Energy and pseudomomentum of propagating disturbances on the beta-plane

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The far-field amplitude of the waves generated by a steadily propagating radially symmetric disturbance on the beta-plane is calculated using Lighthill's method. From this can be obtained the fluxes of quantities such as wave energy which are radiated away from the disturbance. The radiated wave power is computed for a variety of forms of the disturbance. The rate of change of pseudomomentum in the system is also calculated: the component parallel to the motion of the disturbance is the radiated wave power divided by velocity. Results are compared to previous work and some physical issues are discussed.

1. Introduction

The behavior of vortices in the ocean has been the subject of much interest in oceanography since the realisation in the 1970s of their ubiquity (MODE Group, 1978), while vortices in the atmosphere have been studied for even longer. The particular problem of the evolution of a vortex under the influence of the Earth's curvature, as modelled by the beta-effect, has been an enduring source of interest, and has been extensively pursued in the atmospheric literature, notably with the aim of improving the forecast of hurricane motion. The evolution of a vortex on the beta-plane combines anisotropic and dispersive Rossby waves with the nonlinear features typical of fluid systems. At their simplest, the latter simplify to the conservation of absolute vorticity.

The problem of the motion of vortices on the beta-plane has been extensively studied (some of the many studies are Adem, 1956; Chan and Williams, 1987; Reznik and Dewar, 1994; Llewellyn Smith, 1997, with the last of these containing more background material and references).

Other authors have examined the decay of vortical structures due to the radiation of Rossby waves. Flierl and Haines (1994) studied the decay of modons, which are dipolar structures, while McDonald (1998) looked at the decay of a circular vortex. A natural and more tractable related problem is the simplification to linear Rossby wave dynamics, representing the vortex as a propagating radial disturbance and seeking to understand the waves generated by such forcing. This approach was followed by Korotaev (1988) and was also examined by Stepanyants and Fabrikant (1992). An interesting review of this and other topics is given in Korotaev (1997). The physical quantities that these studes concentrate on are the wave power radiated away by the waves generated by the vortex, and the forces that one may consequently aim to define. These quantities are obviously important in understanding the fully nonlinear decay of vortices due to wave radiation. The same results are derived here using a simpler method and subsequently generalized to arbitrary vortex profiles, and the physical significance of the problem is examined.

The far-field wave amplitude for the model problem of the response of the linearized beta-plane equations to the propagation of a symmetric vorticity disturbance is derived in § 2. This result is then used in § 3 to construct fluxes of quantities such as the wave energy away from the forcing. The results of Stepanyants and Fabrikant (1992) and Korotaev (1997) are considered in § 4 in the light of our calculations, and show how they in fact correspond to the rate of change of pseudomomentum. Section 5 summarizes and concludes.

2. Far-field radiation from moving sources

2.1. *Lighthill's method for uniformly propagating sources*

The asymptotic behavior of forced linear wave systems has been extensively studied since the classical work of Kelvin and others. Lighthill (1978) provides a comprehensive and physically enlightening presentation of waves in fluids, and in particular includes a presentation of the behavior of linear systems for large times and distances. It complements and synthesizes previous papers, but note should also be taken of Lighthill (1990).

An unforced linear partial differential equation in two spatial dimensions with constant coefficients may be written as

$$P\left(i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y},\right)\phi = 0. \tag{2.1}$$

The existence of a plane-wave solution $\phi = \phi_0 \exp\left(i\boldsymbol{k}\cdot\boldsymbol{r} - i\sigma t\right)$ is equivalent to the existence of a solution to the dispersion relation

$$P(\sigma, l, m) = 0, \tag{2.2}$$

where $\boldsymbol{k} = (l, m)$. The corresponding causal solution to the forced problem, with localized forcing term $f(\boldsymbol{r} - \boldsymbol{U}t)$ in (2.1), may be expressed as a Fourier integral

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F(\boldsymbol{k})\exp\{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{U}t)\}}{P(\boldsymbol{U}\cdot\boldsymbol{k}, l, m)}\,\mathrm{d}l\,\mathrm{d}m, \tag{2.3}$$

where $F(\boldsymbol{k})$ is the Fourier transform of the forcing function $f(\boldsymbol{r})$, defined by

$$F(\boldsymbol{k}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{r})\exp\{-i\boldsymbol{k}\cdot\boldsymbol{r}\}\,\mathrm{d}x\,\mathrm{d}y. \tag{2.4}$$

Lighthill's method gives an expression for the behavior of ϕ in its far field. There, the amplitude of ϕ is asymptotically

$$\frac{(2\pi)^{\frac{3}{2}}}{|\kappa|^{\frac{1}{2}} r^{\frac{1}{2}}}\frac{|F(\boldsymbol{k})|}{|\nabla P(\boldsymbol{U}\cdot\boldsymbol{k}, l, m)|}, \tag{2.5}$$

where $r = |\boldsymbol{r} - \boldsymbol{U}t|$, ∇ is the gradient operator in $\boldsymbol{k}$ space, and κ is the curvature of the surface

$$P(\boldsymbol{U}\cdot\boldsymbol{k}, l, m) = 0. \tag{2.6}$$

We observe the characteristic inverse-square-root decay of two-dimensional wavefields, except where the curvature vanishes: those portions of the curves correspond to caustics or to plane wave propagation, and (2.5) has to be adjusted appropriately in these cases.

