Non-Interactive Simulation of Joint Distributions: Maximal Correlation and the Hypercontractivity Ribbon

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UC Berkeley

October 3, 2012

Wireless Foundations

Non-interactive Simulation

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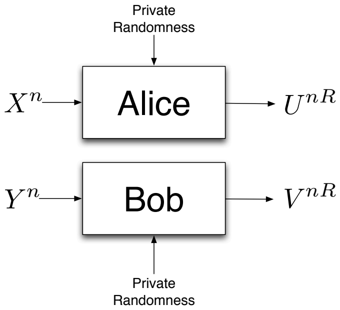

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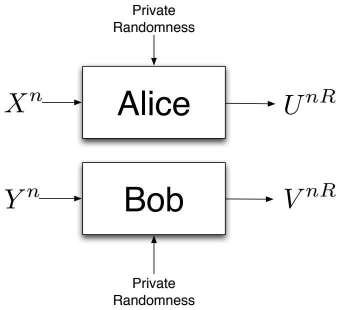

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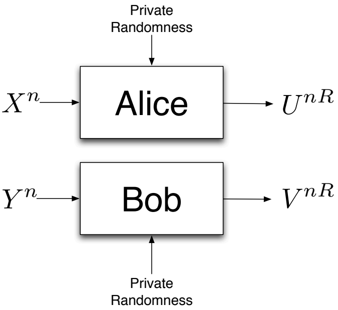

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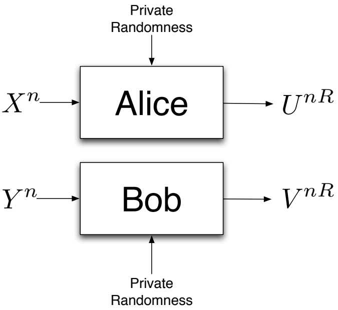

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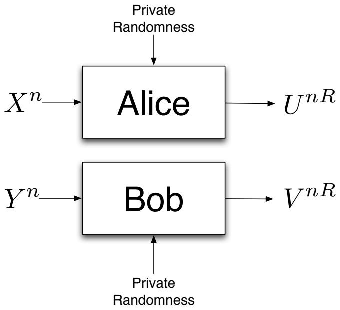

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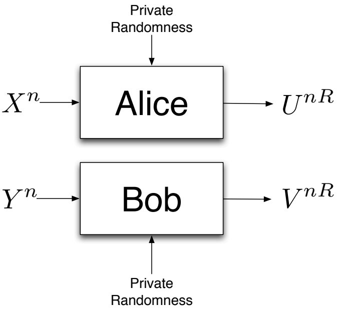

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- When $X = Y =$ fair coin flip, $\frac{1}{R^*} = C(U;V)$ [Wyner '75]
- Seems hard in general; even to determine if $R^* > 0$.

Single-sample simulation - setup for this talk

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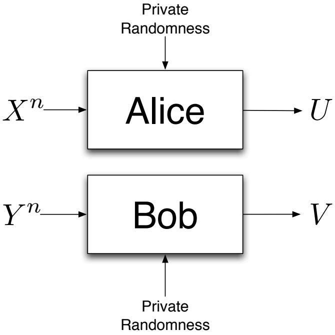

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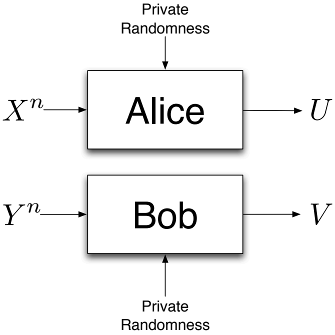

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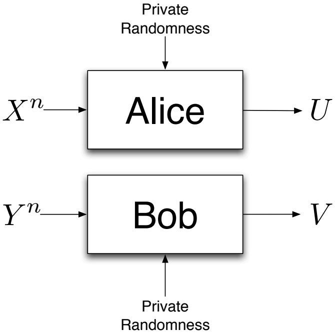

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- Determine if *one sample* with distribution $\approx Q(u,v)$ can be simulated
- Possible iff $Q(u,v)$ belongs to closure of marginals of $U - X^n - Y^n - V$ for some n

