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Linear Relaying for Gaussian Diamond Networks

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***Abstract*—Linear relaying for the Gaussian diamond network is studied as a natural extension of the amplify–forward relaying strategy by Schein and Gallager. A single-letter optimal rate is characterized, which is shown to be achieved by time sharing between four amplify–forward strategies at different power levels. This *linear relaying capacity* has a bounded gap from the cutset bound when the network is symmetric, but in general has an unbounded gap. The main idea of the proof is to transform a multiletter rate expression into an infinite-dimensional optimization problem, the relaxation of which matches the performance of time-shared amplify–forward. A similar proof technique can be applied to other relay networks with layered structure such as the N-relay Gaussian diamond network and the receiver frequency-division Gaussian relay channel.**

I. Introduction

Amplify–forward (AF) relaying, which was introduced by Schein and Gallager [1] for 4-node diamond networks and was later popularized by Laneman, Tse, and Wornell [2] for 3-node relay channels, is one of the basic relaying schemes for Gaussian relay networks. In this simple "analog-to-analog" interface, a relay scales the received signal and transmits it symbol-by-symbol. The performance–complexity tradeoff of AF is often excellent, making it one of the most practical relaying schemes. For example, for the single-antenna real Gaussian relay channel, AF achieves within 1 bit from the cutset bound for all signal-to-noise ratios (SNRs). Consequently, AF has been studied for various relay network models in the literature; see, for example, [3], [4], [5], [6], [7].

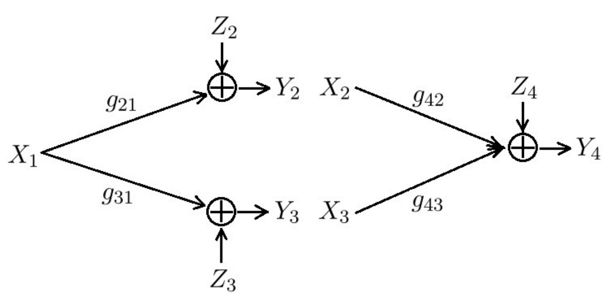

Fig. 1. Gaussian Diamond Network Model.

In this paper we consider the discrete-time Gaussian diamond network depicted in Fig. 1. Here g_{jk} denotes the channel gain from node k to node j. The relationship between channel inputs and outputs follows

$$\begin{aligned} Y_2 &= g_{21}X_1 + Z_2, \\ Y_3 &= g_{31}X_1 + Z_3, \\ Y_4 &= g_{42}X_2 + g_{43}X_3 + Z_4, \end{aligned} \tag{1}$$

where X_1, X_2, X_3 are the inputs at nodes $1, 2, 3$, respectively, Y_2, Y_3, Y_4 are the outputs at nodes $2, 3, 4$, respectively, and $Z_j \sim \mathrm{N}(0, 1)$, $j \in \{2, 3, 4\}$, are independent noise components. Node 1 wishes to communicate the message M to node 4 with help of nodes 2 and 3. For block length n and rate R, we specify a code $(2^{nR}, n)$ with an encoding function $x_1^n(m)$, $m \in [1 : 2^{nR}] := \{1, ..., 2^{\lceil nR \rceil}\}$, satisfying the average power constraint

$$\frac{1}{n}\sum_{i=1}^{n} x_{1i}^2(m) \le P,$$

relaying functions $x_{2i}(y_2^{i-1})$ and $x_{3i}(y_3^{i-1})$ for $i \in [1 : n]$, satisfying the expected average power constraints

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} \mathsf{E}(x_{2i}^2(Y_2^{i-1})) &\le P, \\ \frac{1}{n}\sum_{i=1}^{n} \mathsf{E}(x_{3i}^2(Y_3^{i-1})) &\le P, \end{aligned}$$

and a decoding functions $\hat{m} : \mathbb{R}^n \to \{1, ..., 2^{nR}\}$. The probability of error $P_e^{(n)}$ is defined as

$$P_e^{(n)} := \mathsf{P}\{\hat{m}(Y_4^n) \ne M\},$$

where M is drawn uniformly from $[1 : 2^{nR}]$. We say that the rate R is achievable if there exists a sequence of $(2^{nR}, n)$ codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$. The capacity C is defined as the supremum of the set of achievable rates R. We sometimes use the notation $C(P)$ or even $C(P; g_{21}, g_{31}, g_{42}, g_{43})$ to stress the parameters of the given network, and henceforth follow a similar convention for other rates (upper and lower bounds on the capacity). We denote by $S_{jk} = g_{jk}^2 P$ the received SNR for the signal from node k to node j.

