at the center of the filament with index 1, that two vortices are always at a finite distance from each other,

$$u_1 = \frac{\mathrm{d}W_1}{\mathrm{d}y_1}, \quad v = -\frac{\mathrm{d}W_1}{\mathrm{d}x_1},$$
$$W_1 = -\frac{1}{\pi}\sum(m_2 \log\rho_{12} + m_3 \log\rho_{13} + \cdots), \tag{A.5}$$

where ρ_{12}, ρ_{22}, ... are the distances of the center of gravity of filament 1 to the centers of filaments 2, 3, ... The equations which can be formed in this fashion can be written

$$m_1\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}P}{\mathrm{d}y_1}, \quad m_2\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\mathrm{d}P}{\mathrm{d}y_2},$$
$$m_1\frac{\mathrm{d}y_1}{\mathrm{d}t} = -\frac{\mathrm{d}P}{\mathrm{d}x_1}, \quad m_2\frac{\mathrm{d}y_2}{\mathrm{d}t} = -\frac{\mathrm{d}P}{\mathrm{d}x_2},$$
$$P = -\frac{1}{\pi}\sum m_1 m_2 \log\rho_{12} \tag{A.6}$$

where the sum is to be taken over all combinations of two different indices.

[...]

Date	Author	Submitted
1952	Legendre	6/23/1952
6/1953	Adams	3/23/1953
2/1954	Edwards	11/10/1953
?	Cheng Tech. Rep.	?
3/1954	Cheng JAS forum	12/1/1953
10/1954	BM JAS	1/1954 (presented)
5/5/1955	BM Tech. Rep.	
4/1955	Cheng JAS	6/11/1954
5/1956	Rott	1/1/1956

Table B.1: Timeline of the development of the BM equation. Publication and submission or presentation dates are from the articles (JAS: Journal of the Aeronautical Sciences, later Journal of the Aerospace Sciences).

Appendix B. The history of the Brown–Michael equation

The 1950s saw active research on the lift acting on delta wings. A remarkable series of papers deriving BME appeared in rapid succession, initially considering two-dimensional sections going down the delta wing, and studying two-dimensional dynamics in each section. The chronology is presented in Table B.1. There is also a review of vortex sheet roll-up from delta wings by Legendre dating from 1966 [84] and a mention in a report on EUROMECH meeting 471 by Riley from 1974 [85].

After the work of Legendre in 1952 and Adams in 1953 [86, 87], Edwards in 1954 [88] was the first to derive what is essentially BME from vorticity considerations for the delta-wing case. This approach should work in the general two-dimensional case. He uses circulation theorems taking into account the vorticity being fed into the vortex by the cut. Cheng in the 1954 JAS forum piece [89] "Remarks on Nonlinear Lift and Vortex Separation" is the first to consider time dependence. The Cheng Technical Report cannot be found now.

Brown and Michael's 1954 JAS article "Effect of Leading-Edge Separation on the Lift of a Delta Wing" [90] is in fact a precursor to their widely-cited technical report [91]. This considers the steady problem for the delta wing problem which introduces some extra terms that go away in two dimensions.

Cheng's 1955 JAS paper "Aerodynamics of a Rectangular Plate with Vortex Separation in Supersonic Flow" [92] attacks the steady delta-wing problem. Extensive discussion of BM equation, including the following: "Since the exact boundary condition requires continuity of the pressure across the free vortex sheet, the equivalent condition in the simplified model shall then be the vanishing of the total force on the vortex system, which is in reality the fulfillment of the exact boundary condition by the *mean value*. [...] In order to render the vortex system *dynamically free*, this force shall be balanced by the one acting on the vortex core at $r = \epsilon$, which is essentially a "Joukowski Force." [...]."

In 1956, Rott, while talking about vortex sheet shedding, discusses the 'single vortex' approximation [93]. He attributes force balance argument to the Brown and Michael technical report and to Edwards (1954). He quotes Cheng (albeit with the wrong year: 1955 rather than 1954) as saying this equation can be applied to any flows with vortex generation, even without similarity. This is really the origin of the BME.

