

Two-unicast is hard

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Abstract—Consider the k -unicast network coding problem over an acyclic wireline network: Given a rate vector k -tuple, determine whether the network of interest can support k unicast flows with those rates. It is well known that the one-unicast problem is easy and that it is solved by the celebrated max-flow min-cut theorem. The hardness of k -unicast problems with small k has been an open problem. We show that the two-unicast problem is as hard as any k -unicast problem for $k \geq 3$. Our result suggests that the difficulty of a network coding instance is related more to the magnitude of the rates in the rate tuple than to the number of unicast sessions. As a consequence of our result and other well-known results, we show that linear coding is insufficient to achieve capacity, and non-Shannon inequalities are necessary for characterizing capacity, even for two-unicast networks.

I. INTRODUCTION

Wireline networks are a simple class of general stochastic networks. These networks have links between nodes that are unidirectional, orthogonal and noise-free, and hence, are a simplification of real-world channels, with aspects such as broadcast, superposition, interference, noise absent. Understanding the capacity regions of directed wireline networks is the first step towards understanding the capacity of general networks.

The seminal work of Ahlswede-Cai-Li-Yeung [1] introduced the problem of communicating a common message from one source to many destinations and showed that coding operations in the network are necessary to achieve capacity. Furthermore, they characterized the capacity of the network by the cutset bound.

However, the problem of characterizing the capacity region of a multiple-unicast network with multiple messages - each demanded by a single destination - has been shown to be a difficult problem on many accounts. (A more general traffic pattern where some messages are required to be decoded at multiple destinations can be converted to an equivalent multiple-unicast network using the construction in [2].) [3] showed that a simple class of network coding schemes - linear network coding - while sufficient to achieve capacity for the multicast problem, cannot achieve capacity for a general k -unicast network when $k \geq 10$. [4] showed that for $k \geq 6$, outer

bounds obtained by Shannon-type information inequalities don't suffice to characterize the capacity region of k -unicast networks in general and so-called non-Shannon information inequalities are necessary. Indeed, [5] showed that solving the general network coding problem requires knowledge of the closure of the entire entropic cone $\bar{\Gamma}_n^*$.

The counterexample networks in the above series of works all have either many different sources or many different destinations. A natural question to ask is whether the difficulty of the problem stems from this. A candidate for the simplest unsolved problem is the two-unicast problem - the problem of communication between two sources and two destinations, with each source having a message for its own destination. The only complete characterization result in the literature dealing with the two-unicast network capacity is [6] which characterizes the necessary and sufficient condition for achieving $(1, 1)$ in a two-unicast network with all links having integer capacities.

The contrast of this success with the $(1, 1)$ rate pair in two-unicast networks against the intractability of the general k -unicast problem [3]–[5], is not entirely surprising since many different problems enjoy a simplicity with 2 users that is not shared by the corresponding problems with 3 or more users. For instance, in two-unicast *undirected* networks, the cutset bound is tight while this is not the case for three-unicast undirected networks [7]. The capacity of two-user interference channels is known to within one bit [8] but no such result is known for three-user interference channels. Although the two-unicast problem for general integer rates (R_1, R_2) remains open, it is often believed that the two-unicast problem enjoys a similar simplicity as other two-user information theoretic problems.

There are many existing results that aim to characterize the general achievable rate region for the two-unicast problem (not limited to the $(1, 1)$ case in [6]) and/or the k -unicast problem with small k . For example, [9], [10], [11] study capacity of two-unicast, three-unicast and k -unicast networks respectively, from a source-destination cut-based analysis. The authors of [12] present an edge-

reduction lemma using which they compute an achievable region for two-unicast networks. In a subsequent work [13], they show that the Generalized Network Sharing bound defined in [14] gives necessary and sufficient conditions for achievability of the $(N, 1)$ rate pair in a special class of two-unicast networks (networks with Z-connectivity and satisfying certain richness conditions). Unfortunately, none of the above results is able to fully characterize the capacity region for general two-unicast networks even for the second simplest instance of the rate pair $(1, 2)$, let alone the capacity region for three-unicast or k -unicast networks (for small k). Such a relatively frustrating progress prompts us to re-examine the problem at hand and investigate whether the lack of new findings is actually due to the inherent hardness of the two-unicast problem.