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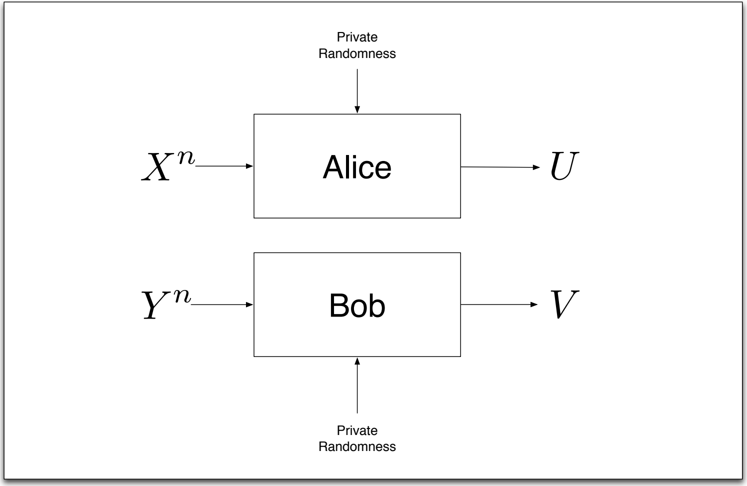

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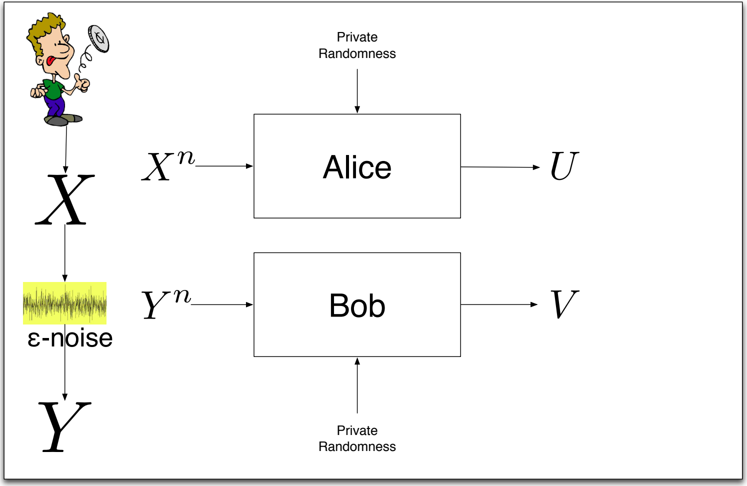

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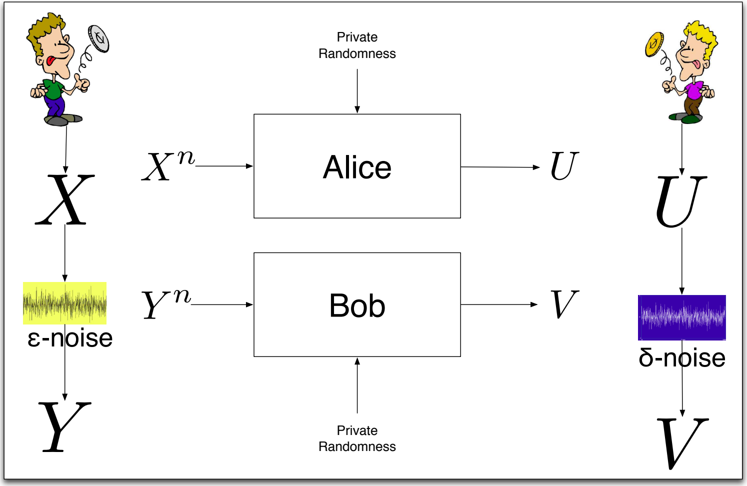

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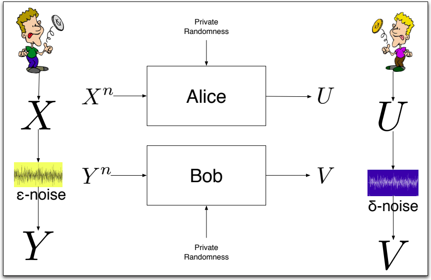
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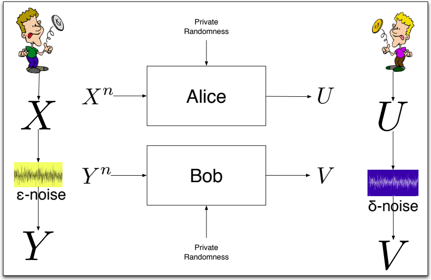