The wavenumber of waves found in any direction of physical space is given by the solution $\boldsymbol{k}$ of (2.6) when the oriented normal to P, i.e. ∇P in $\boldsymbol{k}$-space, is in the physical direction away from the origin. Some directions may not have any waves propagating along them away from the origin. Some may correspond to more than one point of (2.6).

In that case, the amplitude of the various contributions must be evaluated separately from (2.5).

2.2. *The Rossby wave case*

Lighthill (1967) considers the linear Rossby wave equation, calling it the 'beta-plane ocean'. The notation here follows this paper with however ϵ in place of β. Lighthill (1978) uses (k,l,m) in place of (l,m,n), B and ω instead of P and σ, and $-\boldsymbol{U}$ rather than $\boldsymbol{U}$. For the two-dimensional linear Rossby wave system, the governing polynomial is

$$P(\sigma,l,m) = \sigma(l^2+m^2) + \epsilon l, \tag{2.7}$$

corresponding to the forced linear equation of the particular form

$$\frac{\partial}{\partial t}\nabla^2\psi + \epsilon\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial t}q(\boldsymbol{r},t). \tag{2.8}$$

This is just the linearized vorticity equation on the beta-plane. For a radially symmetric structure propagating on the beta-plane, the right-hand side forcing term is

$$\frac{\partial}{\partial t}q(|\boldsymbol{r}-\boldsymbol{U}t|), \tag{2.9}$$

where $q(r)$ describes the radially symmetric vorticity disturbance. The velocity of the vortex is $\boldsymbol{U} = U(\cos\alpha, \sin\alpha)$, where α is the angle of propagation of the vortex with the x-axis. Then the appropriate Fourier transform of (2.9) is

$$F(\boldsymbol{k}) = -i\boldsymbol{U}\cdot\boldsymbol{k}Q(k) = -iUk\cos(\varphi-\alpha)Q(k), \tag{2.10}$$

where $k^2 = l^2+m^2$ and the polar angle in $\boldsymbol{k}$-space is defined by

$$\boldsymbol{k} = k(\cos\varphi, \sin\varphi). \tag{2.11}$$

The function $Q(k)$ is the Fourier transform of the spatial vorticity disturbance q.

The geometric factors in the expression (2.5) come from the wavenumber curve $P(\boldsymbol{U}\cdot\boldsymbol{k}, l, m)$. The curve $P=0$ corresponds to the equation

$$U(l\cos\alpha + m\sin\alpha)(l^2+m^2) + \epsilon l = 0. \tag{2.12}$$

which may be rewritten in polar form as

$$k[Uk^2\cos(\varphi-\alpha) + \epsilon\cos\varphi] = 0. \tag{2.13}$$

The point $k=0$ does not generate any radiation so the points on the wavenumber curve that contribute to the far-field radiation satisfy

$$k = \left(-\frac{\epsilon\cos\varphi}{U\cos(\varphi-\alpha)}\right)^{\frac{1}{2}} \tag{2.14}$$

when

$$\cos\varphi\cos(\varphi-\alpha) < 0. \tag{2.15}$$

The wavenumber curve $P=0$ is shown in figure 1 for different values of α. All the calculations in this section will take place on the locus $P=0$, and consequently much use will be made of (2.14). The range of angles φ for which (2.14) has a solution corresponds to angles in wavenumber space, and does not indicate in which directions waves are emitted in physical space.

