PHYSICS OF FLUIDS VOLUME 13, NUMBER 10 OCTOBER 2001

Three-dimensional acoustic scattering by vortical flows. I. General theory

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(Received 19 September 2000; accepted 5 June 2001)

When an acoustic wave is incident on a three-dimensional vortical structure, with length scale small compared with the acoustic wavelength, what is the scattered sound field that results? A frequently used approach is to solve a forced wave equation for the acoustic pressure, with nonlinear terms on the right-hand side approximated by the bilinear product of the incident wave and the undisturbed vortex: we refer to this as the ''acoustic analogy'' approximation. In this paper, we show using matched asymptotic expansions that the acoustic analogy approximation always predicts the leading-order scattered sound field correctly, provided the Mach number of the vortex is small, and the acoustic wavelength is a factor of order M^{-1} larger than the scale of the vortex. The leading-order scattered field depends only on the vortex dipole moment. Our analysis is valid for acoustic frequencies of the same order or smaller than the vorticity of the vortex. Over long times, the vortex may become significantly disturbed by the incident acoustic wave. Additional conditions are derived to maintain validity of the acoustic analogy approximation over times of order M^{-1}, long enough for motion of the vortex to be significant on the length scale of the acoustic waves.
 [DOI: 10.1063/1.1401814]

I. INTRODUCTION

The problem of acoustic scattering by localized vortical flows has a long history,[1–9] and continues to attract attention in a variety of fields, including acoustics, fluid dynamics,[10–14] astrophysics,[15] and superfluids.[16,17] The development of experimental techniques to measure vortex structures by acoustic scattering has been responsible for recent interest in the problem.[18–20]

The ''Born'' limit, where the wavelength of the incoming sound wave is much longer than the size of the vortex, is amenable to analytic progress. To a first approximation, the vortex is shaken backward and forward by the longitudinal sound wave, which is uniform on the scale of the vortex; at the next level of approximation, the vortex responds to the large-scale compression and straining motion induced by the wave. Consequently, sound is radiated and the incoming wave may be viewed as being scattered by the vortex.

Acoustic scattering by a radially symmetric vortex in two dimensions has been a major component of previous work. In most studies, the problem is formulated as a forced wave equation. Following Lighthill's pioneering work,[21] the fluid equations are rewritten with the acoustic wave operator acting on the density perturbation on the left-hand side, and all the remaining (nonlinear) terms, which take the form of a quadrupole source, on the right. The quadrupole source is expressed mathematically as $\partial^2 T_{ij}/\partial x_i \partial x_j$, where the x_i are the independent Cartesian coordinates, and T_{ij} is a tensor. For small-Mach-number flows, $T_{ij} \approx \rho_0 u_i u_j$, where ρ_0 is the mean density, and the u_i are velocity components. The scattered wave field is then computed on the assumption that T_{ij} may be approximated by the bilinear product of the velocities of the incoming plane wave (u_i^w, say) and the known vortical flow (u_i^v, say), so that $T_{ij} \approx \rho_0(u_i^v u_j^w + u_i^w u_j^v)$. We shall call this approach the acoustic analogy approximation, since the problem is analogous to one in which an acoustic wave operator is provided with a known source. This terminology was used by Lighthill to refer to acoustic wave radiation by vortical flows, in which the relative weakness of the radiation implies that the vortical flow can be regarded as known, to leading order, and contains precisely the information required to evaluate the nonlinear source term.

The acoustic analogy approximation has been used to obtain results for scattering by point vortices[22] and distributed vortices[8] in two dimensions, and results obtained using this approximation agree well with numerical aeroacoustic calculations.[11] It has been claimed[23] that it leads to a singularity in the scattered field in the forward scattering direction, but, in fact, this singularity can be removed by considering a region of parabolic shape about the forward axis.[9,10,14]

A more serious criticism of the acoustic analogy approximation is that no reason can be given *a priori* to explain why this approximation to the nonlinear terms T_{ij} should give the correct scattered sound field. The difficulty lies in the fact that the vortex and the sound wave interact, so the time-dependent velocity field in the vortex differs significantly from the superposition of the velocity field of the basic vortex and the velocity field of the incident sound

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wave. Moreover, this difference occurs at precisely the order of calculation required to determine the scattered sound field.

Reference 14 (hereafter FLS) developed a rational asymptotic expansion procedure that clearly showed the nature of the scattering in the farfield. Two regions are required in this analysis: an inner, vortical region, and an outer, wave region. A solution is developed in the vortical region, in response to forcing by the incident acoustic wave. The analysis shows how this response in the vortical region forces a scattered wave in the wave region, and hence a complete leading-order solution for scattering in the Born limit is obtained. It is shown in FLS that the leading-order scattered wave field is, in fact, correctly predicted by the forced wave equation approach, at least for steady two-dimensional vortices with circulation. Moreover, the scattered field depends only on the circulation, and parameters of the incident acoustic wave, but contains no information about the detailed structure of the vortex.

The acoustic analogy approximation has also been employed in three dimensions.[18,24–27] In this case there is clearly no singularity in the scattered field, but the question of the validity of the acoustic analogy approximation remains. In this paper we ask the following question: under what circumstances is the scattered field predicted correctly when the acoustic analogy approximation is used to evaluate T_{ij}.

We answer this question by proceeding along the lines of FLS, deriving the leading-order scattered field far from the vortex via a rational asymptotic expansion procedure. The basic equations and expansion procedure are outlined in Sec. II. The full equations of motion are solved in Secs. III and IV in the inner and outer regions, respectively. The acoustic analogy approximation to the scattered sound field can be derived directly from the analysis presented, and it is shown that, in fact, the acoustic analogy approximation does not neglect any terms that contribute to the leading-order scattered sound. This is discussed in Sec. V, and conclusions are presented in Sec. VI.

In Part II of this paper,[28] we calculate sound scattering by Hill's spherical vortex in the axisymmetric case, where the incident sound wave is parallel to the axis of symmetry and direction of propagation of the vortex. The inner and outer problems are solved in closed form, and compared to the results derived in this paper.

II. STATEMENT OF PROBLEM

For simplicity, we consider the problem of the scattering of acoustic waves by a vortex in a homentropic ideal gas. The momentum and continuity equations and the equation of state are

$$\rho_a\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p_a, \tag{1}$$

$$\frac{\partial \rho_a}{\partial t}+\nabla\cdot(\rho_a\boldsymbol{u})=0, \tag{2}$$

$$\frac{p_a}{p_0}=\left(\frac{\rho_a}{\rho_0}\right)^{\gamma}, \tag{3}$$

where here all the symbols take their usual meanings, and p_a and ρ_a denote the absolute pressure and density, respectively, as distinct from perturbation pressure and density to be defined below. We shall assume that the vortex has spatial scale L. We also define the vorticity $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$.

We assume that the vorticity decays rapidly with distance, so that $|\boldsymbol{\omega}|=O[(r/L)^{-\infty}]$ as $r\to\infty$, meaning that $|\boldsymbol{\omega}|$ decays faster than any inverse power of (r/L), where r is a measure of distance from the center of the vortex. We shall take the velocity induced by the vorticity to be of magnitude U. We then define the Mach number $M\equiv U/c_0$, where $c_0\equiv(\gamma p_0/\rho_0)^{1/2}$ is the speed of sound. Throughout this paper we take the Mach number to be small.

We shall refer to the region of scale L centered on the vortex as the vortical region, since it is only in this region that the vorticity is significantly different from zero. If the velocity is scaled on U, and the length scaled on L, which is the nondimensional scaling appropriate to the vortical region, the governing equations are

$$(1+M^2\rho)\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p, \tag{4}$$

$$M^2\left(\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})\right)+\nabla\cdot\boldsymbol{u}=0, \tag{5}$$

$$1+\gamma M^2 p=(1+M^2\rho)^{\gamma}, \tag{6}$$

where the relations,

$$p_a=1+\gamma M^2 p, \quad \rho_a=1+M^2\rho, \tag{7}$$

imply that pressure and density depart from their uniform background values by $O(M^2)$, consistent with near-incompressible motion.[29]

From (4) and (5) we can form the "Lighthill equation,"

$$-M^2\frac{\partial^2\rho}{\partial t^2}+\frac{\partial^2 T_{ij}}{\partial x_i\partial x_j}=-\nabla^2 p, \tag{8}$$

where the Lighthill stress tensor is given by

$$T_{ij}=(1+M^2\rho)u_iu_j. \tag{9}$$

In the two-dimensional problem solved in FLS, the undisturbed vortex (i.e., in the absence of the acoustic waves) was taken to be steady, and so in that problem the vortical region is fixed in space. In three dimensions, however, there are very few examples of localized vortices that are fixed in space. The classic example of Hill's spherical vortex translates at a constant velocity, and we wish to include this example within our analysis. We also wish to allow the vortex to move through a significant number of wavelengths of the incident acoustic wave, so our analysis must remain valid for times $O(M^{-1})$. We shall therefore employ, in the vortical region, a spatial coordinate $\boldsymbol{\xi}$, defined such that

$$\boldsymbol{\xi}=\boldsymbol{x}-\boldsymbol{x}_c(t), \tag{10}$$

where $\boldsymbol{x}_c$ is the center of the vortical region. The definition of the center of the vortical region is an arbitrary one and, as we shall see, the exact definition that we use does not affect the results that we obtain. However, we shall require that in our definition, the center of the vortical region $\boldsymbol{x}_c$ move with the vortex, so that the vorticity remains confined to a region in which $|\boldsymbol{\xi}|$ is of order unity. For example, we may use the definition of vortex centroid given by[30]

$$\boldsymbol{x}_v = \frac{1}{2I^2}\int \boldsymbol{x}'[\boldsymbol{x}'\times\boldsymbol{\omega}(\boldsymbol{x}')\cdot\boldsymbol{I}]\, d^3\boldsymbol{x}', \tag{11}$$

where

$$\boldsymbol{I} \equiv \frac{1}{2}\int \boldsymbol{x}'\times\boldsymbol{\omega}(\boldsymbol{x}')\, d^3\boldsymbol{x}', \tag{12}$$

and $I = |\boldsymbol{I}|$.

In coordinates $\boldsymbol{\xi}$ and t, Eqs. (4)–(6) for the flow in the vortical region are

$$(1+M^2\rho)\left(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p, \tag{13}$$

$$M^2\left(\frac{\partial \rho}{\partial t} - \boldsymbol{v}\cdot\nabla\rho + \nabla\cdot(\rho\boldsymbol{u})\right) + \nabla\cdot\boldsymbol{u} = 0, \tag{14}$$

$$1+\gamma M^2 p = (1+M^2\rho)^\gamma, \tag{15}$$

where now ∇ is a gradient with respect to $\boldsymbol{\xi}$, and

$$\boldsymbol{v} = \frac{d\boldsymbol{x}_c}{dt}. \tag{16}$$

Note that in (13)–(15) the coordinates are $(\boldsymbol{\xi},t)$, but the velocity $\boldsymbol{u}$ is the velocity relative to an observer at a fixed location in $\boldsymbol{x}$.