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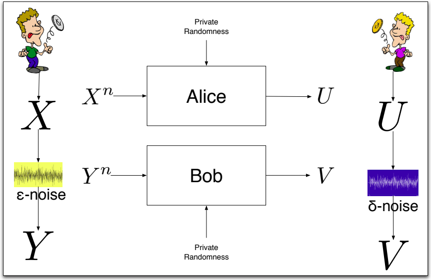

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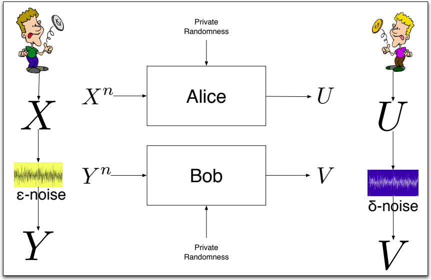

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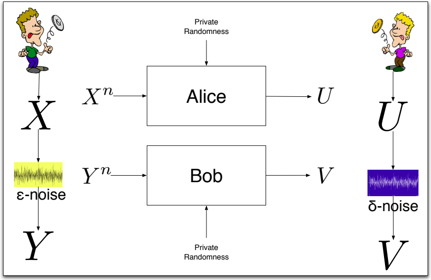

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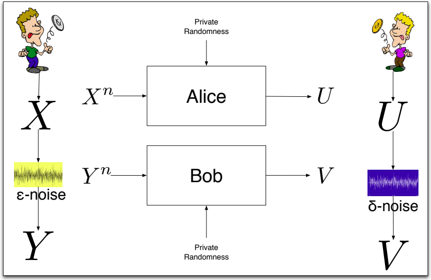

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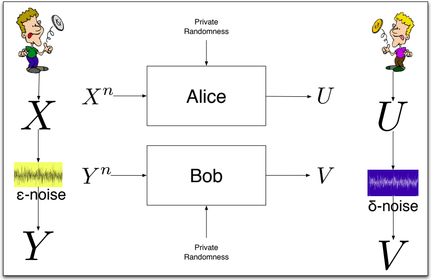
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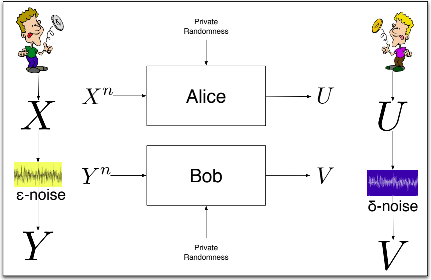

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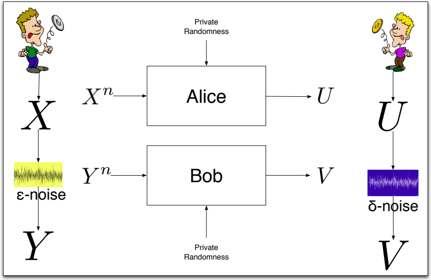

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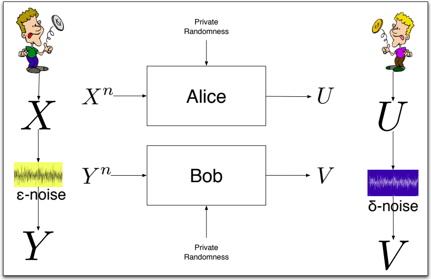

- Use Data Processing Inequality: $I(X^n;Y^n) \geq I(U;V)$
- Not useful because $I(X^n;Y^n) = nI(X;Y)$ grows with n
- Use a different measure of correlation

Maximal correlation $\rho(X;Y)$

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over all mean zero, unit variance

$$f(X),\ g(Y)$$

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Tensorization [Witsenhausen '75]

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Data Processing Inequality

If $U = \phi(X^n), V = \psi(Y^n)$, then

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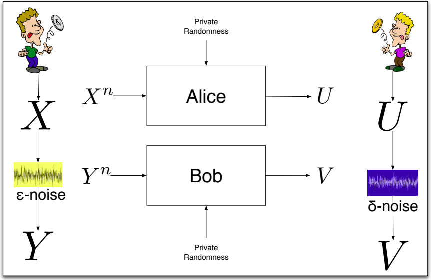
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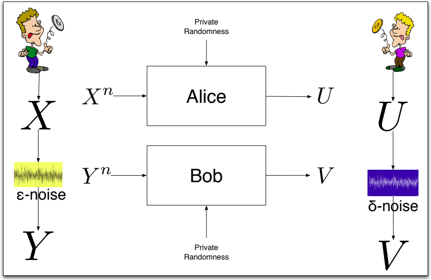