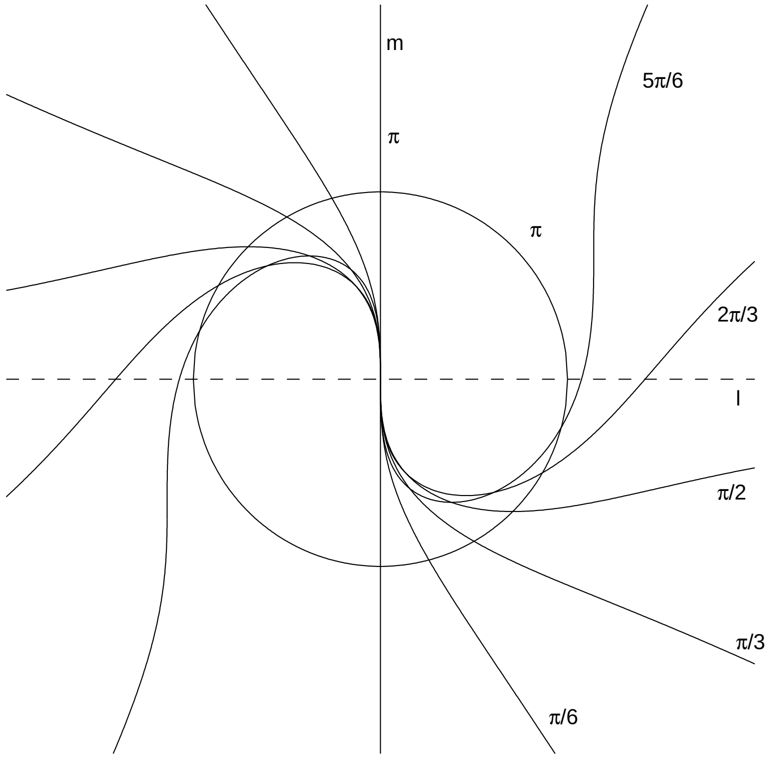

FIGURE 1. Wavenumber curves $P(\boldsymbol{U}\cdot\boldsymbol{k}, l, m) = 0$ generated by steadily propagating disturbances for different values of α (solid curves). The value next to each curve gives the relevant value of α; for $\alpha = \pi$, the wavenumber curves are the m-axis and the circle $k = 1$. The unit of length is $(U/\epsilon)^{1/2}$.

The gradient of the locus $P = 0$ is

$$\nabla P = \begin{pmatrix} U(l^2+m^2)\cos\alpha + 2Ul(l\cos\alpha + m\sin\alpha) + \epsilon \\ U(l^2+m^2)\sin\alpha + 2Um(l\cos\alpha + m\sin\alpha) \end{pmatrix}. \tag{2.16}$$

This may be rewritten in a more geometric form as

$$\nabla P = \begin{pmatrix} Uk^2\cos\alpha - \epsilon\cos 2\varphi \\ Uk^2\sin\alpha - \epsilon\sin 2\varphi \end{pmatrix}; \tag{2.17}$$

this shows that the group velocity, which may be obtained from (2.7),

$$\boldsymbol{c}_g \equiv \nabla\sigma = -\frac{\nabla P}{\partial P/\partial\sigma} = -\frac{\nabla P}{k^2}, \tag{2.18}$$

can be viewed as the vector sum of a velocity in the direction of motion of the vortex and a velocity oriented at twice the angle φ. The most convenient form of ∇P for subsequent calculation is

$$\nabla P = -\frac{\epsilon}{\cos(\varphi-\alpha)} \begin{pmatrix} \cos\varphi\cos\alpha + \cos 2\varphi\cos(\varphi-\alpha) \\ \cos\varphi\sin\alpha + \sin 2\varphi\cos(\varphi-\alpha) \end{pmatrix}. \tag{2.19}$$

The modulus of $|\nabla P|$ is then

$$|\nabla P| = \frac{\epsilon}{|\cos(\varphi-\alpha)|}\times$$
$$\times[\cos^2\varphi+\cos^2(\varphi-\alpha)+2\cos\varphi\cos(\varphi-\alpha)\cos(2\varphi-\alpha)]^{\frac{1}{2}} \tag{2.20}$$
$$= \frac{\epsilon\Phi(\varphi,\alpha)^{\frac{1}{2}}}{|\cos(\varphi-\alpha)|}. \tag{2.21}$$

The only other term that remains to be calculated in (2.5) is the curvature of P. A general expression for the curvature of the curve $P = 0$ in two dimensions is

$$\kappa = \frac{1}{|\nabla P|^3}[P_{ll}P_m^2 + P_{mm}P_l^2 - 2P_{lm}P_lP_m]. \tag{2.22}$$

The higher derivatives of P may be written as follows:

$$P_{ll} = 2Uk\cos(\varphi-\alpha) + 4Uk\cos\alpha\cos\varphi, \tag{2.23a}$$
$$P_{mm} = 2Uk\cos(\varphi-\alpha) + 4Uk\sin\alpha\sin\varphi, \tag{2.23b}$$
$$P_{lm} = 2Uk\sin(\alpha+\varphi). \tag{2.23c}$$

Therefore, the curvature is given by

$$\kappa = \frac{U^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}}\frac{\cos(\varphi-\alpha)}{\Phi(\varphi,\alpha)^{\frac{3}{2}}}[-\cos\varphi\cos(\varphi-\alpha)]^{\frac{1}{2}}[3+4\cos(2\varphi-2\alpha)+\cos(4\varphi-2\alpha)]. \tag{2.24}$$

For a radially symmetric propagating disturbance, as considered here, the wavefield amplitude in the far field is given from (2.5), (2.10), (2.21) and (2.24) as

$$A = \frac{(2\pi)^{\frac{3}{2}}U^{\frac{1}{4}}\Phi^{\frac{1}{4}}}{\epsilon^{\frac{1}{4}}r^{\frac{1}{2}}}\frac{[-\cos\varphi\cos^3(\varphi-\alpha)]^{\frac{1}{4}}|Q(k)|}{|3+4\cos(2\varphi-2\alpha)+\cos(4\varphi-2\alpha)|^{\frac{1}{2}}} \tag{2.25}$$

The phase information, which is required to construct the actual far-field behaviour, is ignored here.

