A new dual to the Gács-Körner common information defined via the Gray-Wyner system

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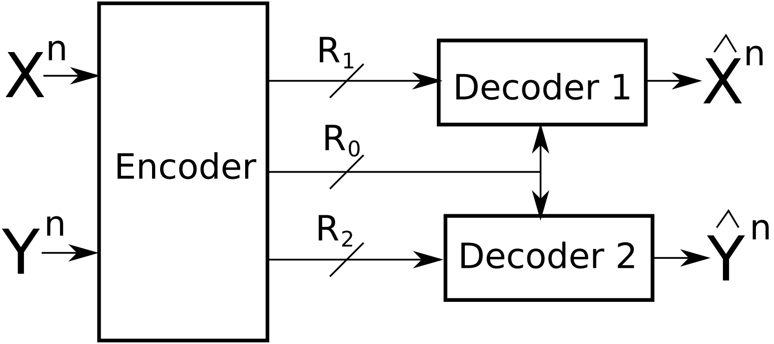
September 30, 2010

Outline

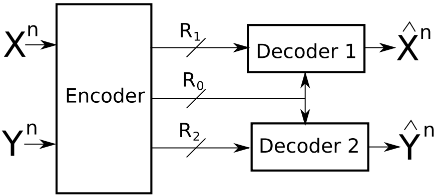
- Gray-Wyner system
- Gács-Körner common information
- Main Results - New Dual
- A side-information problem

The Gray-Wyner system

$$(X_i, Y_i) \sim Q(x, y) \text{ i.i.d. } 1 \leq i \leq n$$



Trivial Outer Bound



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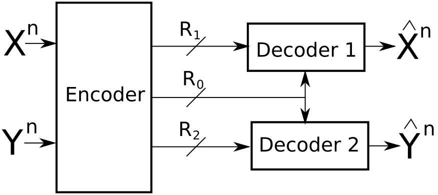

$$
\begin{aligned}
R_0, R_1, R_2 &\geq 0 \\
R_0 + R_1 &\geq H(X) \\
R_0 + R_2 &\geq H(Y) \\
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\end{aligned}
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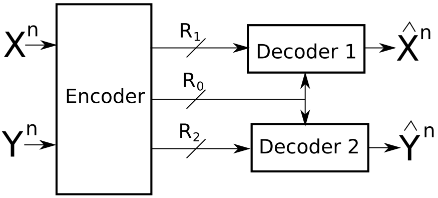

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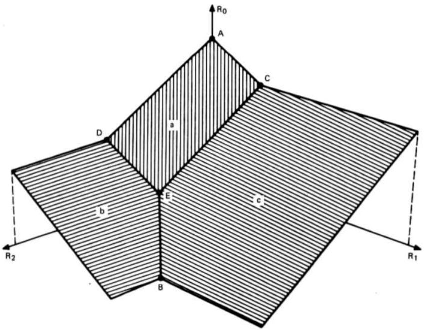

Figure: [Gray-Wyner '74]

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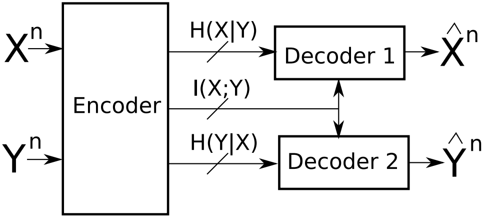

Figure: Point E

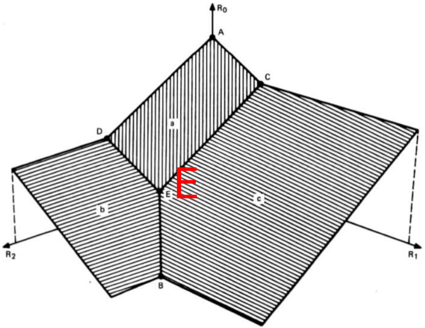

Figure: [Gray-Wyner '74]

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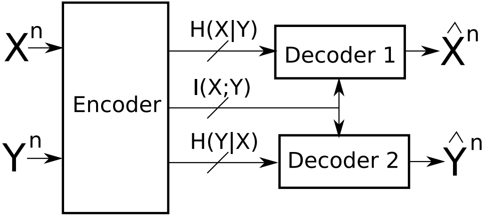

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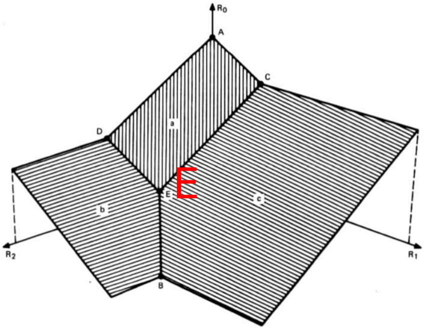

Figure: [Gray-Wyner '74]

Point E is not achievable!

