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Information Flooding

Yu Xiang, Lele Wang, and Young-Han Kim
Department of Electrical and Computer Engineering
University of California, San Diego
La Jolla, CA 92093, USA
Email: {yxiang,lew001,yhk}@ucsd.edu

***Abstract*—This paper studies the problem of broadcasting a common message over a relay network as the canonical platform to investigate the utilities and limitations of traditional relay coding schemes. For a few special classes of networks, such as the 3-node relay channel and the 4-node diamond network, the decode–forward coding scheme by Cover and El Gamal, and its generalization to networks by Xie and Kumar, and Kramer, Gastpar, and Gupta achieve the cutset bound, establishing the capacity. When the network has cycles, however, decode–forward is suboptimal in general and is outperformed by partial decode–forward, compress–forward, or more generally, interactive relaying built upon these $*$–forward coding schemes. In particular, it is demonstrated via a simple example that a coding scheme based on interactive computing by Orlitsky and Roche, and its infinite-round generalization by Ma and Ishwar can strictly outperform existing noninteractive or finite-round interactive coding schemes. Roughly speaking, when the network is to be flooded with information, it is more efficient for the relays to spray tiny droplets of the information back and forth than to splash a huge amount at a time.**

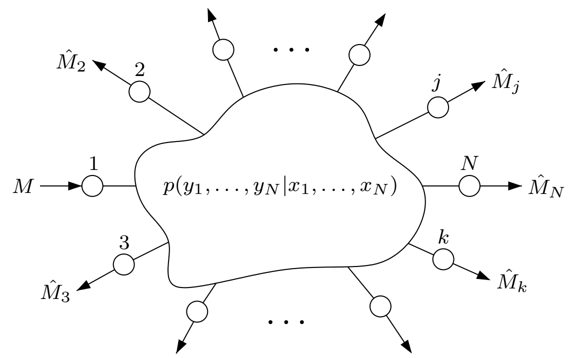

Fig. 1. Common message broadcasting over a noisy network.

I. Introduction

Consider the discrete memoryless network (DMN) model $(\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N,\ p(y^N|x^N),\ \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_N)$ that consists of N sender-receiver alphabet pairs $(\mathcal{X}_k, \mathcal{Y}_k)$, $k \in [1:N] := \{1, 2 \ldots, N\}$, and a collection of channel conditional pmfs (probability mass functions) $p(y^N|x^N) := p(y_1, y_2, \ldots, y_N|x_1, x_2, \ldots, x_N)$. Suppose that source node 1 wishes to communicate a common message M to the rest of the network, as depicted in Figure 1. Compared to the unicast (one node wishes to recover M) or multicast (some nodes wishes to recover M), this problem is relatively simpler since every node in the network has the symmetric goal of recovering the same message.

When the nodes in the network cannot adapt their transmissions based on its received sequence (that is, no relaying or feedback is allowed), then the problem reduces to common message communication over a broadcast channel and the capacity is

$$C_{\text{BC}} = \max_{p(x_1), x_2, \ldots, x_N} \min_{k \in [2:N]} I(X_1; Y_k).$$

Now suppose that each node in the network can adapt its transmission based on the received sequence (that is, relaying is allowed) and thus help other nodes recover the message as well. Despite its relative simplicity, this problem still captures the essential richness of relaying over networks. This paper attempts to demonstrate the inherent complexity in relaying by studying the information flow questions on broadcasting:

- What is the capacity?
- What is the optimal relaying coding scheme that achieves the capacity?

Throughout this paper, we closely follow the notation in [1]. In particular, a random variable is denoted by an upper case letter (e.g., X, Y, Z) and its realization is denoted by a lower case letter (e.g., x, y, z). The shorthand notation X_j^n is used to denote a tuple of random variables $(X_{j1}, \ldots, X_{jn})$, and x_j^n is used to denote their realizations. Given be a tuple of random variables $(X_1, \ldots, X_N)$ and $\mathcal{A} \subseteq [1:N]$, the subtuple of random variables with indices from $\mathcal{A}$ is denoted by $X(\mathcal{A}) := (X_j : j \in \mathcal{A})$. For every positive real number m, the shorthand notation $[1:2^m]$ is used to denote the set of integers $\{1, \ldots, 2^{\lceil m \rceil}\}$.

