A new dual to the Gács-Körner common information defined via the Gray-Wyner system

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September 30, 2010
Outline

- Gray-Wyner system
- Gács-Körner common information
- Main Results - New Dual
- A side-information problem
The Gray-Wyner system

$$(X_i, Y_i) \sim Q(x, y) \text{ i.i.d. } 1 \leq i \leq n$$
Trivial Outer Bound
Trivial Outer Bound

\[ R_0, R_1, R_2 \geq 0 \]
\[ R_0 + R_1 \geq H(X) \]
\[ R_0 + R_2 \geq H(Y) \]
\[ R_0 + R_1 + R_2 \geq H(X, Y) \]
Trivial Outer Bound

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Figure: [Gray-Wyner ’74]
Trivial Outer Bound

Figure: Point E

Figure: [Gray-Wyner ’74]
Trivial Outer Bound

Point E is not achievable!

Figure: [Gray-Wyner '74]
Gray-Wyner Theorem

Theorem [Gray-Wyner ’74]

\[ \mathcal{R} = \text{closure of} \]
\[ \bigcup_{p(w|x,y)} \{ R_0 \geq I(X,Y;W), R_1 \geq H(X|W), R_2 \geq H(Y|W) \} \]

- \[ |\mathcal{W}| \leq |\mathcal{X}| \cdot |\mathcal{Y}| + 2 \]
Gács-Körner common information
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\((X_i, Y_i) \sim Q(x, y)\) i.i.d. \(1 \leq i \leq n\)
Gács-Körner common information

\((X_i, Y_i) \sim Q(x, y)\) i.i.d. \(1 \leq i \leq n\)

\[ X^n \xrightarrow{\text{Function 1}} U_n \]

\[ Y^n \xrightarrow{\text{Function 2}} V_n \]
Gács-Körner common information

\((X_i, Y_i) \sim Q(x, y) \text{ i.i.d. } 1 \leq i \leq n\)

\[ \Pr(U_n \neq V_n) \leq \epsilon_n, \ \epsilon_n \to 0. \]
Gács-Körner common information

\((X_i, Y_i) \sim Q(x, y)\) i.i.d. \(1 \leq i \leq n\)

\[
\begin{align*}
X^n &\xrightarrow{\text{Function 1}} U_n \\
Y^n &\xrightarrow{\text{Function 2}} V_n
\end{align*}
\]

\[\Pr (U_n \neq V_n) \leq \epsilon_n, \ \epsilon_n \to 0.\]

How large can \(\frac{1}{n} H(U_n)\) be?
Gács-Körner common information

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**Theorem [Gács-Körner ’72]**

\[K(X; Y) = \max_{W = f(X) = g(Y)} H(W)\]
Gács-Körner in Gray-Wyner
Gács-Körner in Gray-Wyner

Not Achievable
Gács-Körner in Gray-Wyner

Not Achievable
Gács-Körner in Gray-Wyner

Not Achievable

Achievable
A dual to Gács-Körner common information

Figure: [Gray-Wyner ’74]
A dual to Gács-Körner common information

For what values of $R_0$ is the outer bound in $(R_1, R_2)$ tight?

Figure: [Gray-Wyner '74]
A dual to Gács-Körner common information

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Ans: For $R_0 \in [0, \alpha] \cup [\beta, \infty)$
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$$\alpha = K(X; Y)$$

Figure: [Gray-Wyner '74]
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Ans: For $R_0 \in [0, \alpha] \cup [\beta, \infty)$

\[ \alpha = K(X; Y) \]

\[ \beta = U(X; Y), \text{ the proposed dual} \]