In this paper, we show that solving the two-unicast problem for general rate pairs is as hard as solving the k -unicast problem for any $k \geq 3$. Thus, the two-unicast problem is the “hardest network coding problem”. We show that given any multiple-unicast rate tuple, there is a rate tuple with suitable higher rates but fewer sources that is more difficult to ascertain achievability of. In particular, we show that solving the very well studied but notoriously hard k -unicast problem with unit-rate (eg. [3], [4]) is no harder than solving the two-unicast problem with rates $(k-1, k)$. Furthermore, by coupling our results with those of [3], [4], we can construct a two-unicast network for which linear codes are insufficient to achieve capacity, and a two-unicast network for which non-Shannon inequalities can provide tighter outer bounds than Shannon-type inequalities alone.

The rest of the paper is organized as follows. In Sec. II, we set up preliminaries and notation. We state and prove our main result in Sec. III. We discuss implications for two-unicast networks in IV. We conclude with some possible extensions in Sec. V.

II. PRELIMINARIES

Definition: A k -unicast network is a directed acyclic graph with k sources s_1, s_2, \dots, s_k and corresponding k destinations d_1, d_2, \dots, d_k , and an assignment of non-negative integer capacity C_e to each edge e . The sources have no incoming edges and the destinations have no outgoing edges. The set of edges entering into and leaving node v will be denoted by $\text{In}(v)$ and $\text{Out}(v)$ respectively.

We say that the non-negative integer rate tuple (R_1, R_2, \dots, R_n) is *zero-error achievable* (or simply *achievable* in the rest of this paper) for a k -unicast network if there exists a positive integer N (called block length), a finite alphabet \mathcal{A} and encoding functions:

- For $e \in \text{Out}(s_i)$, $f_e : \mathcal{A}^{NR_i} \mapsto \mathcal{A}^{NC_e}$, $1 \leq i \leq k$,

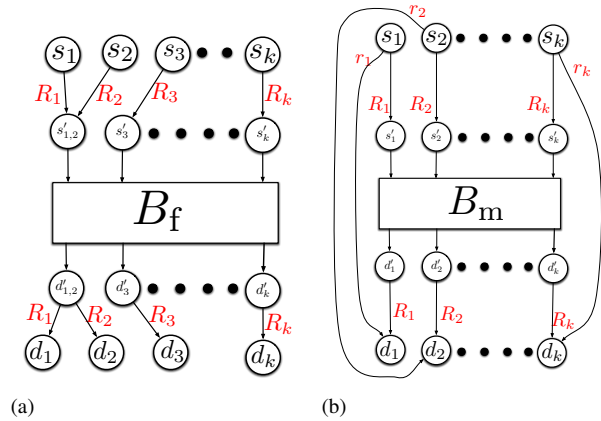


Fig. 1. A pictorial proof for the Fusion and Monotonicity properties of \preceq . The label in red denotes edge capacity. In (a), $(R_1 + R_2, R_3, \dots, R_k)$ is achievable in the network block B_f iff (R_1, R_2, \dots, R_k) is achievable in the extended network. In (b), (R_1, R_2, \dots, R_k) is achievable in the network block B_m iff $(R_1 + r_1, R_2 + r_2, \dots, R_k + r_k)$ is achievable in the extended network.