We are now ready to define the common-message broadcasting problem formally. A $(2^{nR}, n)$ broadcast code for the DMN $p(y^N|x^N)$ consists of

- a message set $[1:2^{nR}]$,
- a source encoder that assigns a symbol $x_{1i}(m, y_1^{i-1})$ to each message $m \in [1:2^{nR}]$ and received sequence y_1^{i-1} for $i \in [1:n]$,
- a set of relay encoders, where encoder $k \in [2:N]$ assigns a symbol $x_{ki}(y_k^{i-1})$ to every received sequence y_k^{i-1} for $i \in [1:n]$, and
- a set of decoders, where decoder $k \in [2:N]$ assigns $\hat{m}_k$ to each y_k^n.

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We assume that the message M is uniformly distributed over the message set. The average probability of error is defined as $P_e^{(n)} = \mathsf{P}\{\hat{M}_k \neq M \text{ for some } k \in [2:N]\}$. A rate R is said to be achievable if there exists a sequence of $(2^{nR}, n)$ broadcast codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$. The broadcast capacity of the DMN is the supremum of all achievable rates.

El Gamal [2] established the cutset upper bound on the capacity:

$$C \leq \max_{p(x^N)} \min_{k\in[2:N]} \min_{\mathcal{S}:\, 1\in\mathcal{S},\, k\in\mathcal{S}^c} I(X(\mathcal{S}); Y(\mathcal{S}^c) \mid X(\mathcal{S}^c)). \quad (1)$$

Xie and Kumar [3] and Kramer, Gastpar, and Gupta [4] generalized the decode–forward coding scheme for the relay channel by Cover and El Gamal [5] and established the network decode–forward lower bound on the capacity:

$$C \geq \max_{p(x^N)} \min_{k\in[1:N-1]} I(X^k; Y_{k+1} | X_{k+1}^N). \quad (2)$$

These two bounds coincide and establish the broadcast capacity when the network is degraded, i.e.,

$$p(y_{k+2}^N | x^N, y^{k+1}) = p(y_{k+2}^N | x_{k+1}^N, y_{k+1}) \quad (3)$$

for $k \in [1:N-2]$ (up to relabeling of nodes).

In Section II, we discuss two other special cases—3-node relay channels and layered networks—for which the decode–forward lower bound is tight. Decode–forward, however, is suboptimal for general networks. In Sections III through VI, we demonstrate gradually through simple examples that optimal relaying can be more sophisticated than simple decode–forward and require partial decode–forward, compress–forward, or interactive relaying built upon these $*$–forward coding schemes. Our discussion will culminate with the binary broadcast relay channel example for which not only interactive communication between relays strictly outperforms the existing noninteractive coding schemes, but also the number of communication rounds needs to go to infinity to fully enjoy the benefit of interaction.

II. Decode–Forward Is Sometimes Optimal

It is already mentioned that the decode–forward coding scheme is optimal when the network is degraded; see (3). Another case in which decode–forward is natural is when the network is *acyclic*, i.e.,

$$p(y^N | x^N) = \prod_{k=1}^{N} p(y_k | x^k, y^{k-1})$$

(up to relabeling of nodes). For this case, node k does not receive any signal from its downstream (nodes $j \in [k+1:N]$). Thus it is natural to decode its received signal at once and forward the recovered message downstream. In the following, we revisit a few special classes of acyclic networks for which this decode–forward coding scheme is optimal.

A. Relay Channel

We first consider the *relay channel* $p(y_2, y_3 | x_1, x_2)$ [6], [5] depicted in Figure 2. It is well known that *decode–forward* is optimal and the capacity is

$$C = \max_{p(x_1, x_2)} \min\big\{I(X_1; Y_2 | X_2),\, I(X_1, X_2; Y_3)\big\}.$$

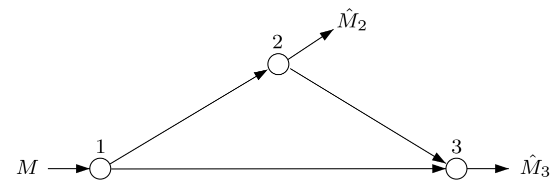

Fig. 2. Relay channel.