Gray-Wyner Theorem

Theorem [Gray-Wyner '74]

$\mathcal{R} =$ closure of

$\cup_{p(w|x,y)}\{R_0 \geq I(X,Y;W), R_1 \geq H(X|W), R_2 \geq H(Y|W)\}$

- $|\mathcal{W}| \leq |\mathcal{X}| \cdot |\mathcal{Y}| + 2$

Gács-Körner common information

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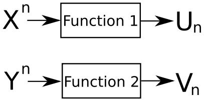

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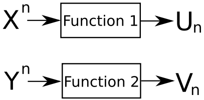

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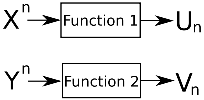

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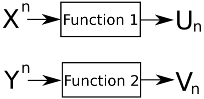

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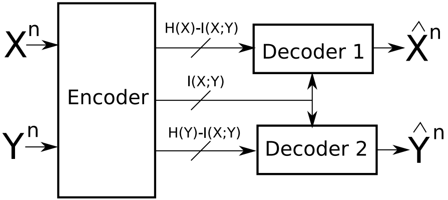
How large can $\frac{1}{n}H(U_n)$ be?

Theorem [Gács-Körner '72]

$$K(X;Y) = \max_{W=f(X)=g(Y)} H(W)$$

Gács-Körner in Gray-Wyner



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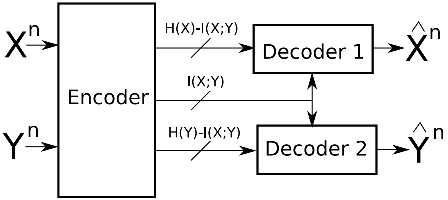

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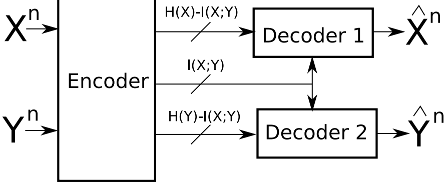

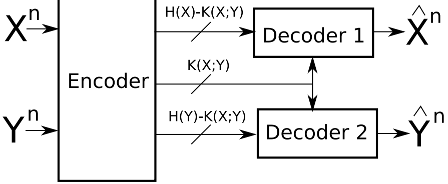

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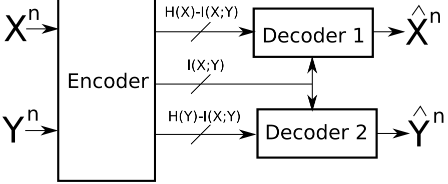

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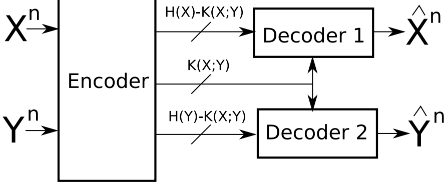

Achievable

A dual to Gács-Körner common information

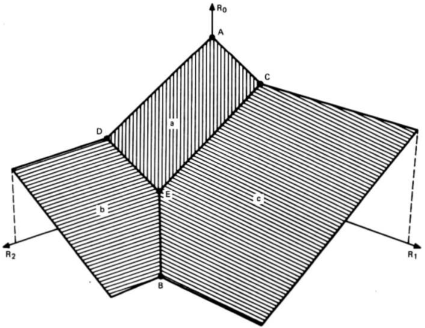

Figure: [Gray-Wyner '74]

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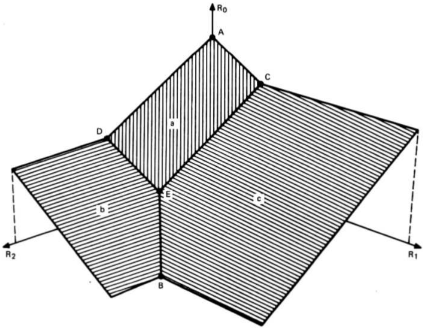

Figure: [Gray-Wyner '74]

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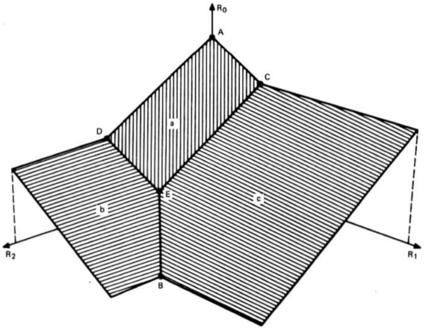

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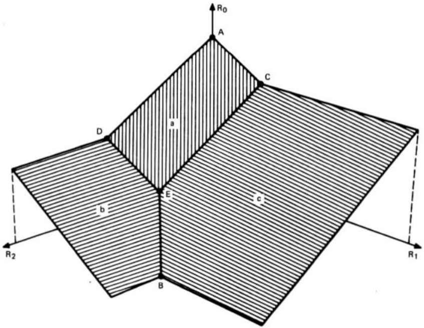