Equations (13)–(15) are a complete set of equations for a homentropic ideal gas, but in the following analysis it proves convenient to use the vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times[(\boldsymbol{u}-\boldsymbol{v})\times\boldsymbol{\omega}], \tag{17}$$

and the Lighthill equation (8), which, in the vortical region, is written as

$$-M^2\left(\frac{\partial}{\partial t} - \boldsymbol{v}\cdot\nabla\right)^2\rho + \frac{\partial^2 T_{ij}}{\partial\xi_i\partial\xi_j} = -\nabla^2 p. \tag{18}$$

We shall use (14), (17), (18) in developing the asymptotic expansion of the flow in the vortical region.

Acoustic waves are incident upon the flow in the vortical region. We shall take the frequency ω of these waves to be of the same order of magnitude, U/L, as the magnitude of the vorticity in the vortical region. Our analysis is hence valid for unsteady vortices, but also applies to steady vortices. The ratio of length scales between the inner and outer region is $L/\lambda = M\omega/2\pi$. Then the assumption of small Mach number of the flow in the vortical region implies that the acoustic waves must have wavelength $O(LM^{-1})$ (Refs. 14, 29), and so the vortical region takes the role of an inner region, surrounded by an outer, wave region, of length scale LM^{-1}.

In the wave region the appropriate spatial variable is $\boldsymbol{X} \equiv M\boldsymbol{x}$. Equations (4)–(6) are then rewritten, using the wave-region spatial variable $\boldsymbol{X}$. The result is

$$(1+M^2H)\left(\frac{\partial \boldsymbol{U}}{\partial t} + M^2\boldsymbol{U}\cdot\nabla\boldsymbol{U}\right) = -\nabla P, \tag{19}$$

$$\frac{\partial H}{\partial t} + \nabla\cdot\boldsymbol{U} + M^2\nabla\cdot(\boldsymbol{U}H) = 0, \tag{20}$$

$$1+\gamma M^2 P = (1+M^2 H)^\gamma. \tag{21}$$

The gradient operator acting on a wave-region quantity corresponds to differentiation with respect to $\boldsymbol{X}$. Nondimensional fields in the wave region are represented by capital letters, except for the density ρ, which is denoted there by H. The velocity field has been scaled by a factor of M in (19), (20).

Now, the vorticity in the wave region is assumed to be smaller than any power of M, and we may hence use a velocity potential Φ in the wave region such that $\boldsymbol{U} = \nabla\Phi$. Using (21), the momentum equation (19) may then be integrated once to yield the wave-region Bernoulli equation,

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}M^2\nabla\Phi\cdot\nabla\Phi + \frac{1}{\gamma-1}M^{-2}[(1+M^2H)^{(\gamma-1)}-1] = 0. \tag{22}$$

This equation and (20) together constitute a complete set of equations for flow in the wave region.

Throughout we shall assume that the amplitude of the incident acoustic wave is sufficiently small that quantities quadratic in the amplitude of the incident wave can be neglected. In the present nondimensional variables, we shall therefore take the pressure $P^{(i)}$ associated with the incident wave to be given by

$$P^{(i)} = \delta e^{i(kX-\omega t)}, \tag{23}$$

with the wave propagating in the positive X direction, and Eqs. (21) and (22) imply

$$\omega = k, \tag{24}$$

with both ω and k positive without loss of generality.

Equations (19)–(21) apply everywhere outside the vortical region, whose location we denote by $\boldsymbol{X} = \boldsymbol{X}_c(t)$. Here we make no assumption on the magnitude of $\boldsymbol{X}_c$, so our analysis is valid for times $O(M^{-1})$, over which the vortex may move through several wavelengths of the incident acoustic wave. However, the vortex does move with a velocity that is characteristic of the velocity in the vortical region, U, which is a factor M smaller than the sound speed, and so

$$M\boldsymbol{V} \equiv \frac{d\boldsymbol{X}_c}{dt} = O(M). \tag{25}$$

Again, we shall expand the expression for $\boldsymbol{V}$, but the expression for $\boldsymbol{X}_c$ is not expanded. Note that it is not necessary for $d\boldsymbol{X}_c/dt$ to be equal to $M\,d\boldsymbol{x}_c/dt$, and, in fact, $d\boldsymbol{X}_c/dt$ will differ from $M\,d\boldsymbol{x}_c/dt$ at $O(M^2\delta)$. There is no technical difficulty in having a difference between $\boldsymbol{X}_c$ and $M\boldsymbol{x}_c$, provided this difference is accounted for when deriving matching conditions between the two regions. Note also that the flow in the wave region is expressed in terms of the spatial variable $\boldsymbol{X}$: no transformation to a moving frame is employed in the wave region.

We now develop the solutions in the inner vortical region and the surrounding wave region, on the assumption that both $M \ll 1$ and $\delta \ll 1$. The solution is thus represented as a double asymptotic series in M and δ (although with a special treatment of the vorticity, as discussed below).

III. THE SOLUTION IN THE VORTICAL REGION

The solution in the vortical region is now expressed as an asymptotic expansion in M and δ:

$$\boldsymbol{u}=\boldsymbol{u}_0+M^2\boldsymbol{u}_2+\cdots+\delta\boldsymbol{u}_{01}+M\,\delta\boldsymbol{u}_{11}+M^2\,\delta\boldsymbol{u}_{21}+M^3\,\delta\boldsymbol{u}_{31}+\cdots, \tag{26}$$

$$p=p_0+M^2p_2+\cdots+\delta p_{01}+M\,\delta p_{11}+M^2\,\delta p_{21}+M^3\,\delta p_{31}+\cdots, \tag{27}$$

$$\rho=\rho_0+M^2\rho_2+\cdots+\delta\rho_{01}+M\,\delta\rho_{11}+M^2\,\delta\rho_{21}+M^3\,\delta\rho_{31}+\cdots. \tag{28}$$

Powers of M^2 are required initially in the expansion of the basic flow, as could be predicted from the form of Eqs. (14)–(15). The terms linear in δ increase in single powers of M due to matching conditions with the incident acoustic wave.

The vorticity itself must not be expanded, however, because the corresponding evolution equations for the higher-order contributions to the vorticity are likely to have solutions that grow exponentially in time,[31,32] and so the expansion will become disordered at times $O[\ln(1/M)]$. Instead, the vorticity is taken to satisfy the induction equation (17), with the induction velocity $\boldsymbol{u}$ and the velocity of the centroid $\boldsymbol{v}$ computed to whatever order is required. The question of the existence of solutions of this induction equation is an unsolved problem. The truncation at leading order is equivalent to the three-dimensional incompressible Euler equations, for which finite-time singularity is conjectured, for some initial conditions,[33] but not proved. Our approach here is a practical one. We do not necessarily require the solution to exist for long times. If it does not, then the analysis presented here is valid for times $O(1)$. On the other hand, if the solution to the vortical flow does exist for long times (e.g., a steady solution, such as Hill's vortex), then our analysis is capable of capturing this solution, and perturbations to it, over times of at least $O(M^{-1})$.

In order to maintain long-time validity of the solution, the location $\boldsymbol{x}_c$ of the centroid of the vortex is not expanded, but its velocity is, so that

$$\boldsymbol{v}=\boldsymbol{v}_0+\delta\boldsymbol{v}_{01}+\cdots. \tag{29}$$

Thus, the evolution of $\boldsymbol{x}_c$ is treated similarly to the evolution of $\boldsymbol{\omega}$, with the expression for $\boldsymbol{x}_c$ not expanded, but the center of the vortex $\boldsymbol{x}_c$ traveling with the velocity $\boldsymbol{v}$ expanded to whatever order is required. We shall choose $\boldsymbol{v}_{01}$ and subsequent δ-dependent terms as is convenient, while the higher-order terms independent of δ are not needed.

Because we do not expand $\boldsymbol{\omega}$, all velocities except $\boldsymbol{u}_0$ must be irrotational, and so only their divergence is required to determine the velocity at each successive order. As we shall see, a perturbation expansion procedure can be established using (14), (15), and (18) to determine the pressure, density, and divergence at successive orders. This procedure differs slightly from that followed by FLS, in which the vorticity is also expanded. The expansion of vorticity is possible in that paper because the basic flow is a steady state that does not support neutral modes at the incident frequency. In this paper, however, we consider general, unsteady vorticity distributions, and so the nonexpansion of vorticity is crucial to retaining well-ordered expansions over a long time.

A. The undisturbed vortical flow

At leading order [i.e., $O(1)$], the continuity equation (14) is

$$\nabla\cdot\boldsymbol{u}_0=0. \tag{30}$$

Thus, given the vorticity $\boldsymbol{\omega}$, we may obtain the leading-order velocity $\boldsymbol{u}_0$ in terms of the vorticity using the Biot–Savart integral,

$$\boldsymbol{u}_0=-\frac{1}{4\pi}\int \boldsymbol{\omega}(\boldsymbol{\xi}')\times\nabla_\xi|\boldsymbol{\xi}-\boldsymbol{\xi}'|^{-1}d^3\boldsymbol{\xi}'. \tag{31}$$

This simply represents the relationship between velocity and vorticity in the three-dimensional incompressible Euler equations. We have assumed that the vorticity is $O(r^{-\infty})$ as $r\to\infty$, where henceforth $r\equiv|\boldsymbol{\xi}|$ denotes distance from the center of the vortical region, and so the integral in (31) converges.

In order to determine matching conditions between the vortical region and the wave region, the asymptotic behavior of flows in the vortical region must be determined in the limit $r\to\infty$.

From (31), we have

$$\boldsymbol{u}_0=-\frac{1}{4\pi}\int \boldsymbol{\omega}(\boldsymbol{\xi}')\times\nabla_\xi\big(r^{-1}-\boldsymbol{\xi}'\cdot\nabla_\xi r^{-1}+\tfrac{1}{2}\boldsymbol{\xi}'\boldsymbol{\xi}':\nabla_\xi\nabla_\xi r^{-1}\big)d^3\boldsymbol{\xi}'+O(r^{-5}). \tag{32}$$

The first term in the expansion (32) vanishes because, by the divergence theorem and the fact that $\nabla\cdot\boldsymbol{\omega}=0$, we have

$$0=\int d^3\boldsymbol{\xi}\frac{\partial}{\partial\xi_i}[\xi_j\omega_i(\boldsymbol{\xi})]=\int d^3\boldsymbol{\xi}\,\omega_j(\boldsymbol{\xi}). \tag{33}$$

The next two terms in (32) can be expressed as gradients of a scalar potential, because the vorticity vanishes as $r\to\infty$. The first of these two is discussed in standard texts,[30] but the second is often not treated. The analysis is presented in Appendix A. The result is that

$$\boldsymbol{u}_0=\frac{1}{4\pi}\nabla(\boldsymbol{I}\cdot\nabla r^{-1}+\boldsymbol{J}:\nabla\nabla r^{-1})+O(r^{-5}),\ \text{as}\ \ r\to\infty, \tag{34}$$

where

$$\boldsymbol{J}=-\frac{1}{3}\int \boldsymbol{\xi}'[\boldsymbol{\xi}'\times\boldsymbol{\omega}(\boldsymbol{\xi}')]d^3\boldsymbol{x}'. \tag{35}$$

Note that (34) implies

$$\boldsymbol{u}_0=O(r^{-3}),\quad \text{as}\ \ r\to\infty; \tag{36}$$

this result is crucial to establishing the convergence of certain integrals that arise as the analysis proceeds.