Expanding the results in [1], Schein [8] studied upper and lower bounds on the capacity and associated coding schemes. By relaxing the standard cutset bound [9], it can be readily

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shown that the capacity is upper bounded as

$$\begin{aligned} C &\le R_{\rm CS} \\ &:= \min\Big\{ \mathsf{C}\big((g_{21}^2+g_{31}^2)P\big), \\ &\qquad\quad \mathsf{C}\big((g_{42}+g_{43})^2P\big), \\ &\qquad\quad \mathsf{C}(g_{21}^2P)+\mathsf{C}(g_{43}^2P), \\ &\qquad\quad \mathsf{C}(g_{31}^2P)+\mathsf{C}(g_{42}^2P) \Big\} \\ &= \min\Big\{ \mathsf{C}(S_{21}+S_{31}), \\ &\qquad\quad \mathsf{C}\big((\sqrt{S_{42}}+\sqrt{S_{43}})^2\big), \\ &\qquad\quad \mathsf{C}(S_{21})+\mathsf{C}(S_{43}), \\ &\qquad\quad \mathsf{C}(S_{31})+\mathsf{C}(S_{42}) \Big\}, \end{aligned} \tag{2}$$

where $\mathsf{C}(x) := (1/2)\log(1+x)$ is the Gaussian capacity function.

By setting $x_{2i} = c_2\, y_{2,i-1}$ and $x_{3i} = c_3\, y_{3,i-1}$, $i \in [1:n]$, for some constants c_2 and c_3, AF induces a standard point-to-point Gaussian channel from X_1 to Y_4 (with one-unit delay at the output). Note that using all the available power at both relays is not optimal in general [8].

Hence we can establish a lower bound on the capacity as

$$C \ge R_{\rm AF} := \max_{0\le P_2,P_3\le P} R_{\rm AF}(P,P_2,P_3)$$

where

$$R_{\rm AF}(P_1,P_2,P_3) := \mathsf{C}\left(\frac{\left(\sqrt{\frac{g_{42}^2P_2\cdot g_{21}^2P_1}{g_{21}^2P_1+1}}+\sqrt{\frac{g_{43}^2P_3\cdot g_{31}^2P_1}{g_{31}^2P_1+1}}\right)^2}{1+\frac{g_{42}^2P_2}{g_{21}^2P_1+1}+\frac{g_{43}^2P_3}{g_{31}^2P_1+1}} \right), \tag{3}$$

As a simple example, the symmetric setting is considered, where it is optimal to use all the available power at both relays [8]. The rate $R_{\rm AF}(P)$ achieved by AF as a function of the power P is convex when P is small and concave when P is large; see Fig. 2. Consequently, we can improve upon AF by time sharing between sending nothing ($P = 0$) and AF at a larger power. This *bursty AF* scheme [10], which can be also viewed as linear relaying (with time-varying scaling coefficients), outperforms the pure AF for small P.

Thus motivated, we study *linear relaying*, that is, general linear operations (with memory) at the relays, as a natural extension of AF and bursty AF. More precisely, the relay encoders are assumed to be of the form

$$x_{2i} = \sum_{j=1}^{i-1} b_{ij}^{(2)} y_{2j}, \quad i \in [1:n],$$

$$x_{3i} = \sum_{j=1}^{i-1} b_{ij}^{(3)} y_{3j}, \quad i \in [1:n],$$

or in vector notation

$$\begin{aligned} x_2^n &= B_2\, y_2^n, \\ x_3^n &= B_3\, y_3^n, \end{aligned} \tag{4}$$

where B_2 and B_3 are $n \times n$ strictly lower triangular matrices. The *linear relaying capacity* $C_{\rm L}$ of the Gaussian diamond network is defined as the supremum of achievable rates with linear relaying.