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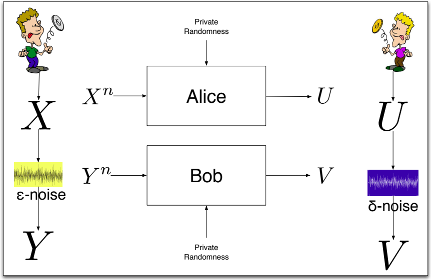

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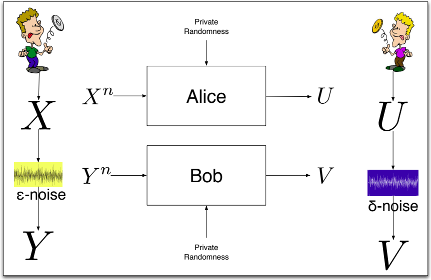

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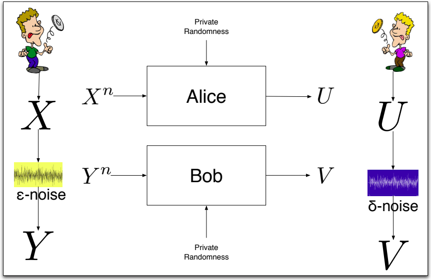

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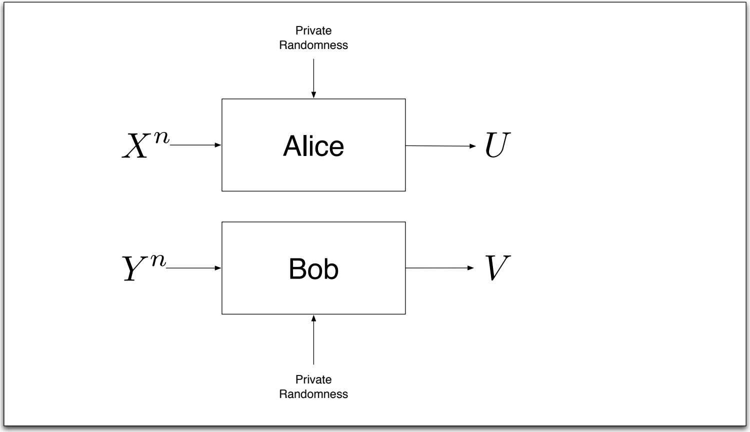

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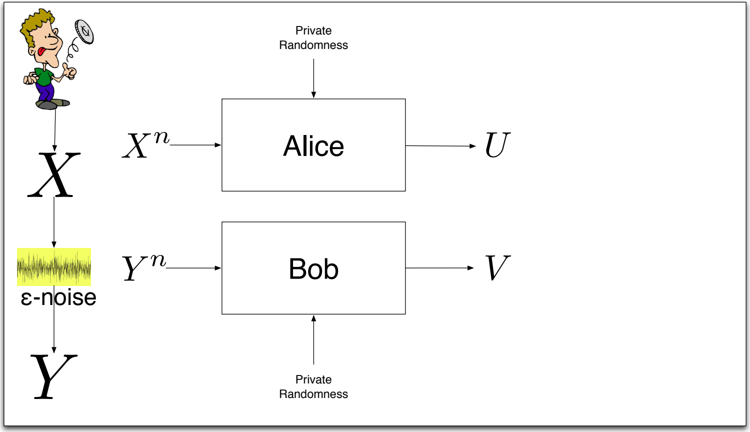
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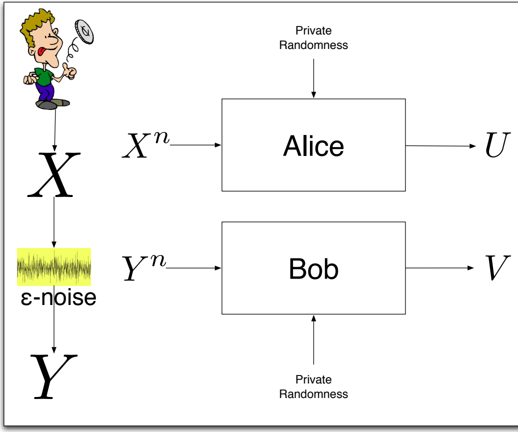