For delta-function forcing, the Fourier transform Q becomes $1/(2\pi)^2$ in (2.4). Figure 2 is a plot of the resultant wavefield amplitude for $\alpha = 2\pi/3$ as a function of φ, the angle of the wave vector. The function $A(\varphi)$ has singularities at locations where the curvature κ vanishes. These are not physical singularities, but correspond rather to caustics where the assumptions used to derive (2.25) are invalid. A more detailed investigation reveals an Airy function transition across the caustics.

Lighthill (1967) obtained the picture of figure 2, but did not calculate any of the factors in (2.5) explicitly.

3. Energy balances

3.1. *Wave-energy flux*

Lighthill (1978) provides a general methodology for calculating the power radiated from a source in an anisotropic wave system. The following discussion is restricted to two-dimensional systems. The directional distribution of wave energy is then (Lighthill, 1978, chapter 4, equation 431)

$$W = W_0|F|^2\left|\frac{\partial P}{\partial n}\right|^{-2}\left(\frac{4\pi^3}{\kappa r}\right), \tag{3.1}$$

where $|\partial P/\partial n| = |\nabla P|$. The quantity W_0 is the function required to obtain the appropriate wave-energy density from the far-field amplitude; A factor of $\frac{1}{2}$ has already been

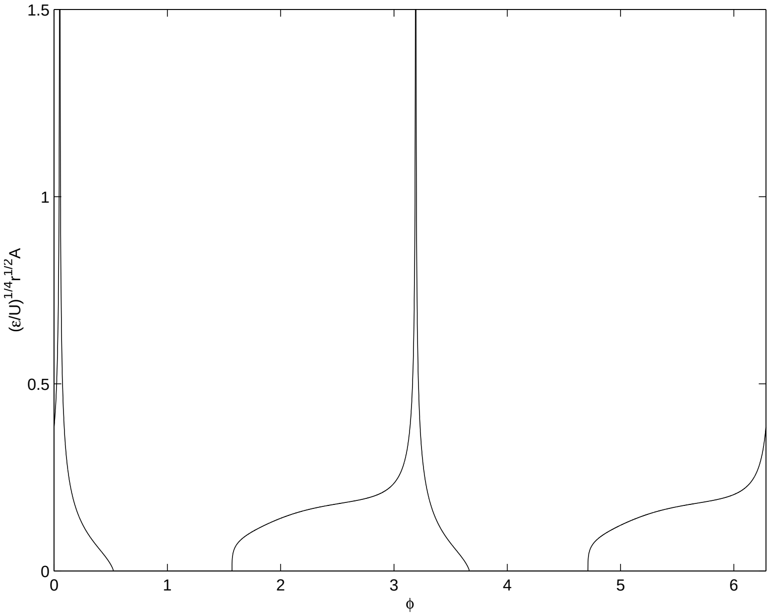

FIGURE 2. Nondimensional far-field wave amplitude $(\epsilon/U)^{1/4}r^{1/2}A$ against φ for $\alpha = 2\pi/3$. The singularities in A correspond to caustics in the wavefield proper.

included in it from integrating over a wave period. The energy flux relative to the source is then $\boldsymbol{I} = W\boldsymbol{c}_g$.

The total power output is therefore obtained by integrating (3.1) along the wavenumber curve, giving

$$P_{\mathrm{W}} = 4\pi^3 \int W_0 |F|^2 \left|\frac{\partial P}{\partial \sigma}\right|^{-1} \left|\frac{\partial P}{\partial n}\right|^{-1} \mathrm{d}s, \tag{3.2}$$

noting that the curvature factor cancels on integration and using (2.18). The expression (3.2) has no singularities at caustics, and hence no detailed analysis is required near caustics.

The variable of integration in (3.2) can be changed to the angle in wavenumber space, φ, using the relations

$$\mathrm{d}s^2 = \mathrm{d}k^2 + k^2\mathrm{d}\varphi^2 = \mathrm{d}\varphi^2 \left[k^2 + \left(\frac{\mathrm{d}k}{\mathrm{d}\varphi}\right)^2\right] \tag{3.3}$$

and

$$\frac{\mathrm{d}k}{\mathrm{d}\varphi} = -\frac{\partial P/\partial \varphi}{\partial P/\partial k} \tag{3.4}$$

which hold on the wavenumber curve $P = 0$. Therefore

$$\mathrm{d}s^2 = k^2\mathrm{d}\varphi^2 \left(\frac{\partial P}{\partial k}\right)^{-2} \left[\left(\frac{\partial P}{\partial k}\right)^2 + \frac{1}{k^2}\left(\frac{\partial P}{\partial \varphi}\right)^2\right] \tag{3.5}$$

$$= k^2\mathrm{d}\varphi^2 \left(\frac{\partial P}{\partial k}\right)^{-2} \left(\frac{\partial P}{\partial n}\right)^2. \tag{3.6}$$

The partial derivative $\partial P/\partial k$ is given from (2.13) by

$$\frac{\partial P}{\partial k} = -2\epsilon \cos\varphi, \tag{3.7}$$

while $\partial P/\partial \sigma = k^2$ has already been used in the definition of $\boldsymbol{c}_g$. Combining all these results, we obtain

$$P_{\mathrm{W}} = \frac{4\pi^3 U^{\frac{3}{2}}}{\epsilon^{\frac{1}{2}}} \int_{\pi/2}^{\pi/2+\alpha} W_0 |Q|^2 \left(-\frac{\cos^3(\varphi-\alpha)}{\cos\varphi}\right)^{\frac{1}{2}} \mathrm{d}\varphi, \tag{3.8}$$

where the range of integration has been halved by symmetry.