To determine the leading-order pressure p_0, we use the Lighthill equation (18), which gives

$$\frac{\partial^2}{\partial\xi_i\partial\xi_j}(u_{0i}u_{0j}) = -\nabla^2 p_0. \tag{37}$$

Here, u_{0i} are the components of $\boldsymbol{u}_0$, which is determined from the vorticity by (31). Thus, (37) shows that p_0 can be determined from $\boldsymbol{\omega}$, and, using the free-space Green's function for the Laplacian, the solution of (37) for p_0 can be written as

$$p_0 = \frac{1}{4\pi}\int d^3\boldsymbol{\xi}' |\boldsymbol{\xi} - \boldsymbol{\xi}'|^{-1} \frac{\partial^2}{\partial\xi_i'\partial\xi_j'}[u_{0i}(\boldsymbol{\xi}')u_{0j}(\boldsymbol{\xi}')]. \tag{38}$$

To determine p_0 in the limit $r\to\infty$, it is convenient to rewrite (38) as

$$p_0 = \frac{1}{4\pi}\int d^3\boldsymbol{\xi}' u_{0i}(\boldsymbol{\xi}')u_{0j}(\boldsymbol{\xi}') \frac{\partial^2}{\partial\xi_i\partial\xi_j}|\boldsymbol{\xi} - \boldsymbol{\xi}'|^{-1}. \tag{39}$$

This is valid since $u_{0i}u_{0j} = O(r^{-6})$ as $r\to\infty$. In the limit $r\to\infty$, this gives

$$p_0 = \frac{\partial^2}{\partial\xi_i\partial\xi_j}\left(\frac{1}{r}\right)\frac{1}{4\pi}\int d^3\boldsymbol{\xi}' u_{0i}(\boldsymbol{\xi}')u_{0j}(\boldsymbol{\xi}') + O(r^{-4}). \tag{40}$$

Thus,

$$p_0 = O(r^{-3}), \quad \text{as } r\to\infty. \tag{41}$$

We now proceed to calculate the velocity and pressure at $O(M^2)$. Expanding (14) at $O(M^2)$, we have

$$\frac{\partial\rho_0}{\partial t} - \boldsymbol{v}_0\cdot\nabla\rho_0 + \nabla\cdot(\rho_0\boldsymbol{u}_0) + \nabla\cdot\boldsymbol{u}_2 = 0, \tag{42}$$

and the equation of state gives

$$p_0 = \rho_0. \tag{43}$$

Equation (42) is an equation for $\nabla\cdot\boldsymbol{u}_2$. Moreover, $\boldsymbol{u}_2$ is irrotational, and so, writing $\boldsymbol{u}_2 = \nabla\phi_2$, Eq. (42) is a Poisson equation for ϕ_2, provided $\boldsymbol{u}_0$, ρ_0, and $\partial\rho_0/\partial t$ are known.

Now, p_0 is computed from (38), in which $\boldsymbol{u}_0$ in the integrand is obtained from the vorticity via (31). Therefore, $\boldsymbol{u}_0$ and ρ_0 are known. To evaluate $\partial\rho_0/\partial t$ we use the fact that $\rho_0 = p_0$, and therefore we evaluate $\partial\rho_0/\partial t$ by taking the time derivative of the integral representation (38). To do this, we must evaluate $\partial\boldsymbol{u}_0/\partial t$, which is done taking the time derivative of the expression (31). This integral depends only upon the vorticity, and so by (17) its time derivative may be evaluated in principle simply by replacing $\partial\boldsymbol{\omega}/\partial t$ by $\nabla\times[(\boldsymbol{u}-\boldsymbol{v})\times\boldsymbol{\omega}]$.

This general structure is applied to all evaluations of time-derivative terms. All the fields are regarded as functionals of the vorticity, with different functionals $\mathcal{F}_{(\cdot)}$ for the different fields, and we introduce the notation $\{\boldsymbol{u}_0, p_0, \ldots\} = \{\mathcal{F}_{\boldsymbol{u}_0}(\boldsymbol{\omega}), \mathcal{F}_{p_0}(\boldsymbol{\omega}), \ldots\}$ to represent this. The rate of change of any of these fields is thus ultimately determined by the evolution of $\boldsymbol{\omega}$, or more precisely by $\partial\boldsymbol{\omega}/\partial t$, through

$$\frac{\partial\mathcal{F}}{\partial t} = \frac{\delta\mathcal{F}}{\delta\boldsymbol{\omega}}\cdot\frac{\partial\boldsymbol{\omega}}{\partial t}, \tag{44}$$

where $\delta\mathcal{F}/\delta\boldsymbol{\omega}$ is a functional derivative. It follows that, although a quantity may exist only at a single order in the asymptotic expansion, its time derivatives will exist at that order *and all higher orders*. Thus, if a functional $\mathcal{F}$ is itself expanded, in powers of M and δ, so that

$$\mathcal{F} = \mathcal{F}_0 + \delta\mathcal{F}_{01} + \cdots, \tag{45}$$

then the time derivative of that field is calculated according to

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial t} &= \frac{\partial\mathcal{F}_0}{\partial t} + \delta\frac{\partial\mathcal{F}_{01}}{\partial t} + \cdots \\ &= \frac{\delta\mathcal{F}_0}{\delta\boldsymbol{\omega}}\nabla\times[(\boldsymbol{u}_0 + \delta\boldsymbol{u}_{01} - \boldsymbol{v}_0 - \delta\boldsymbol{v}_{01} + \cdots)\times\boldsymbol{\omega}] \\ &\quad + \delta\frac{\delta\mathcal{F}_{01}}{\delta\boldsymbol{\omega}}\nabla\times[(\boldsymbol{u}_0 - \boldsymbol{v}_0 + \cdots)\times\boldsymbol{\omega}] + \cdots \\ &= \frac{\delta\mathcal{F}_0}{\delta\boldsymbol{\omega}}\nabla\times[(\boldsymbol{u}_0 - \boldsymbol{v}_0)\times\boldsymbol{\omega}] \\ &\quad + \delta\left\{\frac{\delta\mathcal{F}_{01}}{\delta\boldsymbol{\omega}}\nabla\times[(\boldsymbol{u}_0 - \boldsymbol{v}_0)\times\boldsymbol{\omega}]\right. \\ &\quad \left. + \frac{\delta\mathcal{F}_0}{\delta\boldsymbol{\omega}}\nabla\times[(\boldsymbol{u}_{01} - \boldsymbol{v}_{01})\times\boldsymbol{\omega}]\right\} + \cdots.\end{aligned} \tag{46}$$

We therefore introduce the notation

$$\left.\frac{\partial\mathcal{F}}{\partial t}\right|_{O(M^p\delta^q)} \tag{47}$$

to mean the component of the time derivative of the quantity $\mathcal{F}$ at the order $M^p\delta^q$, where, in this paper, p and q will be non-negative integers. This means that the time derivative $\partial\rho_0/\partial t$ in (42) must be replaced by $\partial\rho/\partial t|_{O(1)} = \partial\rho_0/\partial t|_{O(1)}$. Indeed, not only is this consistent with the asymptotic expansion procedure, but it is, in fact, necessary in order for the expansion procedure to proceed. The quantity $\partial\rho_0/\partial t$ can never be evaluated using an asymptotic expansion procedure of the type developed here, since the full velocity field, to all orders, is never known in any asymptotic expansion procedure; however, $\partial\rho_0/\partial t|_{O(1)}$ can be evaluated at this stage in the procedure because $\boldsymbol{u}_0$ is known from (31). To supplement the notation (47), we shall also use shorthand notation for the leading-order time derivative of the quantity F, namely

$$\frac{\partial_0 F}{\partial t} \equiv \left.\frac{\partial F}{\partial t}\right|_{O(1)}. \tag{48}$$

Using (43), and the notation just introduced, Eq. (42) is written as

$$\nabla^2\phi_2 = -\left.\frac{\partial p_0}{\partial t}\right|_{O(1)} + \boldsymbol{v}_0\cdot\nabla p_0 - \nabla\cdot(p_0\boldsymbol{u}_0). \tag{49}$$

A solution to this equation could be given by using the free-space Green's function for the Laplacian, as in (38).

However, the farfield behavior of the first term on the right-hand side of (49) cannot be determined in this way, because p_0 does not decay sufficiently rapidly in r as $r\to\infty$. Instead (cf. Ref. 29), we invert the Laplacian on the kernel in the integral representation (38) for p_0, and hence

$$\phi_2=-\frac{1}{8\pi}\int d^3\boldsymbol{\xi}'|\boldsymbol{\xi}-\boldsymbol{\xi}'|\frac{\partial^2}{\partial\xi_i'\partial\xi_j'}\left(\frac{\partial_0}{\partial t}-\boldsymbol{v}_0\cdot\nabla_{\xi'}\right)$$
$$\times[u_{0i}(\boldsymbol{\xi}')u_{0j}(\boldsymbol{\xi}')]+\frac{1}{4\pi}\int d^3\boldsymbol{\xi}'$$
$$\times|\boldsymbol{\xi}-\boldsymbol{\xi}'|^{-1}\nabla_{\xi'}\cdot[p_0\boldsymbol{u}_0(\boldsymbol{\xi}')]. \tag{50}$$

Now, we have that $u_0=O(r^{-3})$ as $r\to\infty$. Then, from (13) we have

$$\left(\frac{\partial_0}{\partial t}-\boldsymbol{v}_0\cdot\nabla\right)\boldsymbol{u}_0\sim-\nabla p_0,\quad \text{as}\quad r\to\infty, \tag{51}$$

and so

$$\left(\frac{\partial_0}{\partial t}-\boldsymbol{v}_0\cdot\nabla\right)\boldsymbol{u}_0=O(r^{-4}),\quad \text{as}\quad r\to\infty. \tag{52}$$

It follows that

$$\left(\frac{\partial_0}{\partial t}-\boldsymbol{v}_0\cdot\nabla\right)(u_{0i}u_{0j})=O(r^{-7}),\quad \text{as}\quad r\to\infty. \tag{53}$$

Hence, an expression for ϕ_2 in the limit $r\to\infty$ can be obtained, namely,

$$\phi_2=-\frac{1}{8\pi}\left(\frac{\partial^2}{\partial\xi_i\partial\xi_j}r\right)\int d^3\boldsymbol{\xi}'\left(\frac{\partial_0}{\partial t}-\boldsymbol{v}_0\cdot\nabla_{\xi'}\right)$$
$$\times[u_{0i}(\boldsymbol{\xi}')u_{0j}(\boldsymbol{\xi}')]+O(r^{-2}), \tag{54}$$

and, since $\boldsymbol{u}_2=\nabla\phi_2$, (54) implies

$$\boldsymbol{u}_2=O(r^{-2}),\quad \text{as}\quad r\to\infty. \tag{55}$$

B. The flow in the vortical region to $O(M\delta)$

Because of the long wavelength of the incident acoustic wave, it affects the flow in the vortical region via matching conditions in the limit $r\to\infty$. The matching process carries through in a straightforward manner, while the unexpanded vorticity and the use of different frames in near- and far-fields is not so simple, so the matching will be presented in an informal manner for convenience. At $O(\delta)$, the incident acoustic wave imposes matching conditions:

$$p_{01}\to e^{ikX_c}e^{-i\omega t};\ \rho_{01}\to e^{ikX_c}e^{-i\omega t};\ \boldsymbol{u}_{01}\to\mathbf{0};\quad \text{as}\quad r\to\infty, \tag{56}$$

since $X-X_c=M\xi$, where ξ is the x component of $\boldsymbol{\xi}$.