B. Layered Network

Consider the diamond network $p(y_2, y_3 | x_1) p(y_4 | x_2, x_3)$ [7] depicted in Figure 3. Again, decode–forward is optimal and the capacity is

$$C = \max_{p(x_1)p(x_2,x_3)} \min\big\{I(X_1; Y_2), I(X_1, Y_3),\, I(X_2, X_3; Y_4)\big\}.$$

To prove the converse, simplify the cutset bound in (1) as

$$\begin{aligned} C &\leq \max_{p(x^3)} \min\big\{I(X_1, X_3; Y_2 | X_2),\, I(X_1, X_2; Y_3 | X_3), \\ &\qquad\qquad I(X_1, X_2, X_3; Y_4)\big\} \\ &\overset{(a)}{\leq} \max_{p(x^3)} \min\big\{I(X_1; Y_2),\, I(X_1; Y_3),\, I(X_2, X_3; Y_4)\big\} \\ &\overset{(b)}{=} \max_{p(x_1)p(x_2,x_3)} \min\big\{I(X_1; Y_2),\, I(X_1; Y_3),\, I(X_2, X_3; Y_4)\big\}, \end{aligned}$$

where (a) follows since $(X_2, X_3) \to X_1 \to Y_2$, $(X_2, X_3) \to X_1 \to Y_3$, and $X_1 \to (X_2, X_3) \to Y_4$, respectively, form Markov chains, and (b) follows since the mutual information terms $I(X_1; Y_2), I(X_1; Y_3)$, and $I(X_2, X_3; Y_4)$ depend on the channel input pmf $p(x_1, x_2, x_3)$ only through the marginals $p(x_1)$ and $p(x_2, x_3)$. The achievability follows by simplifying

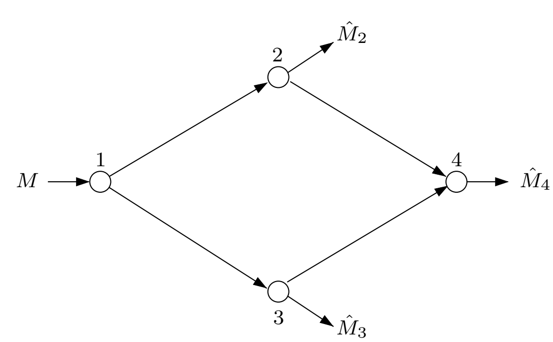

Fig. 3. Diamond network.

the decode-forward lower bound in (2) as

$$\begin{aligned} C &\geq \max_{p(x^3)} \min\big\{ I(X_1;Y_2|X_2,X_3),\ I(X_1,X_2;Y_3|X_3), \\ &\qquad\qquad I(X_1,X_2,X_3;Y_4) \big\} \\ &\overset{(a)}{\geq} \max_{p(x_1)p(x_2,x_3)} \min\{ I(X_1;Y_2|X_2,X_3), I(X_1;Y_3|X_3), \\ &\qquad\qquad I(X_2,X_3;Y_4) \} \\ &\overset{(b)}{\geq} \max_{p(x_1)p(x_2,x_3)} \min\big\{ I(X_1;Y_2),\ I(X_1;Y_3),\ I(X_2,X_3;Y_4) \big\}, \end{aligned}$$

where (a) follows since the maximum is over a smaller set and (b) follows since X_1 is independent of (X_2, X_3).

This result can be easily generalized to *layered network*

$$p(y^N|x^N) = \prod_{l=1}^{\lambda} p(y(\mathcal{L}_l)|x(\mathcal{L}_{l-1}))$$

depicted in Figure 4, where the layers of nodes $\mathcal{L}_0 = \{1\}$ and $\mathcal{L}_j$, $j \in [1:\lambda]$ partition the network, i.e.,

$$\mathcal{L}_0 \uplus \mathcal{L}_1 \uplus \cdots \uplus \mathcal{L}_\lambda = [1:N].$$

The capacity of the layered network is

$$C = \max_{\prod_{l=1}^{\lambda} p(x(\mathcal{L}_{l-1}))} \min_{l \in [1:\lambda]} \min_{j \in \mathcal{L}_l} I(X(\mathcal{L}_{l-1}); Y_j).$$

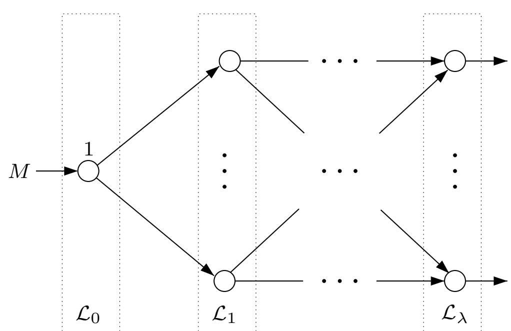

Fig. 4. Layered network.