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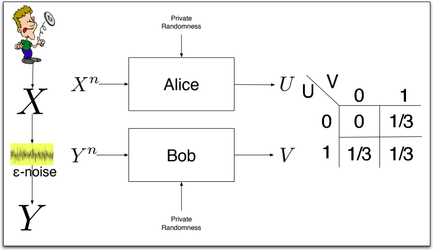

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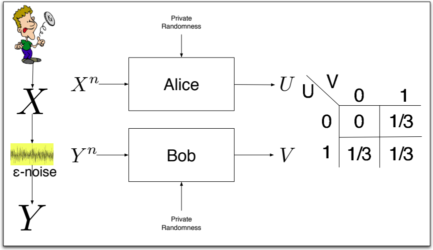

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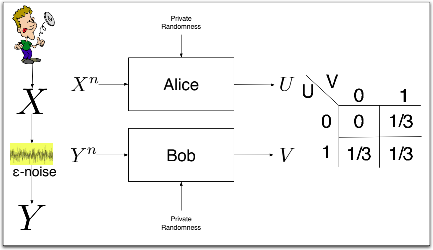

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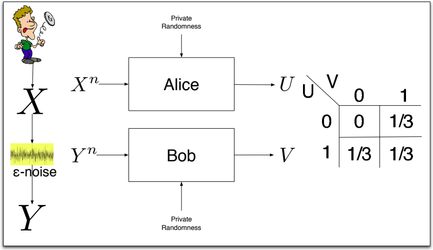

- $\rho(X;Y) \geq \rho(U;V)$ gives impossibility for $\epsilon \geq \frac{1}{4}$
- Is simulation possible for small enough ϵ?
- Impossible for any $\epsilon > 0$: shown using Reverse Hypercontractivity

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Operator T defined by $Tg(x) := \mathbb{E}[g(Y)|X = x]$

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[Bonami '70 - Nelson '73 - Gross '75 - Beckner '75], [Borell '82]

Both kinds of inequalities tensorize!

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Define $(p, q) \in \mathcal{R}_{X;Y} \subset \mathbb{R}^2$ if corresponding hypercontractive or reverse hypercontractive inequalities hold

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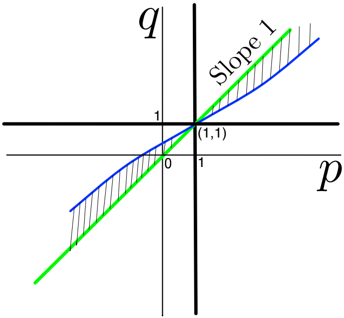

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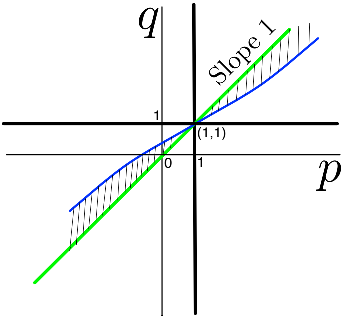
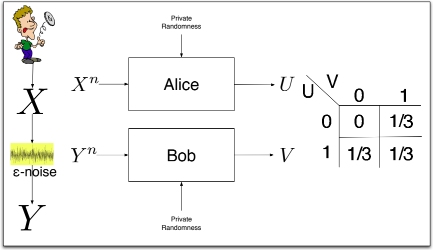


Main Result I

If simulation of (U, V) from (X, Y) is possible, then

$$\mathcal{R}_{X;Y} \subseteq \mathcal{R}_{U;V}$$

Recall Example 2



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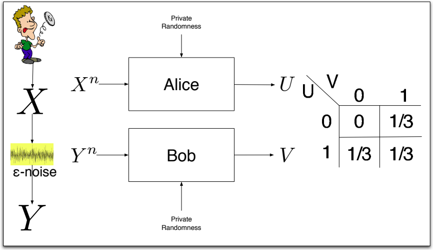

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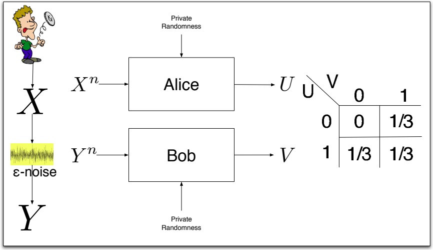

- Does $\mathcal{R}_{X;Y}$ contain $\mathcal{R}_{U;V}$?
- For any $\epsilon > 0$?