After Rott's paper, the equation is used and called the BME, both for the delta-wing and two-dimensional situations. Typical uses are to model steady vortex sheets, in which the vortex sheets are represented by point vortices, but a BM vortex to model the end of the vortex sheet [94, 95]. Jones (2003) should, and might, be doing this as well. A modified approach is suggested by Howe [96] but has not been adopted elsewhere.

References

[1] H. Helmholtz, Uber Integrale der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen, Crelles J. 55 (1858) 25–55.

[2] P. G. Tait, Translation of 'On integrals of the hydrodynamical equations which express vortex motion' by H. Helmholtz, Phil. Mag. 33 (1867) 485–512.

[3] H. Aref, Point vortex dynamics: A classical mathematics playground, J. Math. Phys. 48 (2007) 065401.

[4] H. Aref, The development of chaotic advection, Phys. Fluids 14 (2002) 1315–1325.

[5] B. N. Shashikanth, J. E. Marsden, J. W. Burdick, S. D. Kelly, The Hamiltonian structure of a two-dimensional rigid circular cylinder interacting dynamically with *N* point vortices, Phys. Fluids 14 (2002) 1214.

[6] A. V. Borisov, I. S. Mamaev, S. M. Ramodanov, Dynamic interaction of point vortices and a two-dimensional cylinder, J. Math. Phys. 48 (2007) 065403.

[7] J. Vankerschaver, E. Kanso, J. E. Marsden, The geometry and dynamics of interacting rigid bodies and point vortices, J. Geom. Mech. 2 (2009) 223–266.

[8] L. Cortelezzi, Nonlinear feedback control of the wake past a plate with a suction point on the downstream wall, J. Fluid Mech. 327 (1996) 303–324.

[9] B. Protas, Vortex dynamics models in flow control problems, Nonlinearity 21 (2008) R203–R250.

[10] L. Cortelezzi, A. Leonard, Point vortex model of the unsteady separated flow past a semi-infinite plate with transverse motion, Fluid Dyn. Res. 11 (1993) 263–295.

[11] R. K. Shukla, J. D. Eldredge, An inviscid model for vortex shedding from a deforming body, Theor. Camp. Fluid Dyn. 21 (2007) 343–368.

[12] S. Michelin, S. G. Llewellyn Smith, B. J. Glover, Vortex shedding model of a flapping flag, J. Fluid Mech. 617 (2008) 1–10.

[13] S. Michelin, S. G. Llewellyn Smith, An unsteady point vortex method for coupled fluid-solid problems, Theor. Comp. Fluid Dyn. 23 (2009) 127–153.

[14] S. Michelin, S. G. Llewellyn Smith, Resonance and propulsion performance of a heaving flexible wing, Phys. Fluids 21 (2009) 071902.

[15] S. Michelin, S. G. Llewellyn Smith, Falling cards and flapping flags: understanding fluid-solid interactions using an unsteady point vortex model, Theor. Comp. Fluid Dyn. 24 (2010) 195–200.

[16] N. G. Hogg, H. M. Stommel, The heton, an elementary interaction between discrete baroclinic geostrophic vortices, and its implications concerning eddy heat-flow, Proc. Roy. Soc. Lond. *A* 397 (1985) 1–20.

[17] G. M. Reznik, Dynamics of singular vortices on a beta-plane, J. Fluid Mech. 240 (1992) 405–432.

[18] L. M. Pismen, Vortices in Nonlinear Fields: From Liquid Crystals to Superfluids, from Non-Equilibrium Patterns to Cosmic Strings, Clarendon, Oxford, 1999.

[19] J. P. Hirth, J. Lothe, Theory of dislocations, McGraw-Hill, New York, 1968.

[20] V. V. Meleshko, H. Aref, A bibliography of vortex dynamics 1858–1956, Adv. Appl. Mech. 41 (2007) 197–292.