C. Diamond Network with Direct Link

Now we consider the variant of the diamond network depicted in Figure 5, which is defined as

$$p(y_2, y_3, y_4|x_1, x_2, x_3) = p(y_2, y_3|x_1)p(y_4|x_1, x_2, x_3).$$

For this case, the cutset bound simplifies to

$$\begin{aligned} C \leq \max_{p(x^3)} \min\big\{ & I(X_1;Y_2|X_2),\ I(X_1;Y_3|X_3), \\ & I(X_1,X_2,X_3;Y_4),\ I(X_1;Y_2,Y_3|X_2,X_3) \big\}, \end{aligned}$$

while the decode–forward lower bound simplifies to

$$\begin{aligned} C \geq \max_{p(x^3)} \min\big\{ & I(X_1;Y_2|X_2,X_3),\ I(X_1;Y_3|X_3), \\ & I(X_1,X_2,X_3;Y_4) \big\}. \end{aligned}$$

Thus, it is not known whether decode–forward is optimal for acyclic networks in general, even though it seems to be the only reasonable coding scheme when there is no cycle in the information flow.

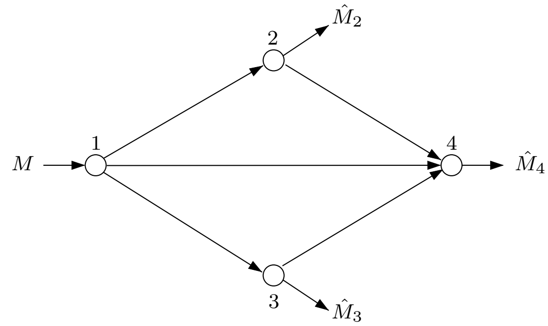

Fig. 5. Diamond network with direct link.

III. Partial Decoding Is Sometimes Necessary

In general, when the network has cycles, it is more advantageous to recover only part of the message at the beginning and recover the rest with the help of other nodes. This idea is best explained by a 3-node *cyclic graphical network* example depicted in Figure 6. Here the network is modeled by a weighted directed cyclic graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = \{1, 2, 3\}$ is the set of nodes, $\mathcal{E} = \{(1,2), (1,3), (2,3), (3,2)\}$ is the set of edges, each of which models an orthogonal communication link that can carry 1 bit per transmission. Note that the corresponding conditional pmf $p(y^3|x^3)$ is given by $X_1 = (X_{12}, X_{13})$, $Y_2 = (X_{12}, X_3)$ and $Y_3 = (X_{13}, X_2)$, where X_{12}, X_{13}, X_2, and X_3 are binary.

It can be easily verified that the cutset bound simplifies to $C \leq 2$ and the decode–forward lower bound simplifies to $C \geq 1$. But by simply *routing* one bit along the path $1 \to 2 \to 3$ and another bit along the path $1 \to 3 \to 2$, we can easily achieve 2 bits per transmission.

This observation can be readily generalized to any graphical networks, for which the capacity is achieved by routing as in the unicast case [8], [9]. Note that unlike the multicast case, network coding [10] is unnecessary for broadcasting. When the network suffers noise, the partial decode–forward coding scheme by Cover and El Gamal [5] and its extension to networks by Aref [11] provide a means of splitting the message into independent parts and forwarding them along multiple paths.

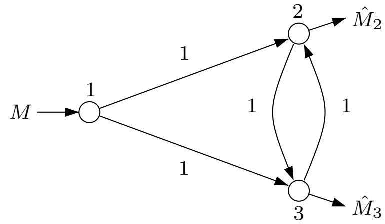

Fig. 6. Cyclic graphical network example.