The solution for the flow fields in the vortical region is then simply

$$p_{01}=\rho_{01}=e^{ikX_c}e^{-i\omega t};\quad \boldsymbol{u}_{01}=\mathbf{0}. \tag{57}$$

Thus, at $O(\delta)$, the pressure and density experience time-dependent but spatially independent oscillations, and there is no flow in the vortical region at this order. Note that p_{01} and ρ_{01} are time dependent in two ways: the $e^{-i\omega t}$ factor due to the incident acoustic wave, and also the factor $e^{ikX_c(t)}$, which arises due to the motion of the vortex through the fluid. Recall that this second time dependence is slow, in the sense that $dX_c/dt=O(M)$. Since $\boldsymbol{u}_{01}=\mathbf{0}$, we also take $\boldsymbol{v}_{01}=\mathbf{0}$.

At $O(M\delta)$, Eq. (14) gives

$$\nabla\cdot\boldsymbol{u}_{11}=0. \tag{58}$$

The conditions on the velocity and pressure, in the limit $r\to\infty$, are

$$\boldsymbol{u}_{11}\to\mathbf{i}e^{ikX_c}e^{-i\omega t};\quad p_{11}\to ik\xi e^{ikX_c}e^{-i\omega t}, \tag{59}$$

where $\mathbf{i}$ is the unit vector in the direction of propagation of the incident acoustic wave.

The flow is irrotational at all orders beyond $O(1)$, and so the solution for the velocity consistent with (58) and (59) is

$$\boldsymbol{u}_{11}=\mathbf{i}e^{ikX_c}e^{-i\omega t}. \tag{60}$$

This is the same as in FLS, but in that paper, there was also a rotational component of the flow at this order, which was obtained by solving a Rayleigh-type equation. In this paper, that component is incorporated into the leading-order flow, because in this paper the leading-order vorticity is advected by the full velocity field, and the vorticity is not expanded in powers of M and δ. The current analysis can be used in the two-dimensional case too, and exactly the same results as FLS are recovered, thereby showing that the $O(M^2\delta)$ scattering prediction of FLS applies to unsteady flow too, as might be expected since the form of the scattering term depends only of the circulation, which is independent of time.

The Lighthill equation (18) at $O(M\delta)$ gives

$$\nabla^2 p_{11}=0, \tag{61}$$

and the solution to (61) consistent with the matching condition (59) is

$$p_{11}=ik\xi e^{ikX_c}e^{-i\omega t}. \tag{62}$$

Note here that $\boldsymbol{u}_{11}$ is independent of the spatial coordinate $\boldsymbol{\xi}$. We shall therefore take

$$\boldsymbol{v}_{11}=\boldsymbol{u}_{11}. \tag{63}$$

Consequently, we can see from (17) that there is no evolution of vorticity relative to the coordinates of the vortical region at $O(M\delta)$. This greatly simplifies the analysis that follows.

C. The flow in the vortical region at $O(M^2\delta)$

At $O(M^2\delta)$, conditions on the velocity and pressure in the limit $r\to\infty$ are

$$\boldsymbol{u}_{21}\to\mathbf{i}ik\xi e^{ikX_c}e^{-i\omega t};\quad p_{21}\to-\tfrac{1}{2}k^2\xi^2e^{ikX_c}e^{-i\omega t}. \tag{64}$$

The velocity is irrotational, but (14) at $O(M^2\delta)$ shows that the expression,

$$\left.\frac{\partial\rho}{\partial t}\right|_{O(\delta)}=\left.\left(\frac{\partial}{\partial t}(\rho_0+\delta\rho_{01}+\cdots)\right)\right|_{O(\delta)}, \tag{65}$$

needs to be evaluated, following the procedure outlined in (46). The $O(\delta)$ truncation of $\partial\boldsymbol{\omega}/\partial t$ in (17) is zero, because there is no $O(\delta)$ velocity field to contribute to the truncation. The truncation of the ρ_{01} term can be calculated explicitly:

$$\frac{\partial}{\partial t}\delta\rho_{01}\Big|_{O(\delta)}=\frac{\partial_0}{\partial t}e^{ikX_c}e^{-i\omega t}=-i\omega e^{ikX_c}e^{-i\omega t}, \tag{66}$$

because, even though X_c varies in time, $dX_c/dt=O(M)$. Therefore Eq. (14) at $O(M^2\delta)$ is

$$\nabla^2\phi_{21}=i\omega e^{ikX_c}e^{-i\omega t}, \tag{67}$$

and so $\boldsymbol{u}_{21}$ is compressible. Equation (67) has the solution

$$\phi_{21}=\tfrac{1}{2}i\omega\xi^2 e^{ikX_c}e^{-i\omega t}, \tag{68}$$

and

$$\boldsymbol{u}_{21}=\nabla\phi_{21}=\mathbf{i}i\omega\xi e^{ikX_c}e^{-i\omega t} \tag{69}$$

verifies that this solution is consistent with matching conditions (64).

Following the same procedure, the equation for the pressure p_{21} is

$$\omega^2 p_{01}+2\frac{\partial^2}{\partial\xi_i\partial\xi_j}\left(u_{0i}u_{21j}+\frac{1}{2}\rho_{01}u_{0i}u_{0j}\right)=-\nabla^2 p_{21}. \tag{70}$$

Using (57) and (69), Eq. (70) can be written as

$$\nabla^2 p_{21}=-2i\omega\frac{\partial u_0}{\partial\xi}e^{ikX_c}e^{-i\omega t}+\nabla^2 p_0\, e^{ikX_c}e^{-i\omega t} -\omega^2 e^{ikX_c}e^{-i\omega t}, \tag{71}$$

where $u_0=\boldsymbol{u}_0\cdot\mathbf{i}$. From (31), we have

$$u_0=-\frac{1}{4\pi}\mathbf{i}\cdot\int d^3\boldsymbol{\xi}'\,\boldsymbol{\omega}(\boldsymbol{\xi}')\times\nabla_\xi|\boldsymbol{\xi}-\boldsymbol{\xi}'|^{-1}, \tag{72}$$

and hence

$$p_{21}=\frac{i\omega}{4\pi}\mathbf{i}\cdot\int d^3\boldsymbol{\xi}'\,\boldsymbol{\omega}(\boldsymbol{\xi}')\times\nabla_\xi\frac{\partial}{\partial\xi}|\boldsymbol{\xi}-\boldsymbol{\xi}'|e^{ikX_c}e^{-i\omega t} -\tfrac{1}{2}k^2\xi^2 e^{ikX_c}e^{-i\omega t}+e^{ikX_c}e^{-i\omega t}p_0. \tag{73}$$

In the limit $r\to\infty$, this gives

$$p_{21}=-\frac{1}{2}k^2\xi^2 e^{ikX_c}e^{-i\omega t}+\frac{i\omega}{4\pi}\boldsymbol{I}\cdot\boldsymbol{\xi}\left(\frac{1}{r^3}-\frac{3\xi^2}{r^5}\right) \times e^{ikX_c}e^{-i\omega t}+O(r^{-3}). \tag{74}$$

D. The flow in the vortical region at $O(M^3\delta)$

At $O(M^3\delta)$, Eq. (14) gives

$$\nabla^2\phi_{31}=-\nabla\cdot(\rho_{11}\boldsymbol{u}_0+\rho_0\boldsymbol{u}_{11})-\frac{\partial\rho}{\partial t}\bigg|_{O(M\delta)} +\boldsymbol{v}_0\cdot\nabla\rho_{11}+\boldsymbol{v}_{11}\cdot\nabla\rho_0. \tag{75}$$

Note the presence of terms dependent on $\boldsymbol{v}_{11}$ in Eq. (75), where $\boldsymbol{v}_{11}$ is given by (63).

We now have

$$\frac{\partial\rho}{\partial t}\bigg|_{O(M\delta)}=-i\omega\rho_{11}+ikv_0\rho_{01}, \tag{76}$$

since there can be change of ρ by the $O(M\delta)$ velocity. However $\boldsymbol{v}_0\cdot\nabla\rho_{11}=ikv_0\rho_{01}$, and $\nabla\cdot(\rho_0\boldsymbol{u}_{11})=\boldsymbol{v}_{11}\cdot\nabla\rho_0$, so (75) simplifies to

$$\nabla^2\phi_{31}=-\nabla\cdot(\rho_{11}\boldsymbol{u}_0)+i\omega\rho_{11}. \tag{77}$$

Note that, at $O(M\delta)$ and $O(M^2\delta)$, the velocities obtained were just the velocity associated with the incident acoustic wave, in the absence of the vortex. However, Eq. (75) shows that there will be an additional contribution to the velocity at $O(M^3\delta)$ because of the $\boldsymbol{u}_0$ and ρ_0 terms.

To solve (77), we divide ϕ_{31} as follows:

$$\nabla^2\phi_{31}^{(i)}=i\omega\rho_{11} \tag{78}$$

and

$$\nabla^2\phi_{31}^{(r)}=-\nabla\cdot(\rho_{11}\boldsymbol{u}_0). \tag{79}$$

Equation (78) has the solution

$$\phi_{31}^{(i)}=-\tfrac{1}{6}\omega k\xi^3 e^{ikX_c}e^{-i\omega t}, \tag{80}$$

and so $\boldsymbol{u}_{31}^{(i)}\equiv\nabla\phi_{31}^{(i)}$ is the velocity due to the incident wave in the absence of the vortex:

$$\boldsymbol{u}_{31}^{(i)}=-\tfrac{1}{2}\omega k\xi^2\mathbf{i}\,e^{ikX_c}e^{-i\omega t}. \tag{81}$$

This matches the incident wave in the limit $r\to\infty$.

The equation for $\phi_{31}^{(r)}$ can be simplified:

$$\nabla^2\phi_{31}^{(r)}=-iku_0 e^{ikX_c}e^{-i\omega t}. \tag{82}$$

We can solve (82) using (72), giving

$$\phi_{31}^{(r)}=\frac{ik}{8\pi}\mathbf{i}\cdot\int d^3\boldsymbol{\xi}'\,\boldsymbol{\omega}(\boldsymbol{\xi}')\times\nabla_x|\boldsymbol{\xi}-\boldsymbol{\xi}'|e^{ikX_c}e^{-i\omega t}. \tag{83}$$

Hence, $\phi_{31}^{(r)}=O(r^{-1})$, and

$$\boldsymbol{u}_{31}^{(r)}=O(r^{-2}),\quad \text{as } r\to\infty. \tag{84}$$

In fact, the result (84) holds whatever the value of $\boldsymbol{v}_{11}$.