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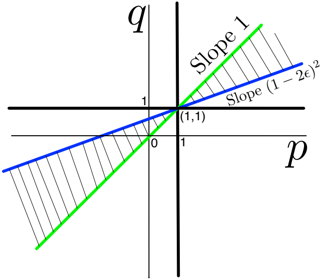

Figure: $\mathcal{R}_{X;Y}$

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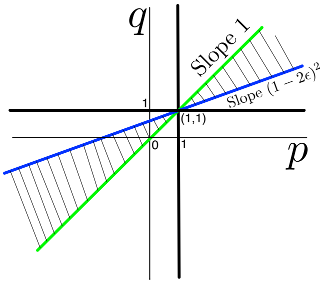

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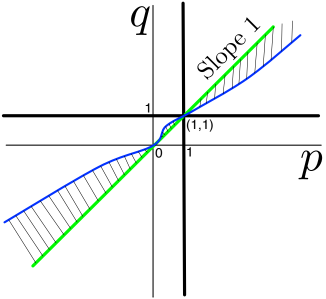

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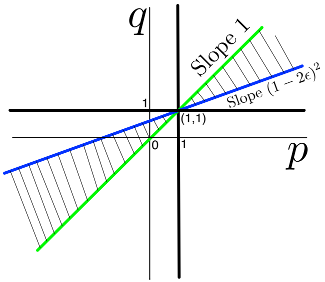

Figure: $\mathcal{R}_{X;Y}$

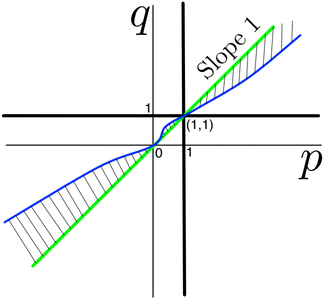

Figure: $\mathcal{R}_{U;V}$

Impossible to simulate for any $\epsilon > 0$

Answer to Example 2

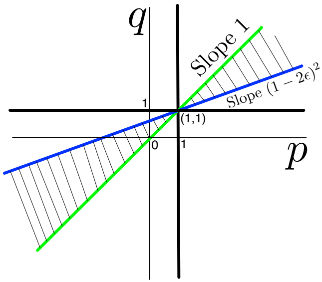

Figure: $\mathcal{R}_{X;Y}$

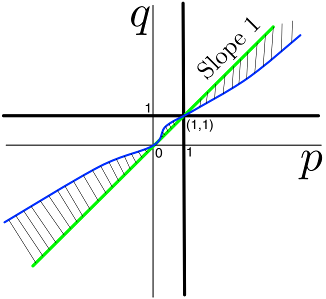

Figure: $\mathcal{R}_{U;V}$

Impossible to simulate for any $\epsilon > 0$

Reverse Hypercontractivity for the win!

Our Main Result II

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Geometric meaning of $\rho(X;Y)$ in $\mathcal{R}_{X;Y}$

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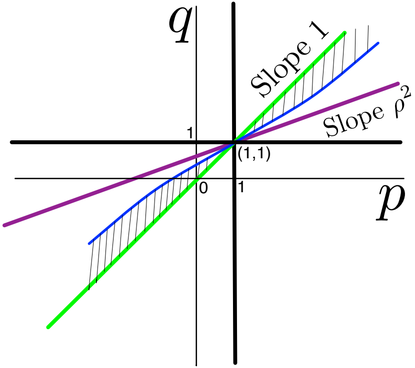