IV. Initial Decoding Is Sometimes Impossible

In some cases, decoding is actually impossible at the beginning and more sophisticated coding schemes are necessary. To illustrate the depth of this problem, throughout the rest of the paper we focus on a simple 3-node cyclic network model depicted in Figure 7, which is commonly referred to as the *broadcast relay channel*. Here the message is sent over a broadcast channel $p(y_2, y_3|x_1)$. In addition, nodes 2 and 3 are connected via two noiseless links of rates R_2 and R_3, respectively, that are orthogonal to the main broadcast channel. Let $C(R)$ be the broadcast capacity as a function of the sum $R = R_2 + R_3$ of the link capacities between nodes 2 and 3.

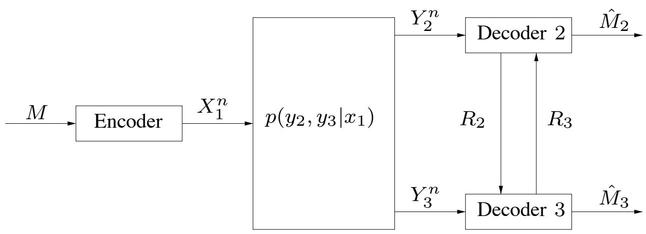

Fig. 7. Broadcast relay channel.

To be more specific, we consider the Gaussian broadcast relay channel depicted in Figure 8. The channel outputs corresponding to the input X_1 are

$$\begin{aligned} Y_2 &= X_1 + Z_2, \\ Y_3 &= X_1 + Z_3, \end{aligned} \tag{4}$$

where Z_2 are Z_3 are jointly Gaussian with zero mean, equal variance $\mathsf{E}(Z_2^2) = \mathsf{E}(Z_3^2) = 1$, and correlation coefficient $\rho = \mathsf{E}(Z_2 Z_3)$. Note that the capacity without the two noiseless links between the two receivers is

$$C(0) = \frac{1}{2} \log(1 + P).$$

In the following, we focus on the case of $\rho = 0$.

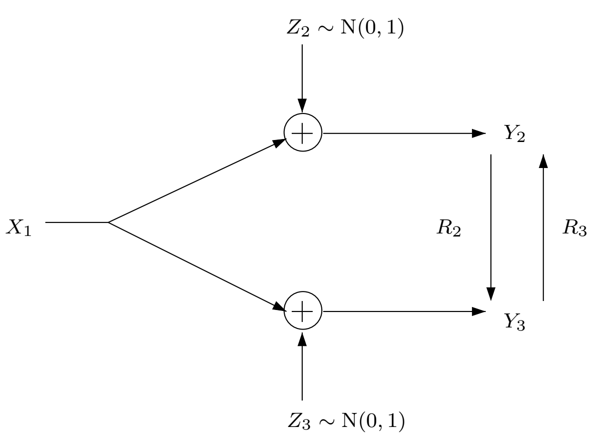

Fig. 8. Gaussian broadcast relay channel.

By the cutset bound, the capacity is upper bounded as

$$C(R) \le C(0) + \frac{R}{2}, \tag{5}$$

where the optimal $R_2 = R_3 = R/2$ by symmetry. In comparison, since both receivers are symmetric, i.e., $F_{Y_2|X_1}(y|x) = F_{Y_3|X_1}(y|x)$, one recovers exactly what the other can recover about the message. Thus, any decoding-based relaying scheme (decode–forward, partial decode–forward, or compute–forward [12]) cannot achieve more than $C(0)$, which tends to zero as $P \to 0$.

Now we consider the compress–forward coding scheme for the relay channel by Cover and El Gamal [5], which can be readily extended to the current setup. It can be easily shown [13] that the corresponding lower bound (with the optimal rate splitting $R_2 = R_3 = R/2$) simplifies to

$$C(R) \ge \max_{F(x_1)F(\hat{y}_2|y_2)F(\hat{y}_3|y_3)} \min\{I_1,\, I_2,\, I_3,\, I_4\}, \tag{6}$$

where

$$\begin{aligned} I_1 &= I(X_1; Y_2, \hat{Y}_3), \\ I_2 &= I(X_1; \hat{Y}_2, Y_3), \\ I_3 &= I(X_1; Y_2) - I(Y_3; \hat{Y}_3 | X_1, Y_2) + R/2, \\ I_4 &= I(X_1; Y_3) - I(Y_2; \hat{Y}_2 | X_1, Y_3) + R/2. \end{aligned}$$