- For $e \in \text{Out}(v)$, $v \neq s_i, 1 \leq i \leq k$, $f_e : \prod_{e' \in \text{In}(v)} \mathcal{A}^{NC_{e'}} \mapsto \mathcal{A}^{NC_e}$,

and decoding functions $f_{t_i} : \prod_{e' \in \text{In}(t_i)} \mathcal{A}^{NC_{e'}} \mapsto \mathcal{A}^{NR_i}$, $1 \leq i \leq k$, so that $\forall (m_1, m_2, \dots, m_k) \in \prod_{j=1}^n \mathcal{A}^{NR_j}$, we have $g_{t_i}(m_1, m_2, \dots, m_k) = m_i$, $\forall i, 1 \leq i \leq k$ where $g_{t_i} : \prod_{j=1}^k \mathcal{A}^{NR_j} \mapsto \mathcal{A}^{NR_i}$ are functions induced inductively by the f_e 's and f_{t_i} 's. If \mathcal{A} is any finite field and all functions are linear operations on vector spaces over the finite field, then we say the rate tuple is *achievable by vector linear coding*.

Definition: For integer rate tuples $\underline{R} = (R_1, R_2, \dots, R_k)$, $\underline{R}' = (R'_1, R'_2, \dots, R'_n)$, we say $\underline{R} \preceq \underline{R}'$ if any algorithm that can determine whether \underline{R}' is achievable (or achievable by vector linear coding) in any given network can be used to determine whether \underline{R} is achievable (respectively achievable by vector linear coding) in any given network.

The following properties may be observed very easily (see Fig. 1):

- If π is any permutation on $\{1, 2, \dots, k\}$, then $(R_1, R_2, \dots, R_k) \preceq (R_{\pi(1)}, R_{\pi(2)}, \dots, R_{\pi(k)})$.
- *Fusion:* $(R_1 + R_2, R_3, \dots, R_k) \preceq (R_1, R_2, R_3, \dots, R_k)$.
- *Monotonicity:* If $R_i, r_i \geq 0$ for each i , then $(R_1, \dots, R_k) \preceq (R_1 + r_1, \dots, R_k + r_k)$.

Note that using properties listed above, we can never obtain $\underline{R} \preceq \underline{R}'$ where the number of non-zero entries in \underline{R}' is strictly less than that in \underline{R} , i.e. these properties still suggest that determining the capacity of a two-unicast network can be strictly easier than determining that of a k -unicast network with $k > 2$. Our main result, Theorem 1, shows that this is not the case.

III. MAIN RESULT

Our main result is the following:

Theorem 1: For $k \geq 2, m \geq 1$, let

$R_1, R_2, \dots, R_k, R_{k+1}, \dots, R_{k+m}, r_1, r_2, \dots, r_m \geq 0$ be non-negative integers such that $\sum_{i=1}^k R_i = \sum_{j=1}^m r_j$. Then,

$$(R_1, R_2, \dots, R_k, R_{k+1}, R_{k+2}, \dots, R_{k+m}) \preceq \left(\sum_{i=1}^k R_i, R_{k+1} + r_1, R_{k+2} + r_2, \dots, R_{k+m} + r_m \right) \quad (1)$$

Proof:

Let us split the number $\sum_{i=1}^k R_i = \sum_{j=1}^m r_j$ into a ‘coarsest common partition’ formed by c_1, c_2, \dots, c_l as shown in Fig. 2. Set $c_0 = 0$. Recursively define c_h as the minimum of

$$\min_{s: \sum_{i'=1}^s R_{i'} > \sum_{u=1}^{h-1} c_u} \sum_{i'=1}^s R_{i'} - \sum_{u=1}^{h-1} c_u \quad (2)$$

and

$$\min_{t: \sum_{j'=1}^t r_{j'} > \sum_{u=1}^{h-1} c_u} \sum_{j'=1}^t r_{j'} - \sum_{u=1}^{h-1} c_u. \quad (3)$$