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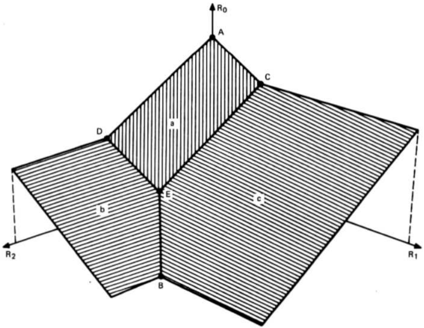

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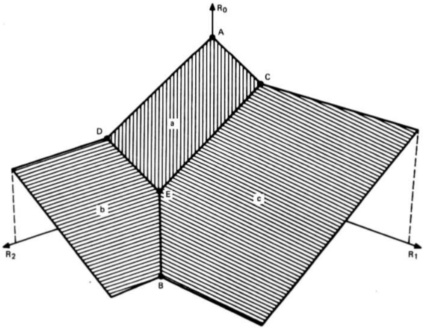

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$$K(X;Y) \leq I(X;Y) \leq U(X;Y)$$

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$$U(X;Y) = \max\left\{ \inf_{\substack{X-Y-W \\ X-W-Y}} I(X,Y;W), \inf_{\substack{W-X-Y \\ X-W-Y}} I(X,Y;W) \right\}$$

Closer look

$$K(X;Y) = \sup_{\substack{W-X-Y \\ X-Y-W}} I(X,Y;W)$$

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$$W = f(X) = g(Y)$$

$$f(X) - X - Y$$

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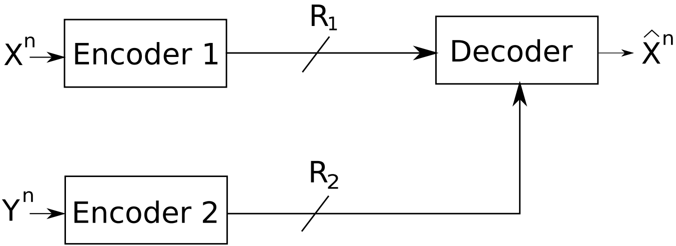
Explicit Characterization

$$\inf_{\substack{X - Y - W \\ X - W - Y}} I(X,Y;W) = H(\Phi_Y)$$

Side-information problem

$$(X_i, Y_i) \sim Q(x, y) \text{ i.i.d. } 1 \leq i \leq n$$



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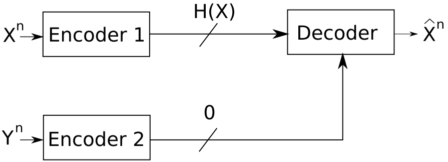

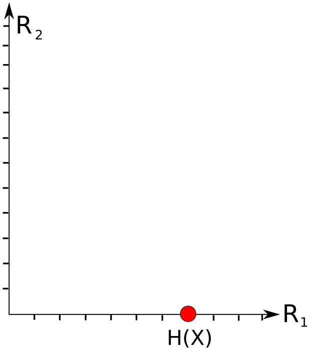

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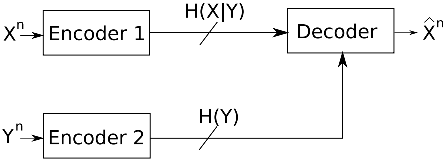

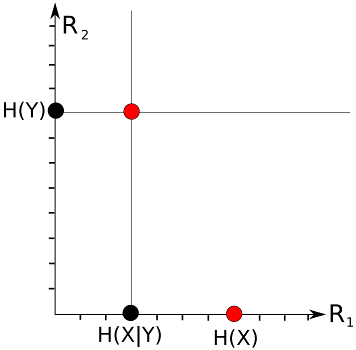

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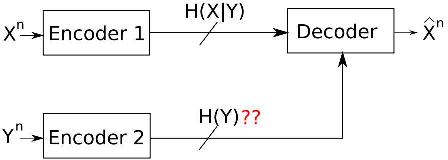

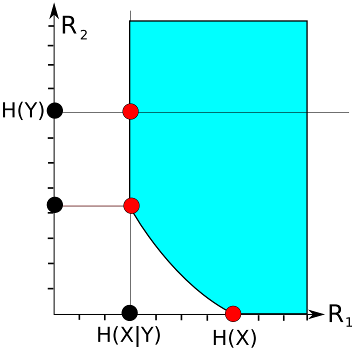

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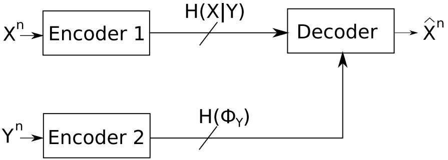

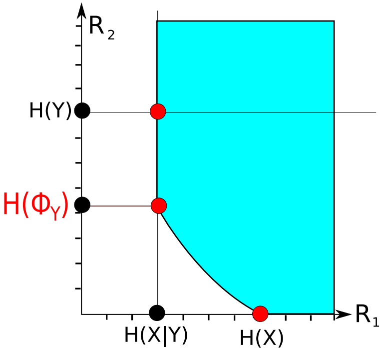

Thank you!
Questions?