Evaluated with the Gaussian input distribution and test channels, this lower bound simplifies to

$$C(R) \ge \frac{1}{2} \log\left(1 + \frac{2P(P+1)(2^R - 1) + P(2P+1)}{(P+1)(2^R - 1) + (2P+1)}\right),$$

which is strictly larger than $C(0)$ for every $R > 0$. Thus, compress–forward strictly improves upon decoding-based relaying schemes. Note that when $\rho = -1$, the corresponding compress–forward lower bound coincides with the cutset bound in (5). This lower bound can be also achieved by the hash–forward coding scheme [14], [15].

V. Interaction Is Sometimes Necessary

In the relay coding schemes we have discussed so far—(partial) decode–forward, compress–forward, compute–forward, hash–forward, each node summarizes its received signal and forwards it to other nodes. It turns out, however, that interactive cooperation between nodes can achieve higher rates, as demonstrated by Draper, Frey, and Kschschang [16] for the broadcast relay channel consisting of two binary erasure channels.

In this section, we adapt their interactive relaying scheme to the Gaussian broadcast relay channel in (4) studied in the previous section. Suppose that node 2 first uses compress–forward to help node 3 recover the message and node 3 then uses decode–forward to help node 2 recover the message. It can be easily shown that this "compress–forward-followed-by-decode–forward" coding scheme yields the following lower bound on the capacity:

$$C(R) \ge \max_{F(x_1)F(\hat{y}_2|y_2)} \min\{I_2,\, I_3'\}, \tag{7}$$

where

$$I_2 = I(X_1; \hat{Y}_2, Y_3),$$
$$I_3' = I(X_1; Y_2) - I(Y_2; \hat{Y}_2 | Y_3) + R.$$

By symmetry, it can be shown that this lower bound strictly improves upon the compress–forward lower bound in (6). Thus, two-round interactive relaying is sometimes better than noninteractive relaying. When evaluated with the Gaussian input distribution and test channels, the lower bound in (7) simplifies to

$$C(R) \geq \max_{\sigma^2} \min\left\{ \frac{1}{2}\log\left(1 + \frac{2P + P\sigma^2}{1+\sigma^2}\right), \right.$$
$$\left. R + \frac{1}{2}\log(1+P) - \frac{1}{2}\log\left(1 + \frac{2P+1}{(P+1)\sigma^2}\right)\right\}.$$

Figure 9 compares the cutset bound and the (partial) decode–forward, compress–forward, and compress–forward decode–forward lower bounds.

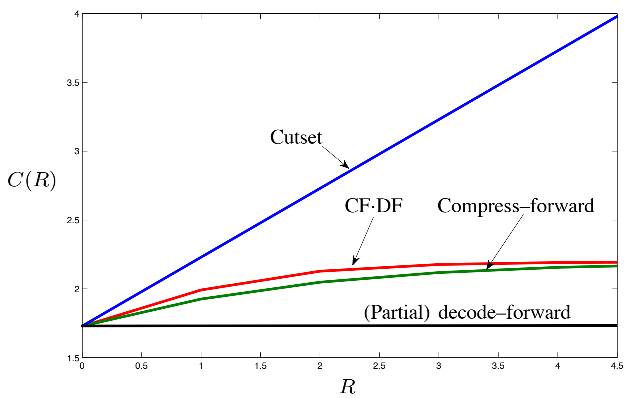

Fig. 9. Comparison of the capacity bounds for the Gaussian broadcast relay channel when $P = 10$.

VI. Infinite Interaction Is Sometimes Necessary

As shown in the previous section, interactive relaying can outperform noninteractive $*$–forward coding schemes. It is natural to ask the following:

- Would more than two rounds of interactive relaying further outperform two rounds of interactive relaying?
- If so, how many rounds would be necessary?

In this section, we study a simple binary broadcast relay channel that consists of two correlated Z channels as depicted in Figure 10, and show that infinite rounds of interactive relaying can strictly outperform known finite-round coding schemes.