The Lighthill equation (18) at $O(M^3\delta)$ is

$$-\left(\frac{\partial}{\partial t}-\boldsymbol{v}_0\cdot\nabla\right)^2\rho_{11}\bigg|_{O(M\delta)}-\left(\frac{\partial}{\partial t}-\boldsymbol{v}_{11}\cdot\nabla\right)^2\rho_0\bigg|_{O(M\delta)} +2\frac{\partial^2}{\partial\xi_i\partial\xi_j}\left(u_{0i}u_{31j}+u_{2i}u_{11j}+\rho_0u_{0i}u_{11j}+\frac{1}{2}\rho_{11}u_{0i}u_{0j}\right) =-\nabla^2 p_{31}. \tag{85}$$

The first term simplifies to leave $\omega^2\rho_{11}$, while the second one is given by

$$-\left(\frac{\partial}{\partial t}-\boldsymbol{v}_{11}\cdot\nabla\right)^2\rho_0\bigg|_{O(M\delta)}=2\boldsymbol{v}_{11}\cdot\nabla\frac{\partial_0\rho_0}{\partial t}-i\omega\boldsymbol{v}_{11}\cdot\nabla\rho_0. \tag{86}$$

Hence, we may define two contributions to p_{31}, namely $p_{31}^{(i)}$ and $p_{31}^{(r)}m$ such that $p_{31}=p_{31}^{(i)}+p_{31}^{(r)}$, and

$$\omega^2\rho_{11}+2\frac{\partial^2}{\partial\xi_i\partial\xi_j}(u_{0i}u_{31j}^{(i)}+u_{2i}u_{11j}+\rho_0u_{0i}u_{11j}+\tfrac{1}{2}\rho_{11}u_{0i}u_{0j}) =-\nabla^2 p_{31}^{(i)} \tag{87}$$

and

$$2\frac{\partial^2}{\partial\xi_i\partial\xi_j}(u_{0i}u_{31j}^{(r)})+2\boldsymbol{v}_{11}\cdot\nabla\frac{\partial_0\rho_0}{\partial t}-i\omega\boldsymbol{v}_{11}\cdot\nabla\rho_0=-\nabla^2 p_{31}^{(r)}. \tag{88}$$

Recall that $\boldsymbol{u}_{31}^{(i)}$ is the velocity that would be present due to the incident acoustic wave, in the absence of the vortex, and $\boldsymbol{u}_{31}^{(r)}$ is the remainder.

We shall show first that $p_{31}^{(r)}$ does not enter the matching conditions at $O(M^4\delta)$. In the limit $r\to\infty$, we have

$$\boldsymbol{u}_0\boldsymbol{u}_{31}^{(r)}=O(r^{-5}), \tag{89}$$

and so by (88) we have

$$p_{31}^{(r)}=\frac{1}{2\pi}\int d^3\boldsymbol{\xi}'(u_{0i}u_{31j}^{(r)})\frac{\partial^2}{\partial\xi_i\partial\xi_j}|\boldsymbol{\xi}-\boldsymbol{\xi}'|^{-1} -\frac{1}{8\pi}\left(2\boldsymbol{v}_{11}\cdot\nabla\frac{\partial_0}{\partial t}-i\omega\boldsymbol{v}_{11}\cdot\nabla\right) \times\int d^3\boldsymbol{\xi}' u_{0i}(\boldsymbol{\xi}')u_{0j}(\boldsymbol{\xi}')\frac{\partial^2}{\partial\xi_i\partial\xi_j}|\boldsymbol{\xi}-\boldsymbol{\xi}'|e^{ikX_c}e^{-i\omega t}. \tag{90}$$

In the limit $r\to\infty$, (90) implies

$$p_{31}^{(r)}=O(r^{-2}),\quad \text{as } r\to\infty. \tag{91}$$

Thus, $p_{31}^{(r)}$ matches to terms in the wave region that are $O(M^5\delta)$ and smaller.

We turn now to $p_{31}^{(i)}$. First, we observe that

$$2\frac{\partial^2}{\partial\xi_i\partial\xi_j}(u_{2i}u_{11j})=2u_{11}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi_i}u_{2i}=2\frac{\partial}{\partial\xi}\nabla^2\phi_2 e^{ikX_c}e^{-i\omega t} \tag{92}$$

and

$$2\frac{\partial^2}{\partial\xi_i\partial\xi_j}(u_{0i}u_{31j}^{(i)})=-2\omega k\left(u_0+\xi\frac{\partial u_0}{\partial\xi}\right)e^{ikX_c}e^{-i\omega t}. \tag{93}$$

Hence, $p_{31}^{(i)}$ satisfies

$$\nabla^2 p_{31}^{(i)}=2\omega k\left(u_0+\xi\frac{\partial u_0}{\partial\xi}\right)e^{ikX_c}e^{-i\omega t}-2\frac{\partial}{\partial\xi}\nabla^2\phi_2 \times e^{ikX_c}e^{-i\omega t}-2\frac{\partial^2}{\partial x_i\partial x_j}\left(\rho_0u_{0i}u_{11j}+\frac{1}{2}\rho_{11}u_{0i}u_{0j}\right) -k^2\rho_{11}. \tag{94}$$

It follows that

$$p_{31}^{(i)}=-\frac{\omega k}{4\pi}\mathbf{i}\cdot\int d^3\boldsymbol{\xi}'\,\boldsymbol{\omega}(\boldsymbol{\xi}')\times\nabla_\xi \times\left[|\boldsymbol{\xi}-\boldsymbol{\xi}'|+\xi\frac{\partial}{\partial\xi}|\xi-\boldsymbol{\xi}'|-\frac{1}{6}\frac{\partial^2}{\partial\xi^2}|\boldsymbol{\xi}-\boldsymbol{\xi}'|^3\right] \times e^{ikX_c}e^{-i\omega t}+\frac{1}{2\pi}\int d^3\boldsymbol{\xi}'\left(\rho_0u_{0i}u_{11j}+\frac{1}{2}\rho_{11}u_{0i}u_{0j}\right) \times\frac{\partial^2}{\partial x_i\partial x_j}|\boldsymbol{\xi}-\boldsymbol{\xi}'|^{-1}-2u_2e^{ikX_c}e^{-i\omega t} -\tfrac{1}{6}ik^3\xi^3e^{ikX_c}e^{-i\omega t}. \tag{95}$$

In the limit $r\to\infty$, we have

$$\rho_0\boldsymbol{u}_0\boldsymbol{u}_{11}=O(r^{-6}),\quad \rho_{11}\boldsymbol{u}_0\boldsymbol{u}_0=O(r^{-5}). \tag{96}$$

It follows that, in the limit $r\to\infty$, the second line in (95) is $O(r^{-3})$, and the third is $O(r^{-2})$. The leading-order contribution to $p_{31}^{(i)}$ in the limit $|\boldsymbol{\xi}|\to\infty$ therefore comes from the fourth line of (95), which is $O(r^3)$. and matches to the incident wave. The next-order contribution comes from the first line of (95), which is $O(r^{-1})$, and so matches to flow at $O(M^4\delta)$ in the wave region. The result is

$$p_{31}^{(i)}=-\frac{1}{6}ik^3\xi^3e^{ikX_c}e^{-i\omega t}-\frac{\omega k}{8\pi}\left(\frac{I_1}{r}+\frac{\xi^2I_1}{r^3}+\frac{\xi\xi_kI_k}{r^3} -\frac{3\xi^3\xi_kI_k}{r^5}\right)e^{ikX_c}e^{-i\omega t}+O(r^{-2}). \tag{97}$$

It follows that only the superscript-(i) components match the flow in the wave region at $O(M^4\delta)$. Moreover, it can be seen that $\boldsymbol{u}_2$ is not required in order to calculate $p_{31}^{(i)}$. Therefore, in the limit $r\to\infty$, the superscript-(i) fields are exactly what would be calculated by the acoustic analogy method. We now use matched asymptotic analysis to obtain the flow in the wave region.

IV. THE OUTER SOLUTION

The solution in the wave region is expressed as an asymptotic expansion in M and δ. We shall see that the asymptotic expansion takes the form

$$\Phi=M^2\Phi_2+M^3\Phi_3+\cdots+\delta\Phi_{01}+M^4\delta\Phi_{41}+\cdots, \tag{98}$$

$$P=M^3P_3+\cdots+\delta P_{01}+M^4\delta P_{41}+\cdots, \tag{99}$$

$$H=M^3H_3+\cdots+\delta H_{01}+M^4\delta H_{41}+\cdots. \tag{100}$$

Here we are using the fact that, in the wave region, $\boldsymbol{U}=\nabla\Phi$ to all orders required.

We start by considering the flow in the wave region in the absence of the incident acoustic wave. This is the component of the flow in the expansions (98)–(100) that is independent of δ.

The leading-order velocity $\boldsymbol{u}_0$ in the vortical region decays as r^{-3} in the limit $r\to\infty$. This corresponds to a velocity potential $O(M^2)$ in the wave region, since the velocity in the wave region has already been scaled by one factor of M. This explains why the expansion for Φ starts at $O(M^2)$. The leading-order pressure p_0 in the vortical region also decays as r^{-3} as $r\to\infty$, and this corresponds to a pressure $O(M^3)$ in the wave region. This explains why the expansions for P and H start at $O(M^3)$.

With the expansions for $\boldsymbol{U}$, P, and H as given, Eq. (20) at $O(M^2)$ gives

$$\nabla^2\Phi_2=0, \tag{101}$$

and (22) at $O(M^2)$ gives

$$\frac{\partial_0 \Phi_2}{\partial t} = 0. \tag{102}$$

To determine Φ_2, we must use asymptotic matching to the flow in the vortical region. Equation (34) gived the form of $\boldsymbol{u}_0$ in the limit $r \to \infty$. Matching to (34), we can see that, to leading order, (101) implies

$$\Phi_2 = \frac{1}{4\pi} \boldsymbol{I} \cdot \nabla \left(\frac{1}{R} \right), \tag{103}$$

where

$$R = |\boldsymbol{X} - \boldsymbol{X}_c(t)|. \tag{104}$$

Equation (102) is satisfied, since $\boldsymbol{I}$ is conserved by the leading-order incompressible dynamics of the vortical region[30] (see also Appendix B), and so

$$\left. \frac{d\boldsymbol{I}}{dt} \right|_{O(1)} = 0. \tag{105}$$

Time dependence is nevertheless present in Φ_2 due to motion of the center $\boldsymbol{X}_c(t)$, and the fact that $\boldsymbol{I}$ is not conserved to all orders in the perturbation expansion, but these are higher-order effects. Hence, although there is a contribution to the velocity in the wave region at $O(M^2)$, there are no propagating acoustic waves at this order.