3.2. *Examples*

The function W_0 defines the kind of energy whose radiated flux is being estimated. For the usual wave-energy density, it is ρk^2, where ρ is some density; this corresponds to the energy density $\frac{1}{2}\rho|\boldsymbol{u}|^2$ in physical space. Then

$$P_{\mathrm{W}} = 4\pi^3 \rho (\epsilon U)^{\frac{1}{2}} \int_{\pi/2}^{\pi/2+\alpha} |Q|^2 [-\cos(\varphi-\alpha)\cos\varphi]^{\frac{1}{2}} \,\mathrm{d}\varphi. \tag{3.9}$$

Thus for point forcing, where $Q = \mathit{\Gamma}/(2\pi)^2$, the wave power is

$$P_{\mathrm{W}} = \frac{\rho \mathit{\Gamma}^2 (\epsilon U)^{\frac{1}{2}}}{4\pi} \int_{\pi/2}^{\pi/2+\alpha} [-\cos(\varphi-\alpha)\cos\varphi]^{\frac{1}{2}} \,\mathrm{d}\varphi \tag{3.10}$$

$$= \frac{\rho \mathit{\Gamma}^2 (\epsilon U)^{\frac{1}{2}}}{4\pi} \left[2E(\sin^2 \tfrac{1}{2}\alpha) - (1+\cos\alpha) K(\sin^2 \tfrac{1}{2}\alpha)\right], \tag{3.11}$$

where $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively, with parameter m. This is the result obtained by Korotaev (1988) from a direct and involved evaluation of the asymptotic behavior of (2.3). However in that work the argument of the elliptic integral is taken to be the modulus $k \equiv \sqrt{m}$.

The wave power can also be calculated for other forcing functions. For the Gaussian forcing $q = q_0 \exp(-r^2/a^2)$, with Fourier transform $Q(k) = \pi q_0 a^2 \exp(-k^2 a^2/4)$, we may define the quantity $d \equiv a(\epsilon/U)^{\frac{1}{2}}$ to be a nondimensional measure of the spatial extent of the forcing. The wave power is then

$$P_{\mathrm{W}} = \frac{\rho \mathit{\Gamma}^2 (\epsilon U)^{\frac{1}{2}}}{4\pi} \int_{\pi/2}^{\pi/2+\alpha} \exp\left(\frac{d^2\cos\varphi}{4\cos(\varphi-\alpha)}\right) [-\cos(\varphi-\alpha)\cos\varphi]^{\frac{1}{2}} \,\mathrm{d}\varphi, \tag{3.12}$$

where $\mathit{\Gamma} = \pi q_0 a^2$ is again the circulation or integrated vorticity of the forcing function q.

Finally, the power can also be calculated for the Rankine forcing $q = q_0 H(a-r)$, where H is the Heaviside step function. With $\mathit{\Gamma} = \pi q_0 a^2$ as before, the Fourier transform of this forcing becomes $Q(k) = 2\mathit{\Gamma} J_1(ka)/ka$. Thus, also using the same definition of d as before, the power flux is

$$P_{\mathrm{W}} = \frac{\rho \mathit{\Gamma}^2 (\epsilon U)^{\frac{1}{2}}}{4\pi} \int_{\pi/2}^{\pi/2+\alpha} \frac{4}{d^2} J_1\left(\left[-\frac{\cos\varphi}{\cos(\varphi-\alpha)}\right]^{\frac{1}{2}} d\right)^2 \left(-\frac{\cos^3(\varphi-\alpha)}{\cos\varphi}\right)^{\frac{1}{2}} \mathrm{d}\varphi. \tag{3.13}$$

The three different powers are shown in figure 3. By symmetry, we need only consider α in the interval $(0, \pi)$. For small d, the curves are almost identical to the point vortex, as must be the case with the scaling adopted for $\mathit{\Gamma}$. The power becomes very small in the