[21] G. Kirchhoff, Vorlesungen Über Mathematische Physik, Teubner, Leipzig, 1876.

[22] E. J. Routh, Some applications of conjugate functions, Proc. Lond. Math. Soc. 12 (1881) 73–89.

[23] J. J. Thomson, A Treatise on the Motion of Vortex Rings, Macmillan, London, 1883.

[24] M. V. Melander, N. J. Zabusky, J. C. McWilliams, Symmetric vortex merger in two dimensions: causes and conditions, J. Fluid Mech. 195 (1988) 303–340.

[25] A. B. Basset, A Treatise on Hydrodynamics, Deighton, Bell and Company, London, 1888.

[26] H. Poincaré, Théorie Des Tourbillons, Carré, Paris, 1893.

[27] N. E. Zhukovskii, Collected papers. Vol II: Hydrodynamics, Gos. izd-vo tekhniko-teoret. lit-ry, Moscow Leningrad, 1949.

[28] Lord Kelvin, Mathematical and Physical Papers, Cambridge University Press, Cambridge, 1910.

[29] W. Thomson, P. G. Tait, Principles of Mechanics and Dynamics, Cambridge University Press, 1912.

[30] H. Villat, Leçons Sur La Théorie Des Tourbillons, Gauthier-Villars, Paris, 1930.

[31] H. Lamb, Hydrodynamics, Cambridge University Press, Cambridge, 6 edition, 1932.

[32] P. P. Ewald, T. Pöschl, L. Prandtl, The physics of solids and fluids, with recent developments, Blackie, London, 2 edition, 1936.

[33] H. Rouse, Fluid Mechanics for Hydraulic Engineers, McGraw-Hill, New York, 1938.

[34] A. Sommerfeld, Lectures on Theoretical Physics, Academic, New York, 1950.

[35] L. M. Milne-Thomson, Theoretical Hydrodynamics,, Macmillan, London, 5 edition, 1968.

[36] C. Truesdell, The Kinematics of Vorticity, Indiana University Press, Bloomington, 1954.

[37] G. C. McVittie, Review of C. Truesdell, The kinematics of vorticity, Bull. Amer. Math. Soc. 61 (1955) 357–358.

[38] R. P. Feynman, R. B. Leighton, M. Sands, The Feynman lectures on physics, Addison-Wesley, Reading, 1970.

[39] N. E. Kochin, I. A. Kibel', N. V. Roze, Theoretical Hydromechanics, Interscience Publishers, New York, 1964.

[40] L. I. Sedov, Two-Dimensional Problems in Hydrodynamics and Aerodynamics, Interscience Publishers, New York, 1965.

[41] L. I. Sedov, A Course in Continuum Mechanics, Wolters-Noordhoff, Groningen, 1971.

[42] L. D. Landau, E. M. Lifshitz, Fluid mechanics, Pergamon Press, London, 1959.

[43] K. Friedrichs, Special Topics in Fluid Dynamics, Gordon and Breach, New York, 1966.

[44] G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, Cambridge, 1967.

[45] D. J. Tritton, Physical Fluid Dynamics, Van Nostrand Reinhold, Wokingham, 1977.

[46] V. V. Kozlov, General Theory of Vortices, Udmersk University Press, Izhevsk, 1998. In Russian. Available at http://ics.org.ru/doc?book=6&dir=r.

[47] A. V. Borisov, I. S. Mamaev, Mathematical methods in the dynamics of vortex structures, Institute of Computer Science, Moscow – Izhevsk, 2005. In Russian. Available at http://ics.org.ru/doc?book=16&dir=r.

[48] A. V. Borisov, V. V. Kozlov, I. S. Mamaev, M. A. Sokolovisky (Eds.), Proceedings of the IUTAM Symposium on Hamiltonian Dynamics, Vortex Structures,Turbulence (Moscow, 25–30 August, 2006), Springer, Dordrecht, 2008.