At the next order, $O(M^3)$, Eqs. (20) and (22) give

$$\frac{\partial_0 H_3}{\partial t} + \nabla^2 \Phi_3 = 0, \tag{106}$$

$$\frac{\partial_0 \Phi_3}{\partial t} + \left. \frac{\partial \Phi_2}{\partial t} \right|_{O(M)} + H_3 = 0, \tag{107}$$

where

$$\left. \frac{\partial \Phi_2}{\partial t} \right|_{O(M)} = \frac{1}{4\pi} \boldsymbol{I} \cdot \nabla \left[\frac{\partial}{\partial \boldsymbol{X}_c} \left(\frac{1}{R} \right) \cdot \boldsymbol{V}_0 \right] = -\frac{1}{4\pi} \boldsymbol{I} \boldsymbol{V}_0 : \nabla \nabla \left(\frac{1}{R} \right). \tag{108}$$

Taking the time derivatives of (106) and (107) is awkward because some $O(M^3)$ terms in the second derivatives of Φ_3 and H_3 come from applying the general procedure of (46) to lower-order terms in H and Φ. Instead, we combine (20) and (22) to give

$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = -M^2 \nabla \Phi \cdot \frac{\partial \nabla \Phi}{\partial t} + M^2 \nabla \cdot (\boldsymbol{U} H) - \frac{\partial H}{\partial t} [(1 + M^2 H)^{\gamma - 2} - 1]. \tag{109}$$

Truncating this equation at $O(M^3)$, we have

$$\frac{\partial_0^2 \Phi_3}{\partial t^2} - \nabla^2 \Phi_3 = - \left. \frac{\partial^2 \Phi_2}{\partial t^2} \right|_{O(M)} = \frac{1}{4\pi} \left(\frac{d_0}{dt} \boldsymbol{I} \boldsymbol{V}_0 \right) : \nabla \nabla \left(\frac{1}{R} \right). \tag{110}$$

A particular solution to (110) is

$$\Phi_3^{(p)} = \frac{1}{4\pi} \boldsymbol{I} M^{-1} \boldsymbol{X}_c : \nabla \nabla \left(\frac{1}{R} \right). \tag{111}$$

However, this solution is unacceptable because, over long times $O(M^{-1})$, the location of the vortex, $\boldsymbol{X}_c$, will, in general, be $O(1)$. This implies that $\Phi_3^{(p)}$ becomes $O(M^{-1})$ and the asymptotic expansion for Φ would then become disordered for times $O(M^{-1})$. Thus, in common with many asymptotic expansion procedures, a multiple-time-scale expansion is required for $\boldsymbol{x}_c$, so that $\boldsymbol{X}_c$ must be represented as $\boldsymbol{X}_c(t, Mt, \ldots)$. We then apply a nonsecularity condition to the right-hand side of (110). Physically, the nonsecularity condition is simply that any mean motion of the vortex must not be contained in the order-unity time scale. Mathematically, this can be stated as

$$\lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \boldsymbol{X}_c(t', Mt, \ldots) \, dt' = 0, \quad \text{for all times } t. \tag{112}$$

In fact, we can remove the leading-order time dependence from $\boldsymbol{X}_c$ entirely by taking a slowly varying definition of the vortex center. Then, although $\boldsymbol{V}_0$ is taken to be $O(1)$ in general, $d\boldsymbol{V}_0/dt$ will be $O(M)$, and the right-hand side of (110) then vanishes. Thus, Φ_3 satisfies the unforced linear wave equation, and the solution consistent with matching to the vortical region in the limit $R \to 0$ is

$$\Phi_3 = \frac{1}{4\pi} \nabla \nabla : \left(\frac{1}{R} \boldsymbol{J}(t - R) \right). \tag{113}$$

The expression (113) is the familiar "Lighthill" radiation from the vortex, in the absence of any incident wave.

The expansion in the vortical region proceeds in powers of M^2, and so it is possible that, in addition to (113), there may also be a monopole term at $O(M^2)$ in the vortical region, which would also match to a term $O(M^3)$ in the wave region. In fact, it can be seen from (54) that there is no term of the monopole type in ϕ_2 in the limit $r \to \infty$, and so there is no monopole in the acoustic wave field at $O(M^2)$. This result is due to Crow.[29] It is also possible to develop the expansion in the wave region at further powers of M, independent of δ, but we shall see that the development to $O(M^3)$ is sufficient here.

An expression for H_3 can then be obtained from (107). The result is

$$H_3 = -\frac{1}{4\pi} \nabla \nabla : \left(\frac{1}{R} \dot{\boldsymbol{J}}(t - R) \right) - \frac{1}{4\pi} \boldsymbol{I} \boldsymbol{V}_0 : \nabla \nabla \left(\frac{1}{R} \right), \tag{114}$$

where

$$\dot{\boldsymbol{J}} = -\frac{1}{3} \int d^3 \boldsymbol{\xi} \boldsymbol{\xi} \{ \boldsymbol{\xi} \times \nabla \times [(\boldsymbol{u}_0 - \boldsymbol{v}_0) \times \boldsymbol{\omega}] \} \tag{115}$$

is the $O(1)$ time derivative of $\boldsymbol{J}$. The form (114) for density or pressure is a well-known result of Möhring,[34] usually written in terms of a third-order time derivative. A lengthy but straightforward calculation shows that (114) matches to p_0 in the limit $r \to \infty$ given by (40).

We now turn to the incident wave and the associated scattered fields. The outer solution at $O(\delta)$ was already chosen to be the incoming acoustic wave, in which

$$H_{01} = P_{01} = e^{i(kX - \omega t)}; \quad \Phi_{01} = \frac{1}{i\omega} e^{i(kX - \omega t)}. \tag{116}$$

Consideration of matching to the vortical region, to p_{21} from (74) and p_{31} from (97), shows that the first contribution to the scattered wave field occurs at $O(M^4\delta)$, and this is the same as the order at which there is forcing by lower-order terms linear in δ in the wave region. Thus, as indicated in (98)–(100), the next order after $O(\delta)$ at which the solution in the wave region is nonzero is $O(M^4\delta)$. The governing equations at this order are

$$\left.\frac{\partial\Phi_2}{\partial t}\right|_{O(M^2\delta)}+\left.\frac{\partial\Phi_3}{\partial t}\right|_{O(M\delta)}+\frac{\partial_0\Phi_{41}}{\partial t}+\boldsymbol{U}_2\cdot\boldsymbol{U}_{01}+P_{41}=0, \tag{117}$$

$$\left.\frac{\partial H_3}{\partial t}\right|_{O(M\delta)}+\frac{\partial_0 H_{41}}{\partial t}+\nabla\cdot(\boldsymbol{U}_2H_{01})+\nabla\cdot\boldsymbol{U}_{41}=0, \tag{118}$$

$$H_{41}=P_{41}. \tag{119}$$

Using the fact that $H_{41}=P_{41}$ and

$$\boldsymbol{U}_2=\frac{1}{4\pi}\nabla\left(\boldsymbol{I}\cdot\nabla\frac{1}{R}\right), \tag{120}$$

(109) may be evaluated at $O(M^4\delta)$, giving the forced wave equation for Φ_{41},

$$\frac{\partial_0^2\Phi_{41}}{\partial t^2}-\nabla^2\Phi_{41}=-\left.\frac{\partial^2\Phi_2}{\partial t^2}\right|_{O(M^2\delta)}-\left.\frac{\partial^2\Phi_3}{\partial t^2}\right|_{O(M\delta)}+\boldsymbol{U}_2\cdot\nabla H_{01}-\boldsymbol{U}_2\cdot\frac{\partial\boldsymbol{U}_{01}}{\partial t}. \tag{121}$$

Now, the second term on the right-hand side of (121) vanishes because there are no $O(M\delta)$ derivatives in this analysis, since $(\boldsymbol{u}_{11}-\boldsymbol{v}_{11})=\boldsymbol{0}$, and so there is no vorticity evolution at $O(M\delta)$.

The first term on the right-hand side of (121) takes a simple form, and is evaluated in Appendix B. The result is that Φ_{41} satisfies

$$\frac{\partial^2\Phi_{41}}{\partial t^2}-\nabla^2\Phi_{41}=-\frac{\omega^2}{4\pi}\boldsymbol{I}\cdot\nabla\left(\frac{1}{R}\right)e^{i(kX_c-\omega t)}+\frac{ik}{2\pi}e^{i(kX-\omega t)}\frac{\partial}{\partial X}\boldsymbol{I}\cdot\nabla\frac{1}{R}. \tag{122}$$

The appropriate solution to (122) is

$$\Phi_{41}=\frac{1}{4\pi}e^{i(kX_c-\omega t)}\boldsymbol{I}\cdot\nabla\left(\frac{1}{R}\right)-\frac{1}{4\pi}e^{i(kX-\omega t)}\boldsymbol{I}\cdot\nabla\frac{1}{R}+\Phi_{41}^H, \tag{123}$$

where Φ_{41}^H satisfies the (unforced) wave equation, and must be determined by matching. Hence Φ_{41}^H may be expressed as

$$\Phi_{41}^H=\left(\frac{A_0}{i\omega}\frac{e^{ikR}}{R}+\frac{A_i}{i\omega}\frac{\partial}{\partial\Xi_i}\frac{e^{ikR}}{R}+\frac{A_{ij}}{i\omega}\frac{\partial^2}{\partial\Xi_i\partial\Xi_j}\frac{e^{ikR}}{R}+\cdots\right)\times e^{ikX_c-i\omega t}, \tag{124}$$

where $\boldsymbol{\Xi}=\boldsymbol{X}-\boldsymbol{X}_c$, $R=|\boldsymbol{\Xi}|$, and the coefficients A_0, A_i, A_{ij}, etc., are constants.

Using (117), the corresponding expression for pressure is

$$P_{41}=\frac{i\omega}{4\pi}e^{ikX-i\omega t}I_j\frac{\Xi_j}{R^3}-\frac{1}{4\pi}e^{ikX-i\omega t}I_j\left(-\frac{\delta_{1j}}{R^3}+\frac{3\Xi\Xi_j}{R^5}\right)+A_0\frac{e^{ikR+ikX_c-i\omega t}}{R}+A_i\frac{\partial}{\partial X_i}\frac{e^{ikR+ikX_c-i\omega t}}{R}+A_{ij}\frac{\partial^2}{\partial X_i\partial X_j}\frac{e^{ikR+ikX_c-i\omega t}}{R}+\cdots, \tag{125}$$

where Ξ is the component of $\boldsymbol{\Xi}$ in the direction of propagation of the incident acoustic wave. We have retained only monopole, dipole, and quadrupole terms in (125); the higher multipole terms are, in fact, zero, as will be made clear by the fact the the terms displayed are sufficient to satisfy the matching conditions derived below.

We are now in a position to determine the scattered field completely by carrying out the matching, which determines A_0, A_i, and A_{ij}. We are free to express the matching condition in terms of either pressure P or velocity potential Φ. Here we use the pressure. We take the limit for small R of the outer expansions, expanding the term e^{ikX}, and match to the leading-order decaying terms from p_{21} and p_{31} rewritten in terms of R. At this point we must also recall that the origin of the vortical-region coordinates differs from the origin of the wave-region coordinates by $O(M\delta)$, because $\boldsymbol{v}_{11}=\boldsymbol{u}_{11}$, whereas $\boldsymbol{V}_{11}=\boldsymbol{0}$. Therefore, when the δ-independent pressure fields p_0, p_2 are expressed in terms of the wave-region coordinates, terms linear in δ occur due to the coordinate shift. However, because $p_0=O(r^{-3})$, and $p_2=O(r^{-1})$, as $r\rightarrow\infty$, these contribute additional terms to the wave region at $O(M^5\delta)$, and higher, but make no contribution at $O(M^4\delta)$.