Define l by $\sum_{u=1}^l c_u = \sum_{i=1}^k R_i$. l satisfies $\max\{k, m\} \leq l \leq k + m - 1$. We will alternately denote c_h by $c_{(i,j)}$ where i and j are the arg min’s in (2) and (3) respectively. We will use \mathcal{I} to denote the indices (i, j) that correspond to $c_{(i,j)} = c_h$ for some h . In the rest of this proof, we will have i, i_0 denote an index belonging to $\{1, 2, \dots, k\}$, j, j_0 denote an index belonging to $\{1, 2, \dots, m\}$ and (i, j) or (i_0, j_0) denote an index belonging to \mathcal{I} . Note that

$$\sum_j c_{(i,j)} = R_i, \quad \sum_i c_{(i,j)} = r_j. \quad (4)$$

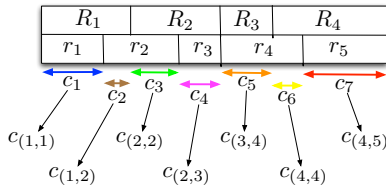


Fig. 2. Splitting of the number $\sum_{i=1}^k R_i = \sum_{j=1}^m r_j$ to obtain $(c_h : h = 1, 2, \dots, l)$

Given a $(k+m)$ -unicast network block B with source-destination pairs (s'_h, d'_h) , $h = 1, 2, \dots, k+m$, we extend it into an $(m+1)$ -unicast network \mathcal{N} as follows:

- Create sources s, s_1, s_2, \dots, s_m and their corresponding destinations d, d_1, d_2, \dots, d_m . Create nodes v_1, v_2, \dots, v_m . Create nodes

$x_{(i,j)}, y_{(i,j)}, z_{(i,j)}, w_{(i,j)}, w^1_{(i,j)}, w^2_{(i,j)}, w^3_{(i,j)}$ for each $(i, j) \in \mathcal{I}$.

- For $j = 1, 2, \dots, m$, create edges of capacity R_{k+j} from s_j to v_j , v_j to s'_{k+j} , and d'_{k+j} to d_j . (see Fig. 3)
- For each $(i, j) \in \mathcal{I}$, create edges of capacity $c_{(i,j)}$ from s to $x_{(i,j)}$, $x_{(i,j)}$ to s'_i , s_j to $y_{(i,j)}$, d'_i to $z_{(i,j)}$, as shown in Fig. 3. and the butterfly edges as shown in Fig. 4.

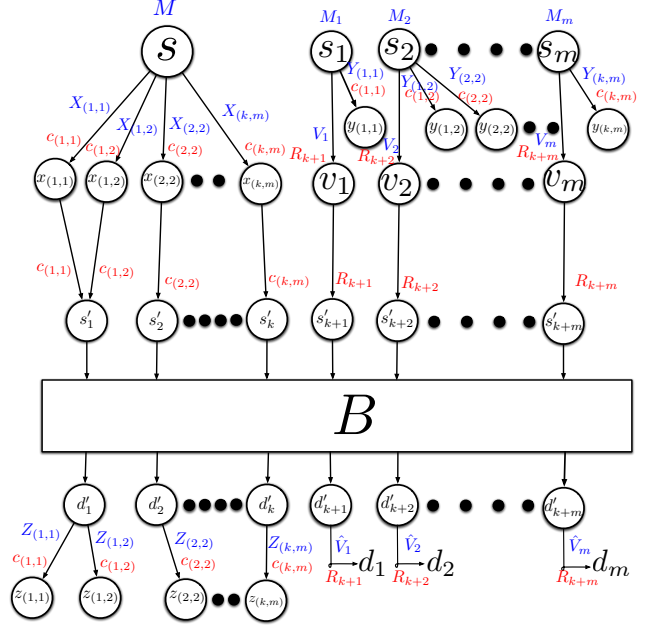


Fig. 3. The $(m+1)$ -unicast network \mathcal{N} constructed around a given $(k+m)$ -unicast network block B . The label in red is the edge capacity and the label in blue is the random variable that flows through that edge. (Network \mathcal{N} shown only partially; completed by Fig. 4)