Figure: [Gray-Wyner '74]
A dual to Gács-Körner common information

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Ans: For $R_0 \in [0, \alpha] \cup [\beta, \infty)$

$$\alpha = K(X; Y)$$

$$\beta = U(X; Y), \text{ the proposed dual}$$

$$K(X; Y) \leq I(X; Y) \leq U(X; Y)$$
Main Results
Main Results

Theorem

\[ K(X; Y) = \sup_{W - X - Y} I(X, Y; W) \]

\[ X - Y - W \]
Main Results

Theorem

\[ K(X; Y) = \sup_{W-X-Y, X-Y-W} I(X, Y; W) \]

Theorem

\[ U(X; Y) = \max \left\{ \inf_{X-Y-W, X-W-Y} I(X, Y; W), \inf_{W-X-Y, X-W-Y} I(X, Y; W) \right\} \]
Closer look

\[ K(X; Y) = \sup_{W \rightarrow X \rightarrow Y} I(X, Y; W) \]

\[ \inf_{X \rightarrow Y \rightarrow W} I(X, Y; W) \]
Closer look

\[ K(X; Y) = \sup_{W-X-Y; X-Y-W} I(X, Y; W) \]

\[ \inf_{X-Y-W; X-W-Y} I(X, Y; W) \]

\[ W = f(X) = g(Y) \]

\[ f(X) - X - Y \]

\[ X - Y - g(Y) \]
Closer look

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\[ f(X) - X - Y \]
\[ X - Y - g(Y) \]

\[ \Phi_Y : \mathcal{Y} \mapsto \mathbb{P}(\mathcal{X}), \quad y \mapsto [p(x|y)] \]
Closer look

\[ K(X; Y) = \sup_{W - X - Y} I(X, Y; W) \]

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\[ f(X) - X - Y \]

\[ X - Y - g(Y) \]

\[ \inf_{X - Y - W} I(X, Y; W) \]

\[ \Phi_Y : \mathcal{Y} \mapsto \mathbb{P}(\mathcal{X}), \ y \mapsto [p(x|y)] \]

\[ X - Y - \Phi_Y \]
Closer look

\[
K(X; Y) = \sup_{W - X - Y} I(X, Y; W) - \inf_{X - Y - W} I(X, Y; W)
\]

\[
W = f(X) = g(Y)
\]

\[
f(X) - X - Y
\]

\[
X - Y - g(Y)
\]

\[
\Phi_Y : \mathcal{Y} \mapsto \mathbb{P}(\mathcal{X}), \quad y \mapsto [p(x|y)]
\]

\[
X - Y - \Phi_Y
\]

\[
X - \Phi_Y - Y
\]
Closer look

\[ K(X;Y) = \sup_{W - X - Y} I(X,Y;W) \]

\[ \inf_{X - Y - W} I(X,Y;W) \]

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\[ f(X) - X - Y \]

\[ X - Y - \Phi_Y \]

\[ X - Y - g(Y) \]

\[ X - \Phi_Y - Y \]

Explicit Characterization

\[ \inf_{X - Y - W} I(X,Y;W) = H(\Phi_Y) \]
Side-information problem

\[(X_i, Y_i) \sim Q(x, y) \text{ i.i.d. } 1 \leq i \leq n\]
Side-information problem
Side-information problem
Side-information problem

\[ X^n \xrightarrow{} \text{Encoder 1} \xrightarrow{H(X|Y)} \text{Decoder} \xrightarrow{} \hat{X}^n \]

\[ Y^n \xrightarrow{} \text{Encoder 2} \xrightarrow{H(Y)} \text{Decoder} \]

Diagram with axes labeled \( R_1 \) and \( R_2 \), showing regions and points labeled \( H(X|Y) \) and \( H(Y) \).
Side-information problem

\[ X^n \xrightarrow{\text{Encoder 1}} H(X|Y) \xrightarrow{\text{Decoder}} \hat{X}^n \]

\[ Y^n \xrightarrow{\text{Encoder 2}} H(\Phi_Y) \]

\[ R_2 \]

\[ H(Y) \]

\[ H(\Phi_Y) \]

\[ R_1 \]

\[ H(X|Y) \]

\[ H(X) \]
Thank you!
Questions?