[49] C. Marchioro, M. Pulvirenti, Vortex Methods in Two-Dimensional Fluid Dynamics, Springer, Berlin, 1984.

[50] M. J. Lighthill, An Informal Introduction to Theoretical Fluid Mechanics, Clarendon Press, Oxford, 1986.

[51] R. Krasny, A vortex-dipole sheet model for a wake, Phys. Fluids *A* 1 (1989) 173–175.

[52] L. Ting, R. Klein, Viscous Vortical Flows, Springer, Berlin, 1991.

[53] L. Ting, R. Klein, O. M. Knio, Vortex Dominated Flows: Analysis and Computation for Multiple Scale Phenomena, Springer, Berlin, 2007.

[54] P. G. Saffman, Vortex Dynamics, Cambridge University Press, Cambridge, 1992.

[55] V. V. Meleshko, M. Y. Constantinov, Dynamics of vortical structures, Naukova dumka, Kiev, 1993. In Russian.

[56] A. J. Chorin, J. E. Marsden, A Mathematical Introduction to Fluid Mechanics, Springer, New York, 3 edition, 1993.

[57] A. J. Chorin, Vorticity and Turbulence, Springer, New York, 1994.

[58] T. E. Faber, Fluid Dynamics for Physicists, Cambridge University Press,

Cambridge, 1995.
[59] J. Katz, A. Plotkin, Low-Speed Aerodynamics, Cambridge University Press, Cambridge, 2 edition, 2001.
[60] P. K. Newton, The N-Vortex Problem: Analytical Techniques, Springer, New York, 2001.
[61] A. Majda, A. L. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002.
[62] J.-Z. Wu, H.-Y. Ma, M.-D. Zhou, Vorticity and Vortex Dynamics, Springer, Berlin, 2006.
[63] S. V. Alekseenko, P. A. Kuibin, V. L. Okulov, Theory of Concentrated Vortices: An Introduction, Springer, Berlin, 2007.
[64] J. M. R. Graham, The forces on sharp-edged cylinders in oscillatory flow at low Keulegan–Carpenter numbers, J. Fluid Mech. 97 (1980) 331–346.
[65] A. A. Fridman, P. Y. Polubarinova, On moving singularities of a flat motion of an incompressible fluid, Geofiz. Sbornik (1928) 9–23.
[66] V. A. Bogomolov, Motion of in ideal fluid of constant density in the presence of sinks, Izv. Akad. Nauk SSSR Meh. Zidk. Gaza 4 (1976) 21–27.
[67] A. E. Novikov, E. A. Novikov, Vortex-sink dynamics, Phys. Rev. E 54 (1996) 3681–3686.
[68] A. V. Borisov, I. S. Mamaev, On the problem of motion of vortex sources on a plane, Regular & Chaotic Dyn. 11 (2006) 455–466.
[69] P. K. Newton, The dipole dynamical system, Discrete Cont. Dyn. Sys. Suppl. (2005) 1–8.
[70] P. G. Saffman, D. I. Meiron, Difficulties with 3-dimensional weak solutions for inviscid incompressible-flow, Phys. Fluids 29 (1986) 2373–2375.
[71] G. Winckelmans, A. Leonard, Weak solutions of the 3-dimensional vorticity equation with vortex singularities, Phys Fluids 31 (1988) 1838–1839.
[72] C. Greengard, E. Thomann, Singular vortex systems and weak solutions of the euler equations, Phys. Fluids 31 (1988) 2810–2813.
[73] S. G. Chefranov, Problem of regularization of the proper energy of point vortex dipoles and the intensification of integral vorticity on extension of the vortex lines, Zh. Eksp. Teor. Fiz. 95 (1989) 547–561.
[74] S. G. Chefranov, Dynamics of point vortex dipoles and spontaneous singularities in 3-dimensional turbulent flows, Zh. Eksp. Teor. Fiz. 93 (1987) 151–158.