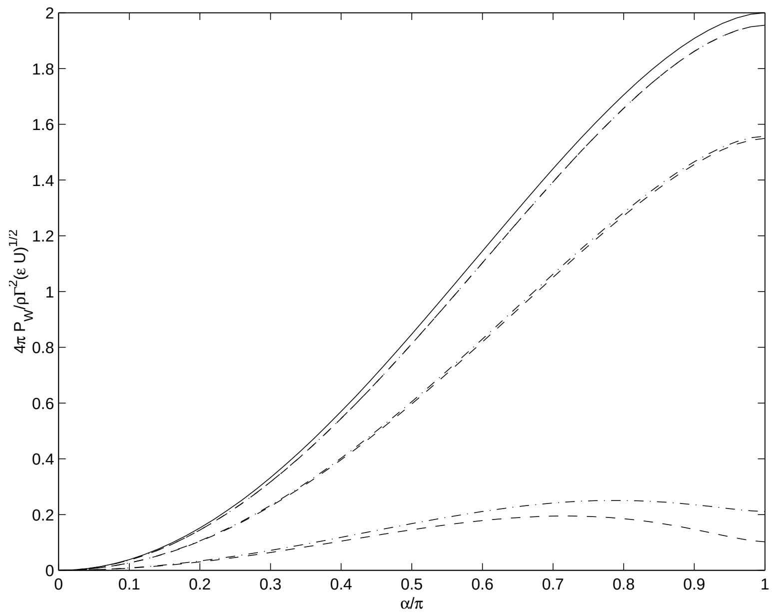

FIGURE 3. Nondimensional power $4\pi P_{\rm W}/\rho\Gamma^2(\epsilon U)^{1/2}$ against the angle α of the trajectory of the vortex for different kinds of forcing. The solid curve corresponds to point forcing, the dashed curves to Gaussian forcing, and the dot-dashed curves to Rankine forcing. For the Gaussian and Rankine forcing, the upper curves correspond to $d = 0.3$ (they are indistinguishable), the middle to $d = 1$ and the lower to $d = 3$.

opposite limit, corresponding to extensively distributed forcing in physical space, since $F \approx \delta(\boldsymbol{k})$ and no radiation is emitted from the point $k = 0$.

4. Momentum fluxes and force balances

4.1. *Wave fluxes and forces*

The linearized governing equations of motion on the beta-plane are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (f_0 + \epsilon y)\mathbf{z} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p, \tag{4.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4.1b}$$

where gradients now refer to spatial differentiation. These equations lead immediately to the flux conservation law

$$\frac{\partial W}{\partial t} = -\nabla \cdot (p\boldsymbol{u}) = -\nabla \cdot \boldsymbol{I}, \tag{4.2}$$

where the energy density is $W = \frac{1}{2}\rho|\boldsymbol{u}|^2$ and the wave flux is $\boldsymbol{I} = p\boldsymbol{u}$. There is hence an easily identifiable physical expression for the wave-energy flux $\boldsymbol{I}$, although there is no simple expression for it in terms of the streamfunction, ψ, alone. This definition of W leads to the spectral quantity $W_0 = k^2$ used above.

We can now use the power radiated by the vortex to define a wave force on the vortex. Power is force multiplied by velocity, and the velocity of the vortex is U. Of course this can only give the component of the force parallel to the displacement of the vortex, $F_{\|}$, since the component of the force perpendicular to the vortex can do no work. Hence the three preceding expressions (3.11), (3.12) and (3.13) for the power give the corresponding force on division by U. This is essentially the procedure underlying the definition of force in Korotaev (1988).

The wave-enstrophy density $Z \equiv \frac{1}{2}(\nabla^2\psi)^2$ could also be calculated using $W_0 = k^4$. In fact, the conservation equation for wave enstrophy can be derived from (2.8) and is given by

$$\frac{\partial Z}{\partial t} = -\nabla \cdot \left[\frac{\epsilon}{2}(\psi_x^2 - \psi_y^2, 2\psi_x\psi_y)\right]. \tag{4.3}$$

There is an obvious geometrical interpretation of the right-hand side of (4.3) as a vector making an angle of 2φ with the x-axis.

Once again, the integrated flux of this quantity could be turned into a quantity with the dimensions of force by diving the result by ϵ. This would give a function of α, with dimensions $\Gamma^2(U/\epsilon)^{1/2}$, as in the previous section. However, it is not clear what the physical meaning of this quantity would be: it depends on the angle of propagation of the vortex, but it is not related to force in the way that is usually understood, i.e. in the sense of being the ratio between rate of change of energy and velocity.

The flux of other quantities can also be calculated using the same procedure, although their physical significance is not clear, since they do not necessarily satisfy a conservation law.

4.2. *Pseudomomentum flux*

Stepanyants and Fabrikant (1992) compute a quantity which they call the force on a steadily translating point forcing. They define the quasiparticle number per unit volume N as $\rho|\psi|^2$ and the quasi-particle-number-density-flux by

$$S(\varphi) = N|\boldsymbol{c}_g|, \tag{4.4}$$

where $\boldsymbol{c}_g$ is the group velocity. The 'force' is then†

$$\boldsymbol{\Pi} = -\int_{\pi/2}^{\pi/2+\alpha} rS(\varphi)\boldsymbol{k}\,\mathrm{d}\varphi; \tag{4.5}$$

we shall not worry any further about the presence of the minus sign. Stepanyants and Fabrikant (1992) obtain the function $S(\varphi)$ from Korotaev (1988):

$$S(\varphi) = \frac{\rho\Gamma^2}{4\pi r}. \tag{4.6}$$

This is inconsistent with the results of the previous two sections which give, for the far field,

$$S(\varphi) = \frac{\rho\Gamma^2 U^{\frac{3}{2}}}{2\pi r\epsilon^{\frac{1}{2}}}\left(-\frac{\cos(\varphi-\alpha)}{\cos\varphi}\right)^{\frac{1}{2}} \frac{\Phi(\varphi,\alpha)}{|3+4\cos(2\varphi-2\alpha)+\cos(4\varphi-2\alpha)|}, \tag{4.7}$$

after using (2.18).