The matching conditions are then expressed as three equations. The first equation, corresponding to terms in R^{-3}, is

$$-\frac{I_j}{4\pi}\left(-\frac{\delta_{1j}}{R^3}+\frac{3\Xi\Xi_j}{R^5}\right)+A_{ij}\left(-\frac{\delta_{ij}}{R^3}+\frac{3\Xi_i\Xi_j}{R^5}\right)=0. \tag{126}$$

This determines the quadrupole coefficient,

$$A_{ij}=\frac{I_j\delta_{i1}}{4\pi}. \tag{127}$$

Using (74), the second equation in the matching, which comes from terms in R^{-2}, is

$$\frac{i\omega}{4\pi}I_j\frac{\Xi_j}{R^3}-A_i\frac{\Xi_i}{R^3}-ik\frac{I_j}{4\pi}\Xi\left(-\frac{\delta_{1j}}{R^3}+\frac{3\Xi\Xi_j}{R^5}\right)=\frac{i\omega}{4\pi}I_k\Xi_k\left(\frac{1}{R^3}-\frac{3\Xi^2}{R^5}\right). \tag{128}$$

This gives the dipole coefficient

$$A_i=\frac{ik}{4\pi}\delta_{1i}I_1. \tag{129}$$

Finally, from the terms in R^{-1} and using (97),

$$-\frac{\omega k}{4\pi}I_j\frac{\Xi\Xi_j}{R^3}+\frac{A_0}{R}-\frac{1}{2}k^2A_{ij}\left(\frac{\delta_{ij}}{R}-\frac{\Xi_i\Xi_j}{R^3}\right)$$
$$+\frac{I_j}{4\pi}\frac{1}{2}k^2\Xi^2\left(-\frac{\delta_{1j}}{R^3}+\frac{3\Xi\Xi_j}{R^5}\right)$$
$$=\frac{\omega k}{8\pi}\left(-\frac{I_1}{R}-\frac{\Xi^2 I_1}{R^3}-\frac{\Xi\Xi_k I_k}{R^3}+\frac{3\Xi^3\Xi_k I_k}{R^5}\right). \tag{130}$$

This gives the monopole coefficient,

$$A_0=0. \tag{131}$$

This completes the matching of terms in the wave region at $O(M^4\delta)$ to all terms in the vortical region that depend explicitly on δ. The result is

$$P_{41}=\frac{1}{4\pi}e^{-i\omega t}\left\{e^{ikX_c}\frac{\partial}{\partial\Xi}\left[(\boldsymbol{I}\cdot\nabla+i\omega I_1)\frac{e^{ikR}}{R}\right]\right.$$
$$\left.+e^{ikX}\left(\frac{\partial}{\partial\Xi}-i\omega\right)\boldsymbol{I}\cdot\nabla\frac{1}{R}\right\}. \tag{132}$$

In the farfield limit $R\to\infty$, the second of the two terms in (132) is $O(R^{-2})$. The first of the two terms contains the radiating waves, and in the limit $R\to\infty$ we have

$$P_{41}=-\frac{\omega^2}{4\pi}\cos\vartheta\left(\frac{\boldsymbol{I}\cdot\boldsymbol{\Xi}}{R}+I_1\right)\frac{e^{i(kR+kX_c-\omega t)}}{R}+O(R^{-2}), \tag{133}$$
$$=-\frac{\omega^2}{4\pi}\cos\vartheta(\cos\vartheta+\cos\mu)I\frac{e^{i(kR+kX_c-\omega t)}}{R}+O(R^{-2}), \tag{134}$$

where here ϑ is the angle subtended at the vortex between the position $\boldsymbol{X}$ and the direction of propagation of the incident acoustic wave, and μ is the angle between the direction of $\boldsymbol{I}$ and the direction of propagation of the incident acoustic wave.

V. RELATIONSHIP TO THE ACOUSTIC ANALOGY APPROXIMATION

The result can readily be shown to agree with the scattered sound field obtained using the acoustic analogy approximation. Equation (17) of Ref. 18 agrees with (133), provided that in the former, the Fourier transform of the vorticity is approximated as

$$\int \boldsymbol{\omega}(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,d^3\boldsymbol{x}\approx\int i\boldsymbol{k}\cdot\boldsymbol{x}\boldsymbol{\omega}(\boldsymbol{x})\,d^3\boldsymbol{x}. \tag{135}$$

This approximation is valid provided the length scale of the vortical region is small compared to the wavelength of the incident acoustic waves, which is precisely the condition required for our analysis to be valid. The applicability of the acoustic analogy approximation to vortices with length scale comparable with the wavelength of the incident waves is an open question, which cannot be addressed by the asymptotic analysis presented here. The present results also hold for waves with wavelength larger than M^{-1}, i.e., which have frequency smaller than the vorticity.

From the analysis presented, it may appear that, to determine the scattered sound, all we require is a leading-order description of the vorticity over the time of interest, since this enables us to determine $\boldsymbol{I}$, and hence P_{41}. Moreover, since $\boldsymbol{I}$ is conserved to leading order, we may apparently use the value of $\boldsymbol{I}$ from the initial, undisturbed vortex.

Note, however, that the effect of scattering is present not only in P_{41}, but also in the lower-order fields. The expression (114) for H_3, and hence P_3, depends on $\dot{\boldsymbol{J}}$, and ultimately upon $\boldsymbol{\omega}$. Now, $\boldsymbol{\omega}$ satisfies an induction equation with an induction velocity up to $O(M^3\delta)$. Hence, the vorticity $\boldsymbol{\omega}$ differs from the vorticity $\boldsymbol{\omega}^{\mathrm{U}}$, say, that would have evolved in the absence of any incident waves. Any resulting difference in pressure P_3 should then be regarded as part of the scattered pressure field. In particular, we see that if $\boldsymbol{J}$ departs by $O(M\delta)$ from the value $\boldsymbol{J}^{\mathrm{U}}$ that would have existed in the absence of the acoustic wave, then this difference will be $O(M\delta)$, and so should be counted as part of the leading-order scattered sound field.

Now, since $\boldsymbol{u}_{01}=\boldsymbol{0}$, and $\boldsymbol{u}_{11}=\boldsymbol{v}_{11}$, there is no evolution of vorticity at $O(\delta)$ and $O(M\delta)$. Over times $O(M^{-1})$, however, it is possible that the velocity field $\boldsymbol{u}_{21}$ may act to lead to differences between $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^{\mathrm{U}}$, and so to differences between $\boldsymbol{J}$ and $\boldsymbol{J}^{\mathrm{U}}$, of order $M\delta$. Such differences would then imply that, over long times, the pressure P_3 would contain a contribution at $O(M\delta)$, and so a contribution to the pressure $O(M^4\delta)$ due to the presence of the acoustic waves. Thus, in order for the leading-order scattered wave field to be represented by (132) over long times, we must assume that $\boldsymbol{u}_{21}$ has no secular effect on $\boldsymbol{J}$ and so

$$|\boldsymbol{J}(t)-\boldsymbol{J}^{\mathrm{U}}(t)|=o(M\delta), \tag{136}$$

over the time interval in question. We also assume that the same structure holds at higher orders, so that the restriction on P_3 is the most restrictive. In general, (136) cannot be guaranteed *a priori*; in Part II,[28] we show that it does apply, under certain assumptions about the magnitude of δ, for the problem of scattering from Hill's spherical vortex. For M and δ independent, the expansion is valid under the conditions we have just mentioned, but the detailed conditions under which it may fail, and, in particular, the extra conditions this may impose on M and δ depend on the particular problem under study.

VI. CONCLUSION

The problem of scattering of an acoustic plane wave by a three-dimensional vortical structure has been solved by a rational expansion of the equations of motion in the limit of long waves, when the acoustic frequency is of the same order as the vorticity. The results also hold for smaller acoustic frequencies. The resulting solution is the same as that obtained by using the acoustic analogy approximation, provided the impulse $\boldsymbol{I}$ is nonzero, and the assumption (136) holds. This is because only the pressure fields p_{01}, p_{11}, p_{21}, and $p_{31}^{(i)}$ from the vortical region are required to determine matching conditions to the wave region at $O(M^4\delta)$, and the right-hand sides of the equations that these pressure fields satisfy are all formed from the product of the velocities of the undisturbed vortex $\boldsymbol{u}^{\mathrm{U}}$ and the incident plane wave, exactly as if the acoustic analogy approximation had been made from the outset. If $\boldsymbol{I}^{\mathrm{U}}=\mathbf{0}$, then the leading-order scattered field is $O(M^5\delta)$, where the details for the structure of the vortex, and not just its impulse, are required in order to determine the scattered sound field. The situation is analogous to that of FLS, where, in two dimensions, the leading-order scattered sound field depends only upon the circulation of the vortex.

For times longer than $O(1)$, there is no difficulty, in principle, in allowing the vortex to move through an order-unity number of incident acoustic wavelengths. The use of the acoustic analogy method for computing the scattered sound field is then valid provided velocities $O(M^2\delta)$ and smaller do not act in a secular fashion, so that the assumption $|\boldsymbol{J}-\boldsymbol{J}^{\mathrm{U}}|=o(M\delta)$ holds over times $O(M^{-1})$. In addition, it must be possible to define a vortex center that propagates smoothly, with order-unity velocity but relatively smaller acceleration. It is certainly possible to meet both of these conditions if the flow in the vortex is both laminar and stable to small-amplitude perturbations. However, for unstable flows, it may be that the dynamics of the vortex is sensitive to the perturbations introduced by the incident acoustic wave, and so (136) may not hold.

A further limitation of the present analysis is that the Lighthill radiation emitted spontaneously by the vortex is formally larger in amplitude—$O(M^3)$ in the farfield—than the scattered wave field which is $O(M^4\delta)$. Lighthill radiation is expected to occur for almost all vortical flows, and so an attempt to measure sound scattered from a vortex is likely to be swamped by noise being emitted by the vortex. It may be possible to select a frequency at which Lighthill radiation is negligible, so that the scattered wave field can be detected, despite the fact that its magnitude is a factor $O(M\delta)$ smaller [and power a factor $O(M^2\delta^2)$ smaller] than that of the Lighthill radiation. Also, if the acoustic frequency is low, the difference in time scales between generated and scattered sound may also make distinguishing the two possible.

Vortices that are stationary, or propagate without a change of shape, do not exhibit Lighthill radiation. One such example is Hill's vortex. In Part II[28] we calculate the scattered sound field using the full asymptotic procedure described here. We show that, for Hill's spherical vortex, the assumption (136) is valid, and so the scattered sound field for Hill's vortex is correctly predicted by the acoustic analogy approximation, even for times $O(M^{-1})$.