[75] S. G. Chefranov, Dynamics of point vortex quadrupoles and elliptic vortices on a plane, Zh. Eksp. Teor. Fiz. 99 (1991) 1149–1165.
[76] V. V. Yanovsky, A. V. Tur, K. N. Kulik, Singularities motion equations in 2-dimensional ideal hydrodynamics of incompressible fluid, Phys. Lett. A 373 (2009) 2484–2487.
[77] K. N. Kulik, A. V. Tur, V. V. Yanovsky, Interaction of point and dipole vortices in an incompressible liquid, Theor. Math. Phys. 162 (2010) 383–400.
[78] O. Agullo, A. Verga, Effect of viscosity on the dynamics of two point vortices: Exact results, Phys. Rev. E 63 (2001) 056304.
[79] R. Nagem, G. Sandri, D. Uminsky, C. E. Wayne, Generalized Helmholtz–Kirchhoff model for two-dimensional distributed vortex motion, SIAM J. Appl. Dyn. Sys. 8 (2009) 160–179.
[80] C. Grotta Ragazzo, J. Koiller, W. M. Oliva, On the motion of two-dimensional vortices with mass, J. Nonlinear Sci. 4 (1994) 375–418.
[81] S. M. Ramodanov, On the motion of two mass vortices in a perfect fluid, in: A. V. Borisov, V. V. Kozlov, I. S. Mamaev, M. A. Sokolovisky (Eds.), Proceedings of the IUTAM Symposium on Hamiltonian Dynamics, Vortex Structures,Turbulence (Moscow, 25–30 August, 2006), Springer, Dordrecht, 2008, pp. 459–468.
[82] N. J. Zabusky, J. C. McWilliams, A modulated point-vortex model for geostrophic beta-plane dynamics, Phys. Fluids 25 (1982) 2175–2182.
[83] A. Barsony-Nagy, J. Er-El, S. Yungster, Compressible flow past a contour and stationary vortices, J. Fluid Mech. 178 (1987) 367–378.
[84] R. Legendre, Vortex sheets rolling-up along leading-edges of delta wings, Progr. Aero. Sci. 7 (1966) 7–33.
[85] N. Riley, Flows with concentrated vorticity: a report on EUROMECH 41, J. Fluid Mech. 62 (1974) 33–39.
[86] R. Legendre, écoulement au voisinage de la pointe avant d'une aile à forte flèche aux incidences moyennes, La Recherche Aéronautique (1952) 3–8.
[87] M. C. Adams, Leading-edge separation from delta-wings at supersonic speeds, J. Aero. Sci. (1953) 430–430.
[88] R. H. Edwards, Leading-edge separation from delta-wings, J. Aero. Sci. 21 (1954) 134–135.
[89] H. K. Cheng, Remarks on nonlinear lift and vortex separation, J. Aero. Sci. 21 (1954) 212–214.
[90] C. E. Brown, W. H. Michael, Effect of leading-edge separation on the lift of a delta-wing, J. Aero. Sci. 21 (1954) 690–694.
[91] C. E. Brown, W. H. Michael, On slender delta wings with leading-edge separation, Technical Report NACA-TN-3430, NACA, 1955.
[92] H. K. Cheng, Aerodynamics of a rectangular plate with vortex separation in supersonic flow, J. Aero. Sci. 22 (1955) 217–226.
[93] N. Rott, Diffraction of a weak shock with vortex generation, J. Fluid Mech 1 (1956) 111–128.
[94] J. H. B. Smith, Improved calculations of leading-edge separation from slender thin delta wings, Proc. R. Soc. Lond. *A* 306 (1968) 67–90.
[95] D. I. Pullin, Large-scale structure of unsteady self-similar rolled-up vortex sheets, J. Fluid Mech. 88 (1978) 401–430.
[96] M. S. Howe, Emendation of the Brown & Michael equation, with application to sound generation by vortex motion near a half-plane, J. Fluid Mech. 329 (1996) 89–101.