It is apparent from this expression that $S(\varphi)$ is not a constant function of its argument. Appendix B of Korotaev (1997) reproduces the working of Stepanyants and Fabrikant (1992) with however the insertion of an extra factor $\mathrm{d}\theta/\mathrm{d}\varphi$ into the appropriate integrals, where θ is the angular variable in physical space. This recovers the geometric terms of (3.1). Stepanyants and Fabrikant (1992) and Korotaev (1997) both then obtain the

† Stepanyants and Fabrikant (1992) integrate only over the range $\pi/2$ to $\pi/2+\alpha$ (their equation 6), since they require that the 'frequency', $\boldsymbol{U}\cdot\boldsymbol{k}$, be positive. This is appropriate if φ is taken to be the angular range over which radiation is present in the far field, but not if it is considered to be the angular coordinate on the wavenumber curve $P = 0$.

expression (3.11) for the power loss due to the waves for the parallel component of the rate of change of pseudomomentum.

We may note, however, that the quantity whose flux is computed is the pseudomomentum. One version of the argument goes as follows. For a linearized system, the pseudomomentum may be written (McIntyre, 1981)

$$\boldsymbol{q} = \frac{W\boldsymbol{k}}{\sigma}, \tag{4.8}$$

where $\sigma = \boldsymbol{U}\cdot\boldsymbol{k}$ is the Doppler-shifted frequency. Taking components parallel and perpendicular to the trajectory of the vortex gives

$$\boldsymbol{q} = (q_{\|}, q_{\perp}) = \frac{W}{U}(1, \tan(\varphi - \alpha)). \tag{4.9}$$

Following the arguments of § 3.1, i.e. considering the geometric terms carefully, we may calculate the rate of change of the two components of pseudomomentum. The rate of change parallel to the trajectory of the vortex comes from integrating the same flux density as in § 3.1 and is hence $G_{\|} = P_{\mathrm{W}}/U$ again, as was found by Stepanyants and Fabrikant (1992) for the point forcing case. The rate of change perpendicular to the trajectory of the vortex differs only by the tangent term and is hence

$$G_{\perp} = 4\pi^3\rho(\epsilon/U)^{\frac{1}{2}} \int_{\pi/2}^{\pi/2+\alpha} |Q|^2 \left(-\frac{\sin^2(\varphi-\alpha)\cos\varphi}{\cos(\varphi-\alpha)}\right)^{\frac{1}{2}} \mathrm{d}\varphi. \tag{4.10}$$

For point forcing, this becomes

$$G_{\perp} = \frac{\rho\Gamma^2(\epsilon/U)^{\frac{1}{2}}}{4\pi} K(\sin^2\tfrac{1}{2}\alpha)\sin\alpha, \tag{4.11}$$

which again agrees with Stepanyants and Fabrikant (1992). The rate of change of pseudomomentum can also be found for Gaussian and Rankine forcing by computing the appropriate integrals.

Figure 4 shows the form of $G_{\perp}$ for the same types of forcing as before. The amplitude of $G_{\perp}$ decreases with increasing d, but we also note that the sign of $G_{\perp}$ changes for α near π when d is large enough. The sign change occurs around $d = 1.6$.

This calculation leads to a definition for the objects $G_{\|}$ and $G_{\perp}$ which have the dimensions of force. The value of $G_{\|}$ is the same as P_{W}/U, as one would expect, since the two quantities are essentially the same. The perpendicular component, $P_{\perp}$, however, cannot be obtained by energy arguments as argued previously, and must hence be viewed as a rate of change of pseudomomentum.

However we are free to choose a different W_0 in the definition of W and hence obtain different forces. If we take $W_0 = k^4$, what will have been calculated is no longer a rate of change of pseudomomentum, but rather a rate of change of what might be called pseudovorticity, given the form of equations (4.1*a*) and (4.3) and the respective natures of the quantities which are squared on the left-hand side of these equations.

5. Conclusion

We have computed the far-field amplitude of the Rossby waves created by an arbitrary symmetric propagating disturbance using the theory of Lighthill (1978). The result is quite simple and enables the flux to infinity of quantities such as wave energy to be computed. The power radiated has been calculated and, as suggested in Korotaev (1988),