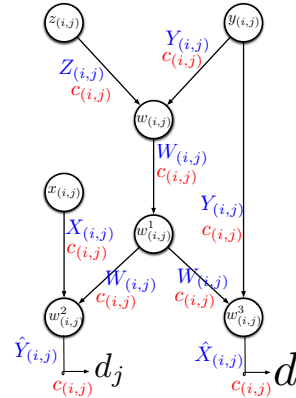


Fig. 4. Butterfly network component in the extended $(m+1)$ -unicast network \mathcal{N} for each $(i, j) \in \mathcal{I}$. Each edge in this component has capacity $c_{(i,j)}$. The label in red is the edge capacity and the label in blue is the random variable that flows through that edge.

We will prove that $(R_1, R_2, \dots, R_{k+m})$ is achievable (or achievable by vector linear coding) in the $(k+m)$ -unicast network B if and only if $(\sum_{i=1}^k R_i, R_{k+1}, \dots, R_{k+m})$ is achievable (respectively achievable by vector linear coding) in the $(m+1)$ -unicast extended network \mathcal{N} .

Suppose $(R_1, R_2, \dots, R_{k+m})$ is achievable in the $(k+m)$ -unicast network block B . Then, we can come up with a ‘butterfly’ coding scheme which proves the achievability of the rate tuple $(\sum_{i=1}^k R_i, R_{k+1} + r_1, R_{k+2} + r_2, \dots, R_{k+m} + r_m)$ in the $(m+1)$ -unicast network \mathcal{N} . This can be done simply by making $X_{(i,j)} = Z_{(i,j)}$ and performing butterfly coding $W_{(i,j)} = X_{(i,j)} + Y_{(i,j)}$ over each butterfly network component.

Now suppose that the rate tuple $(\sum_{i=1}^k R_i, R_{k+1} + r_1, R_{k+2} + r_2, \dots, R_{k+m} + r_m)$ is achievable in the network \mathcal{N} . We will use a lemma whose proof is straightforward and omitted due to lack of space.

Lemma 1: Suppose A, B, C, D are random variables with B, C, D mutually independent and satisfying $H(A|B, D) = 0$. Then,

- a) $H(A|B, C) = 0 \implies H(A|B) = 0$.
- b) $H(B|A, C) = 0 \implies H(B|A) = 0$.

Now, define random variables as shown in Fig. 3, i.e. let M denote the input message at source s and for each $j = 1, 2, \dots, m$, let M_j denote the input message at source s_j . Furthermore, let the random variables V_j, \hat{V}_j for $j = 1, 2, \dots, m$, and $X_{(i,j)}, Y_{(i,j)}, Z_{(i,j)}, \hat{X}_{(i,j)}, \hat{Y}_{(i,j)}$ for $(i, j) \in \mathcal{I}$ be as shown in Fig. 3 and Fig. 4. We will measure entropy with logarithms to the base $|\mathcal{A}|^N$, where \mathcal{A} is the alphabet and N is the block length. M, M_1, M_2, \dots, M_m are mutually independent. $H(M) = \sum_{i=1}^k R_i$, and for each j , $H(M_j) = R_{k+j} + r_j$. Let us denote $A \leftrightarrow B$ if $H(A|B) = H(B|A) = 0$. By observing the tight outgoing rate constraints and encoding at sources s, s_1, s_2, \dots, s_k , and the tight incoming rate constraints and decodability at destinations d, d_1, d_2, \dots, d_k , we can easily conclude

$$M \leftrightarrow \cup_{(i,j)} \{X_{(i,j)}\} \leftrightarrow \cup_{(i,j)} \{\hat{X}_{(i,j)}\}, \quad (5)$$

$$\forall j: M_j \leftrightarrow \cup_i \{Y_{(i,j)}\} \cup \{V_j\} \leftrightarrow \cup_i \{\hat{Y}_{(i,j)}\} \cup \{\hat{V}_j\}, \quad (6)$$