As before, we focus on the capacity $C(R)$ as a function of the sum-rate R of communication between two receivers. In particular, we will focus on the optimal rate of interaction

$$R^* = \min\{R : C(R) = 1\}.$$

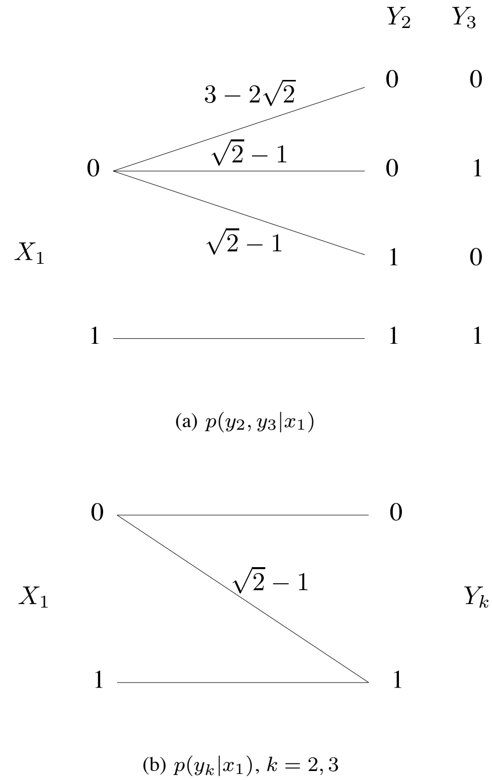

Fig. 10. Two correlated Z channels.

It is easy to see that $C(0) = 0.3941$, which is the capacity of the Z channel, while $C(R) = 1$ for $R \geq 2$, which is the capacity of the DMC from X_1 to (Y_2, Y_3). In other words, $R^* \leq 2$. Note further that $X_1 = Y_2 \cdot Y_3$ and that when $X \sim \text{Bern}(1/2)$, Y_2 and Y_3 are independent and identically distributed $\text{Bern}(1/\sqrt{2})$.

We now compare the existing bounds on the capacity. First, the cutset bound simplifies (under the optimal choice $R_2 = R_3 = R/2$) to

$$C(R)$$
$$\leq \max_{p(x_1)} \min\{I(X_1; Y_2) + R/2,\, I(X_1; Y_2, Y_3)\}$$
$$= \max_{\alpha \in [0:1]} \min\{H((2-\sqrt{2})\alpha) - \alpha H(\sqrt{2}-1) + R/2,\, H(\alpha)\}.$$

In particular, $C(R) < 1$ for $R < 1.2338$; in other words, $R^* \geq 1.2338$.

Since the channel is symmetric as in the Gaussian case, decoding-based coding schemes are useless and the (partial) decode–forward lower bound simplifies as

$$C(R) \geq C(0) = 0.3941.$$

The capacity $C(R)$ lies between two simple bounds as plotted in Figure 11.

While the compress–forward lower bound in (6) can be evaluated only numerically, one extreme point can be calculated analytically. Let R^*_{CF} be the minimum R such that the compress–forward lower bound $C_{\text{CF}}(R) = 1$. Then, the

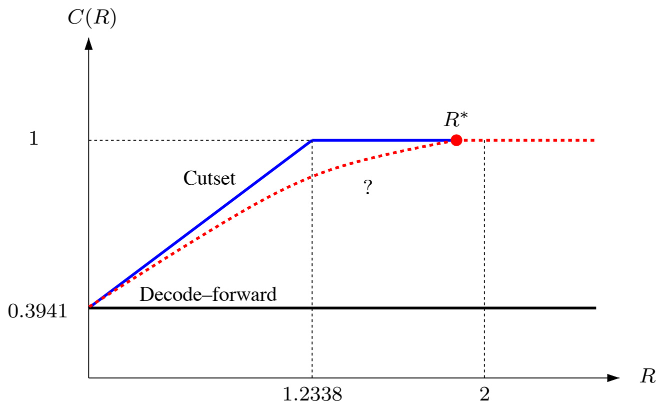

Fig. 11. Optimal $C(R)$ curve.