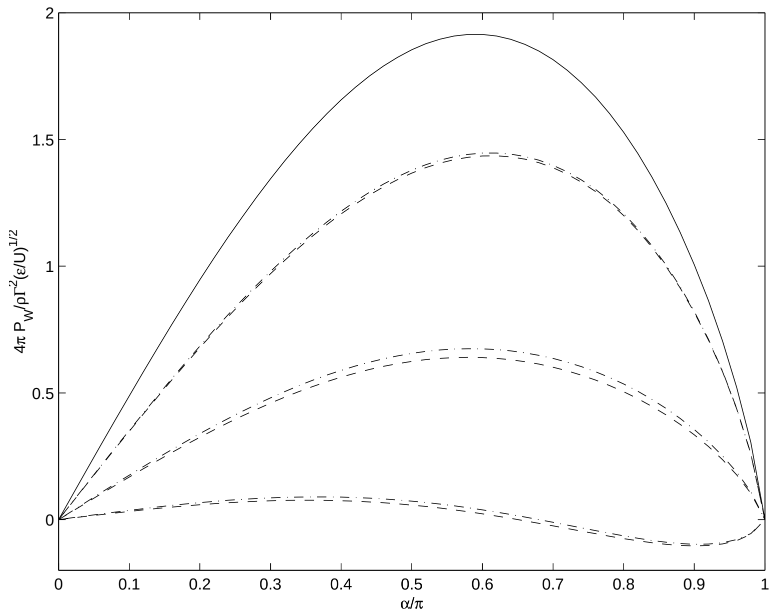

FIGURE 4. Nondimensional pseudomomentum flux perpendicular to the vortex trajectory $4\pi G_\perp P_{\mathrm{W}}/\rho\Gamma^2(\epsilon/U)^{1/2}$ against the angle α of the trajectory of the vortex for different kinds of forcing. The solid curve corresponds to point forcing, the dashed curves to Gaussian forcing, and the dot-dashed curves to Rankine forcing. For the Gaussian and Rankine forcing, the upper line corresponds to $d = 0.3$, the middle to $d = 1$ and the lower to $d = 3$.

can be used to find a quantity which has the dimensions of force. This may be viewed as a force applied on the vortex due to the wave radiation.

The rate of change of pseudomomentum has also been calculated. Its component parallel to the trajectory of the vortex is essentially the power radiated away by the waves in this system. Its perpendicular component has no interpretation in terms of power; it may also change sign for vortices propagating nearly towards the East for widely distributed forcing. It is also possible to compute further quantities such as the rate of change of the enstrophy density, which have the dimensions of power, but whose significance is not immediately apparent.

The results of Stepanyants and Fabrikant (1992) and Korotaev (1997) are consistent with this approach. The present results, however, are more general since they apply to any form of radially propagating disturbance.

There are however some potential objections to the result for the 'force' $(G_\|, G_\perp)$ obtained by consideration of the pseudomomentum:

(a) The physical relevance of the result (4.10) is not immediately apparent (irrespective of whether it corresponds to the quantity that is normally called force). The 'photon analogy' (McIntyre, 1993) says that 'the rate at which momentum is transported from location A to location B, when a wave packet is generated at A, and dissipated at B, is the same as if (a) the fluid were absent, and (b) the wave packet had momentum equal to its pseudomomentum'. However, this is just an analogy, and the actual details need to be checked for different physical systems. These calculations usually require considerations of the $O(a^2)$ behaviour of the system, which has not been attempted here. Hence, while it is undeniable that the waves radiate energy, it is not clear from the preceding calculation that (4.10) has any meaning beyond its definition as a rate of change of pseudomomentum. The same proviso would apply to the rate of change of any pseudovorticity quantity.

(*b*) Conservation of pseudomomentum is intimately connected to invariance of the medium to translation (McIntyre, 1993). Here, however, it is the governing vorticity equation (2.8) which is invariant under translation, while the Coriolis force term $\beta y \boldsymbol{u}$ in the momentum equation (4.1*a*) changes under translation in the y-direction. One could therefore argue that it is the pseudovorticity which should be associated with a wave conservation law and thus that this second wave energy should be the one with a natural interpretation in terms of force.

(*c*) There is no actual vortex in the system, just a moving forcing. While this approach has been used, for example, in internal gravity wave problems (e.g., Gorodtsov, 1994) and is undoubtedly a useful conceptual tool in understanding the response of Rossby waves to a moving disturbance, it is not clear what the relation is between this approach and the physically self-consistent problem of the evolution of a radial vorticity anomaly under the full equations of motion. Papers such as Llewellyn Smith (1997) and McDonald (1998) show that constant translational motion of an unchanging vortex is not a realistic assumption for large times, a result which goes back to Flierl et al. (1983).

(*d*) Even if this forcing is considered to be a vortex, there is no physical object to support pressure, and hence feel a force.† It is therefore unclear what the term force means in this context.

It is revealing that Lighthill (1978) does not mention the word "force" in the discussion about wave power. Nevertheless, the present results are useful in quantifying the energy radiated by waves on the beta-plane as a function of the trajectory of the steadily propagating vortex. This may be of some use in understanding the response in a diagnostic fashion of an evolving nonlinear disturbance as a function of its instantaneous response. As foreseen by Lighthill (1967), the behavior of the linear wave system reveals much about more complicated physical systems, and these results may provide a better understanding of forced waves on the beta-plane.

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† Dolina and Ostrovsky (1990) considered the instability of an eddy due to radiation. However, the eddy is explicitly modelled as a cylinder, which can support pressure forces, surrounding a line vortex.

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