There are two principal differences between the present case of scattering by a three-dimensional vortex and the much studied case of scattering by a two-dimensional vortex filament. One difference is that, in the three-dimensional case, an expression for the scattered field in the wave region exists that is valid for all scattering angles. This contrasts with the two-dimensional case, in which a special parabolic-shaped region exists about the forward scattering direction. The other difference is that in the three-dimensional case the scattered field can be expressed as a sum of monopole, dipole, and quadrupole wave fields, whereas in the two-dimensional case all moments are present in the leading-order scattered field.

Despite these differences, however, it is interesting to note the similarities that exist between the two cases. The leading-order scattered sound field in the MAE framework depends in each case on a single integrated quantity of the vortex, namely, the impulse $\boldsymbol{I}$ in the three-dimensional case and the circulation in the two-dimensional case. Moreover, forcing is present in the wave equation satisfied by the scattered sound, implying that in both cases the velocity due to the vortex is significant in the wave region. This is, at first sight, surprising, since the velocity associated with the vortex decays more rapidly with distance in three dimensions than in two, but scattering occurs in the three-dimensional problem at $O(M^4\delta)$—two orders in M higher than in the corresponding two-dimensional problem—and so in both problems there is significant scattering by flow over $O(1)$ acoustic wavelengths, as well as flow on the scale of the vortex. Finally, the method presented here can be applied to the two-dimensional case and the results match those of FLS, showing that the leading-order result for the scattered sound field holds also for unsteady flows in two dimensions.

The solution to three-dimensional problems is potentially more important to applications and experiments than the two-dimensional ones. The solution (132) takes a simple form and could form the basis for experimental work. As in the two-dimensional problem, in which the dominant response was determined by the circulation, any attempts to use inverse measurements to probe the inner structure using long-wavelength acoustic waves are limited by the fact that the dominant response here is determined entirely by another integrated property of the vortex, namely the vortex dipole moment.

ACKNOWLEDGMENTS

This work was started at the Woods Hole Oceanographic Institution 1999 Summer School for Geophysical Fluid Dynamics. The order of authors is the reverse of the order in FLS, which was determined by a random event. Rupert Ford died suddenly on March 30, 2001, without seeing reviews of the original version of the manuscript, which was revised by Stefan Llewellyn Smith.

APPENDIX A: FARFIELD FOR $\boldsymbol{u}_0$

In this appendix, we show that

$$\boldsymbol{u}_0=-\frac{1}{4\pi}\int \boldsymbol{\omega}(\boldsymbol{\xi}')\times\nabla_{\xi}(r^{-1}-\boldsymbol{\xi}'\cdot\nabla_{\xi}r^{-1} + \tfrac{1}{2}\boldsymbol{\xi}'\boldsymbol{\xi}':\nabla_{\xi}\nabla_{\xi}r^{-1})d^3\boldsymbol{\xi}'+O(r^{-5}) \tag{A1}$$

can be expressed in the form of the gradient of a scalar field, and determine an expression for it.

To treat the first of the two terms in (A1), it proves helpful to note that

$$0=\int d^3\boldsymbol{\xi}\,\frac{\partial}{\partial\xi_i}[\xi_j\xi_k\omega_i(\boldsymbol{\xi})]=\int d^3\boldsymbol{\xi}[\xi_j\omega_k(\boldsymbol{\xi})+\xi_k\omega_j(\boldsymbol{\xi})]. \tag{A2}$$

Hence, we have

$$\frac{1}{4\pi}\epsilon_{ijk}\frac{\partial}{\partial\xi_k}\int d^3\boldsymbol{\xi}'\,\omega_j(\boldsymbol{\xi}')\xi_l'\frac{\partial}{\partial\xi_l}r^{-1}$$
$$=\frac{1}{4\pi}\epsilon_{ijk}\frac{\partial}{\partial\xi_k}\int d^3\boldsymbol{\xi}'\left(\frac{1}{2}\omega_j(\boldsymbol{\xi})\xi_l'-\frac{1}{2}\omega_l(\boldsymbol{\xi}')\xi_j'\right)\frac{\partial}{\partial\xi_l}r^{-1}$$
$$=-\frac{1}{8\pi}\epsilon_{ijk}\frac{\partial}{\partial\xi_k}\int d^3\boldsymbol{\xi}'\{\nabla_\xi r^{-1}\times[\boldsymbol{\xi}'\times\boldsymbol{\omega}(\boldsymbol{\xi}')]\}_j$$
$$=-\frac{1}{8\pi}(\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})$$
$$\times\left(\frac{\partial^2}{\partial\xi_k\partial\xi_l}r^{-1}\right)\int d^3\boldsymbol{\xi}'[\boldsymbol{\xi}'\times\boldsymbol{\omega}(\boldsymbol{\xi}')]_m$$
$$=\frac{1}{4\pi}\frac{\partial}{\partial\xi_i}(\boldsymbol{I}\cdot\nabla\boldsymbol{r}^{-1}),\tag{A3}$$

where $\boldsymbol{I}$ is given by (12).

To treat the second of these two, we use a similar device. We first observe that

$$0=\int d^3\boldsymbol{\xi}\frac{\partial}{\partial\xi_i}[\xi_j\xi_k\xi_l\omega_i(\boldsymbol{\xi})]$$
$$=\int d^3\boldsymbol{\xi}[\xi_j\xi_k\omega_l(\boldsymbol{\xi})+\xi_k\xi_l\omega_j(\boldsymbol{\xi})+\xi_l\xi_j\omega_k(\boldsymbol{\xi})].\tag{A4}$$

Hence

$$-\frac{1}{8\pi}\epsilon_{ijk}\frac{\partial}{\partial\xi_k}\int d^3\boldsymbol{\xi}'\,\omega_j(\boldsymbol{\xi}')\xi_l'\xi_m'\frac{\partial^2}{\partial\xi_l\partial\xi_m}r^{-1}$$
$$=-\frac{1}{8\pi}\epsilon_{ijk}\frac{\partial}{\partial\xi_k}\int d^3\boldsymbol{\xi}'\left(\frac{2}{3}\omega_j(\boldsymbol{\xi}')\xi_l'\xi_m'\right.$$
$$\left.-\frac{1}{3}[\omega_l(\boldsymbol{\xi}')\xi_m'\xi_j'+\omega_m(\boldsymbol{\xi}')\xi_j'\xi_l']\right)\frac{\partial^2}{\partial\xi_l\partial\xi_m}r^{-1}$$
$$=\frac{1}{12\pi}\epsilon_{ijk}\frac{\partial^2}{\partial\xi_k\partial\xi_m}\int d^3\boldsymbol{\xi}'\,\xi_m'\{\nabla_\xi r^{-1}\times[\boldsymbol{\xi}'\times\boldsymbol{\omega}(\boldsymbol{\xi}')]\}_j$$
$$=\frac{1}{4\pi}\frac{\partial}{\partial\xi_i}(\boldsymbol{J}:\nabla\nabla r^{-1}),\tag{A5}$$

where

$$\boldsymbol{J}=-\frac{1}{3}\int\boldsymbol{\xi}'[\boldsymbol{\xi}'\times\boldsymbol{\omega}(\boldsymbol{\xi}')]d^3\boldsymbol{x}'.\tag{A6}$$

Thus

$$\boldsymbol{u}_0=\frac{1}{4\pi}\nabla(\boldsymbol{I}\cdot\nabla r^{-1}+\boldsymbol{J}:\nabla\nabla r^{-1})+O(r^{-5}),\quad \text{as } r\to\infty,\tag{A7}$$

as required.

APPENDIX B: EVALUATION OF $\partial^2\Phi_2/\partial t^2|_{O(M^2\delta)}$

We shall first evaluate $d^2\boldsymbol{I}/dt^2$ to $O(M^2\delta)$. To do this, we have

$$\frac{1}{2}\frac{d}{dt}\int\boldsymbol{x}\times\boldsymbol{\omega}d^3\boldsymbol{x}=\frac{1}{2}\int\boldsymbol{x}\times[\nabla\times\{(\boldsymbol{u}-\boldsymbol{v})\times\boldsymbol{\omega}\}]d^3\boldsymbol{x}$$
$$=\int\boldsymbol{u}\nabla\cdot\boldsymbol{u}d^3\boldsymbol{x}.\tag{B1}$$

This is valid provided $u^2=o(r^{-2})$ as $r\to\infty$, which is valid up to $O(M^2\delta)$. Then

$$\frac{d^2\boldsymbol{I}}{dt^2}=\int\frac{\partial\boldsymbol{u}}{\partial t}\nabla\cdot\boldsymbol{u}\,d^3\boldsymbol{x}+\int\boldsymbol{u}\nabla\cdot\frac{\partial\boldsymbol{u}}{\partial t}\,d^3\boldsymbol{x}.\tag{B2}$$

Now, using $\nabla\cdot\boldsymbol{u}_0=\boldsymbol{0}$ and $\boldsymbol{u}_{01}=\boldsymbol{0}$, we have

$$\left.\frac{d^2\boldsymbol{I}}{dt^2}\right|_{O(M^2\delta)}=\int\frac{\partial_0\boldsymbol{u}_0}{\partial t}\nabla\cdot\boldsymbol{u}_{21}\,d^3\boldsymbol{x}+\int\boldsymbol{u}_0\nabla\cdot\frac{\partial_0\boldsymbol{u}_{21}}{\partial t}\,d^3\boldsymbol{x}.\tag{B3}$$

The first of the two integrals on the right-hand side of (B3) vanishes, because $\boldsymbol{I}_0=\int\boldsymbol{u}_0d^3\boldsymbol{x}=\frac{1}{2}\int\boldsymbol{x}\times\boldsymbol{\omega}d^3\boldsymbol{x}=\boldsymbol{I}$ is conserved to leading order in time. The second can be simplified to give

$$\left.\frac{d^2\boldsymbol{I}}{dt^2}\right|_{O(M^2\delta)}=\omega^2\rho_{01}\boldsymbol{I}.\tag{B4}$$

Now, recalling that R depends on time through $R=|\boldsymbol{X}-\boldsymbol{X}_c|$, the first derivative of Φ_2 with respect to time is

$$\frac{\partial\Phi_2}{\partial t}=\frac{1}{4\pi}\frac{d\boldsymbol{I}}{dt}\cdot\nabla\left(\frac{1}{R}\right)-\frac{1}{4\pi}\boldsymbol{I}M\boldsymbol{V}:\nabla\nabla\left(\frac{1}{R}\right).\tag{B5}$$

Since $\boldsymbol{V}$ is independent of δ by assumption, it follows that the only contribution to $\partial^2\Phi_2/\partial t^2$ at $O(M^2\delta)$ comes from differentiating $\boldsymbol{I}$ again with respect to time in the first term on the right-hand side of (B5), and so

$$\left.\frac{\partial^2\Phi_2}{\partial t^2}\right|_{O(M^2\delta)}=\frac{1}{4\pi}\left.\frac{d^2\boldsymbol{I}}{dt^2}\right|_{O(M^2\delta)}\cdot\nabla\left(\frac{1}{R}\right)=\omega^2\rho_{01}\Phi_2.\tag{B6}$$

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