Our Main Result III

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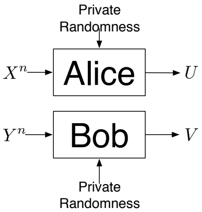
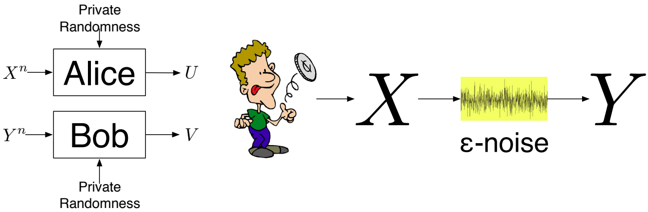
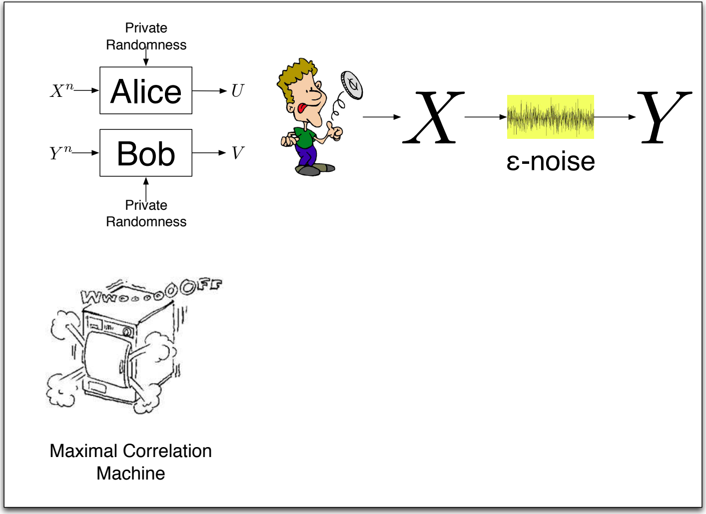


Our Main Result III



Our Main Result III



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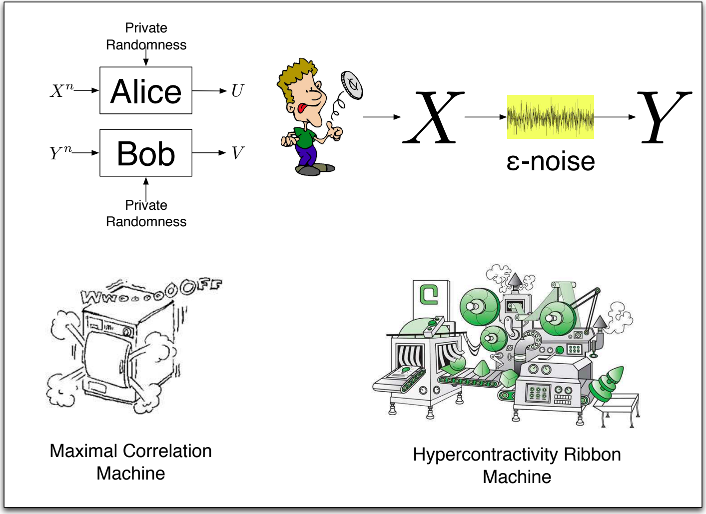
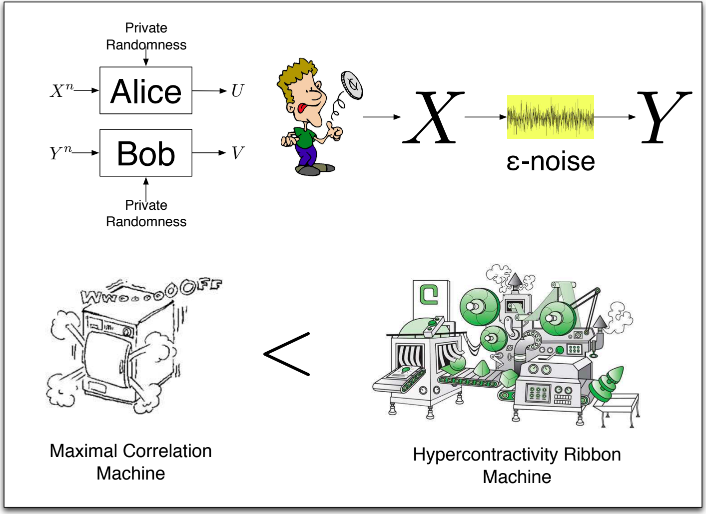


Our Main Result III



Conclusion

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    - What other quantities tensorize? Eg. Gowtham Kumar's Binary Rényi correlation
    - Other techniques to understand the simulation problem?