$$\forall (i, j): H(X_{(i,j)}) = H(\hat{X}_{(i,j)}) = c_{(i,j)}, \quad (7)$$

$$\forall (i, j): H(Y_{(i,j)}) = H(\hat{Y}_{(i,j)}) = c_{(i,j)}, \quad (8)$$

$$\forall j: H(V_j) = H(\hat{V}_j) = R_{k+j}. \quad (9)$$

The random variables in the collection

$$\cup_{(i,j)} (\{X_{(i,j)}\} \cup \{Y_{(i,j)}\}) \cup (\cup_j \{V_j\}) \quad (10)$$

are mutually independent. In particular, (7), (9) (10) imply that the messages received by the sources of the network block s'_h for $h = 1, 2, \dots, k+m$, are mutually independent and the symbol received by s'_h has entropy R_h .

Now, fix any $(i, j) \in \mathcal{I}$.

$$\begin{aligned} & H(W_{(i,j)}|X_{(i,j)}) - H(W_{(i,j)}|M_j, X_{(i,j)}) \\ &= I(W_{(i,j)}; M_j|X_{(i,j)}) \end{aligned} \quad (11)$$

$$= I(X_{(i,j)}, W_{(i,j)}; M_j) - I(X_{(i,j)}; M_j) \quad (12)$$

$$\stackrel{(a)}{=} I(X_{(i,j)}, W_{(i,j)}; M_j) - 0 \quad (13)$$

$$\stackrel{(b)}{\geq} I(\hat{Y}_{(i,j)}; M_j) \quad (14)$$

$$\stackrel{(c)}{=} c_{(i,j)}, \quad (15)$$

where (a) holds because $X_{(i,j)}$ is a function of M and M is independent of M_j , (b) holds because $\hat{Y}_{(i,j)}$ is a function of $(X_{(i,j)}, W_{(i,j)})$, and (c) follows from (6), (8), (9). Combining the inequality chain (11)-(15) with the edge capacity constraint $H(W_{(i,j)}|X_{(i,j)}) \leq H(W_{(i,j)}) \leq c_{(i,j)}$, we obtain

$$H(W_{(i,j)}) = c_{(i,j)}, \quad (16)$$

$$H(W_{(i,j)}|X_{(i,j)}) = c_{(i,j)}, \quad (17)$$

$$H(W_{(i,j)}|X_{(i,j)}, M_j) = 0. \quad (18)$$

By a similar argument, we can show

$$H(W_{(i,j)}|Y_{(i,j)}) = c_{(i,j)}, \quad (19)$$

$$H(W_{(i,j)}|Y_{(i,j)}, M) = 0 \quad (20)$$

From (6), we may rewrite (18) as

$$H(W_{(i,j)}|X_{i,j}, V_j, \cup_{i_0} \{Y_{(i_0,j)}\}) = 0. \quad (21)$$

From (5), we may rewrite (20) as

$$H(W_{(i,j)}|Y_{i,j}, \cup_{i_0, j_0} \{X_{(i_0, j_0)}\}) = 0. \quad (22)$$

Using Lemma 1.a) with $A = W_{(i,j)}, B = \{X_{(i,j)}, Y_{(i,j)}\}, C = \{V_j\} \cup_{i_0} \{Y_{(i_0, j)}\} \setminus \{Y_{(i,j)}\}, D = \cup_{i_0, j_0} \{X_{(i_0, j_0)}\} \setminus \{X_{(i,j)}\}$, and using (10), (21), (22), we obtain

$$H(W_{(i,j)}|Y_{i,j}, X_{(i,j)}) = 0. \quad (23)$$

Now, by the chain rule for entropy,

$$\begin{aligned} & H(Y_{(i,j)}) + H(W_{(i,j)}|Y_{(i,j)}) + H(X_{(i,j)}|W_{(i,j)}, Y_{(i,j)}) \\ &= H(X_{(i,j)}, Y_{(i,j)}) + H(W_{(i,j)}|X_{(i,j)}, Y_{(i,j)}) \end{aligned} \quad (24)$$