inverse problem of finding R^*_{CF} is equivalent to finding the minimum sum-rate of noninteractive communication between nodes 2 and 3 so that each of the nodes can losslessly compute $X_1 = Y_2 \cdot Y_3 \sim \mathrm{Bern}(1/2)$. Thus, we can apply Orlitsky and Roche's result on coding for computing [17] and conclude that

$$\begin{aligned} R^*_{\mathrm{CF}} &= H_{\mathcal{G}}(Y_2|Y_3) + H_{\mathcal{G}}(Y_3|Y_2) \\ &= H(Y_2) + H(Y_3) \\ &= 2H\left(\frac{1}{\sqrt{2}}\right) \\ &= 1.7449, \end{aligned}$$

where $H_{\mathcal{G}}(Y_2|Y_3)$ and $H_{\mathcal{G}}(Y_3|Y_2)$ denote the conditional graph entropies that characterize the minimum rates to compute X_1 at node 3 and node 2, respectively. In other words, $C(R) = 1$ for $R \geq 1.7449$. Note that noninteractive extensions of compress–forward including hash–forward [14], [15], noisy network coding [18], and hybrid coding [19] do not perform better than compress–forward.

Compress–forward can be improved instead by making communication between nodes 2 and 3 interactive. Suppose that node 2 first uses the regular compress–forward to help node 3 recover the message and then node 3 uses a modified version of compress–forward that incorporates the signal from node 2 as side information to help node 2 recover the message; see Kaspi [20] for the origin of this idea in two-way lossy source coding. While this modification does not improve upon the noninteractive compress–forward lower bound in (6) for the Gaussian case (since the quadratic Gaussian rate–distortion function is the same with or without side information at the encoder [21]), it provides strict improvements in general, for example, in the current setup of the binary broadcast relay channel. As before, the inverse problem of finding the corresponding minimum sum-rate $R^*_{\mathrm{CF}^2}$ of this coding scheme can be recast into the problem of finding the minimum sum-rate for computing X_1 at nodes 2 and 3 via two-round communication. Following the results on interactive coding for computing by Orlitsky and Roche [17], and Ma and Ishwar [22], it can be shown that

$$\begin{aligned} R^*_{\mathrm{CF}^2} &= H(Y_2) + H(X_1|Y_3) \\ &= H\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} H\left(\frac{1}{\sqrt{2}}\right) \\ &= 1.4893. \end{aligned}$$

In other words, $C(R) = 1$ for $R \geq 1.4893$.

As for the Gaussian case in Section V, we can adapt the coding scheme by Draper, Frey, and Kschischan [16], in which compress–forward is followed by decode–forward. This interactive relaying scheme in general yields a tighter lower bound than the two-round interactive compress–forward lower bound, since it is more efficient to use full knowledge of the message (decode–forward) for the second-round communication. At the extreme point of the 1-bit message, however, there is no gain since computing X_1 is equivalent to decoding the message itself. Hence, compress–forward followed by decode–forward yields the same upper bound on the minimum sum-rate R^* as

$$\begin{aligned} R^* &\leq 1 - I(X_1; Y_2) + H_{\mathcal{G}}(Y_2|Y_3) \\ &= H(Y_2) + H(X_1|Y_3) \\ &= 1.4893. \end{aligned}$$

Now we further generalize the idea of interactive relaying to q-round interactive compress–forward. Again at the extreme point of the 1-bit message, the inverse problem of finding the minimum sum-rate $R^*_{\mathrm{CF}^q}$ is equivalent to q-round interactive coding for computing, in which nodes 2 and 3 exchange messages in q rounds of communication to losslessly recover X_1. While the exact characterization of this minimum sum-rate for q-round computing seems to be intractable, using ingenious techniques Ma and Ishwar [22], [23] characterized its limiting behavior as

$$\begin{aligned} \lim_{q\to\infty} R^*_{\mathrm{CF}^q} &= (1+p)H(p) + p\log(pe^{1-p})\,\big|_{p=1/\sqrt{2}} \\ &= 1.4346. \end{aligned}$$

They further showed that for the natural coding scheme that achieves this limiting behavior, the corresponding sum-rate R_{CF^q} is strictly larger than $R^*_{\mathrm{CF}^\infty} = \lim_{q\to\infty} R^*_{\mathrm{CF}^q}$. Thus, $C(R) = 1$ for $R \geq 1.4346$, and among all *known* relay coding schemes this can be achieved only by infinite rounds of interactive relaying! Finally, note that as in the two-round case, the infinite-round compress–forward coding scheme can be further improved for rates less than 1 by replacing the last-round compress–forward by decode–forward.

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