Using (10), (19), (7), (8), (23) in (24), we get

$$H(X_{(i,j)}|W_{i,j}, Y_{(i,j)}) = 0. \quad (25)$$

From the encoding constraint at node $w_{(i,j)}$, we have $H(W_{(i,j)}|Y_{i,j}, Z_{(i,j)}) = 0$, and using (25) gives us

$$H(X_{(i,j)}|Y_{i,j}, Z_{(i,j)}) = 0. \quad (26)$$

From the encoding constraint for network block B , we have that

$$H(Z_{(i,j)}|\cup_{j_0} \{V_{j_0}\} \cup_{(i_0, j_0)} \{X_{(i_0, j_0)}\}) = 0. \quad (27)$$

Using Lemma 1.b) with $A = Z_{(i,j)}$, $B = X_{(i,j)}$, $C = Y_{(i,j)}$, $D = \cup_{j_0} \{V_{j_0}\} \cup_{(i_0, j_0)} \{X_{i_0, j_0}\} \setminus \{X_{(i,j)}\}$ and using (10), (26), (27), we obtain

$$H(X_{(i,j)}|Z_{(i,j)}) = 0. \quad (28)$$

From (10), (7), we have that for any $i = 1, 2, \dots, k$, $H(\cup_j \{X_{(i,j)}\}) = R_i$ and (28) implies $H(\cup_j \{X_{(i,j)}\}|\cup_j \{Z_{(i,j)}\}) = 0$. This shows that in the block B , the destination d'_i can decode source s'_i 's message for $i = 1, 2, \dots, k$.

Similar arguments can show that $H(V_j|\hat{V}_j) = 0$ (details omitted due to lack of space). Thus, destination d'_{k+j} can decode s'_{k+j} 's message within block B for $j = 1, 2, \dots, m$. The case of the rate tuples assumed to be achievable by a vector linear coding scheme is identical. This completes the proof. \blacksquare

IV. IMPLICATIONS FOR TWO-UNICAST NETWORKS

The main motivation for Theorem 1 is the following implication:

$$(R_1, R_2, \dots, R_k) \preceq \left(\sum_{i=1}^{k-1} R_i, \sum_{i=1}^k R_i \right), \quad (29)$$

i.e. the general two-unicast problem is as hard as the general multiple-unicast problem. Using the monotonicity property of \preceq , this suggests that the difficulty of determining achievability of a rate tuple for the k -unicast problem is related more to the magnitude of the rates in the tuple rather than the size of k . Moreover, we have

$$\underbrace{(1, 1, 1, \dots, 1)}_{k \text{ times}} \preceq (k-1, k), \quad (30)$$

i.e. solving the k -unicast problem with unit-rate (which is known to be a hard problem for large k [3], [4]) is no harder than solving the two-unicast problem with rates $(k-1, k)$. Furthermore, the construction in our paper along with the network constructions in [3], [4], [2] can be used to show the following. Details are omitted due to lack of space.

Theorem 2: There exists a two-unicast network in which a non-linear code can achieve the rate pair (9, 10) but no linear code can.

Theorem 3: There exists a two-unicast network in which non-Shannon information inequalities can rule out achievability of the rate pair (5, 6), but the tightest outer bound obtainable using only Shannon-type information inequalities cannot.

V. DISCUSSION

The result in this paper only proves a reduction for the notion of zero-error integer rate *exact* achievability. It is an interesting question whether the construction

proposed in the proof of Theorem 1 can also show the reduction for the notions of

- zero-error *asymptotic* achievability (closure of the set of zero-error achievable *fractional* rate tuples); it is known that there may be rate tuples that are not exactly achievable but only asymptotically achievable [15],
- vanishing error asymptotic achievability in the Shannon sense (closure of the set of fractional rate tuples which allow vanishing error probability).

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