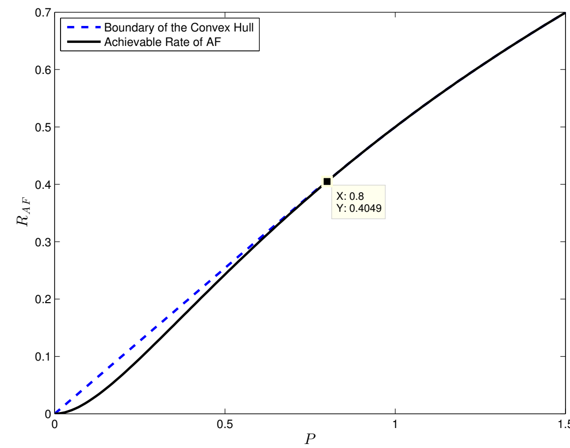

Fig. 2. The plot of $R_{\rm AF}(P)$ (solid black) and its convex envelope (dashed blue) when $g_{21} = g_{31} = g_{42} = g_{43} = 1$. The two curves merge at point $(0.8, 0.4049)$ as P increases.

We are now ready to state our main result.

Theorem 1. *The linear relaying capacity of the Gaussian diamond network is*

$$C_{\rm L} = \max \sum_{k=1}^{4} \alpha_k R_{\rm AF}(P_{1k}, P_{2k}, P_{3k}), \tag{5}$$

where $\alpha_k \ge 0$, $P_{jk} \ge 0$, $j \in [1:3]$, $k \in [1:4]$, *such that* $\sum_{k=1}^4 \alpha_k = 1$ *and* $\sum_{k=1}^4 \alpha_k P_{jk} \le P$, $j \in [1:3]$.

In other words, the optimal linear relaying scheme is equivalent to time sharing among four AF schemes at different power levels.

II. Performance of Linear Relaying Scheme

We compare C_L with the cutset bound $R_{\rm CS}$ under various channel configurations.

A. Symmetric Channel Gains

Assume that $g_{21} = g_{31} := g$ and $g_{42} = g_{43} := h$. Niesen and Diggavi [10] considered this symmetric setting and studied the bursty AF scheme, whereby the nodes communicate for a fraction α of time at power P/α using AF and stay silent for the remaining time; recall Fig. 2. The rate achieved by bursty AF can be expressed as

$$R_{\rm BAF} = \max_{0<\alpha\le 1} \alpha\, \mathsf{C}\left(\frac{(2ghP/\alpha)^2}{1+g^2P/\alpha+2h^2P/\alpha}\right).$$

and satisfies the following performance guarantees in terms of the cutset bound $R_{\rm CS}$.

Proposition 1 ([10, Corollary 3]).

$$\sup_{g,h,P}\big[R_{\rm CS}(P;g,g,h,h)-R_{\rm BAF}(P;g,g,h,h)\big]\le 1.8,$$

$$\sup_{g,h,P}\frac{R_{\rm CS}(P;g,g,h,h)}{R_{\rm BAF}(P;g,g,h,h)}\le 14.$$

We now show that bursty AF is actually the best linear relaying scheme.

Proposition 2.

$$C_{\rm L}(P;g,g,h,h)=R_{\rm BAF}(P;g,g,h,h). \tag{6}$$

Consequently, for the symmetric setting, Proposition 1 also provides performance guarantees for linear relaying in general.

Proof of Proposition 2: . We first note that by symmetry

$$R_{\rm AF}(P_1,\frac{P_2+P_3}{2},\frac{P_2+P_3}{2})\ge R_{\rm AF}(P_1,P_2,P_3)$$

with equality iff $P_2=P_3$. It implies that $P_{2k}=P_{3k}$ for $k=[1:4]$ is a necessary condition for optimal solutions of (5). Hence (5) reduces to

$$C_{\rm L}=\max\sum_{k=1}^{4}\alpha_k R_{\rm AF}(P_{1k},P_{2k},P_{2k}), \tag{7}$$

where $\alpha_k\ge 0$, $P_{jk}\ge 0$, $j\in[1:2]$, $k\in[1:4]$ such that $\sum_{k=1}^4\alpha_k=1$ and $\sum_{k=1}^4\alpha_kP_{jk}=P$, $j\in[1:2]$.

The following lemma, whose proof is deferred to the Appendix, further simplifies (7).

Lemma 1. *The maximum in* (7) *is attained by* $P_{1k}=P_{2k}=0$ *for* $k=[2:4]$ *and* $P_{11}=P_{21}=P/\alpha_1$ *for some* $\alpha_1\in(0,1]$.

This completes the proof of Proposition 2. ■

B. General Channel Gains

In general, linear relaying outperforms bursty AF, and the gap between is bounded. On the other hand, the linear relaying capacity has an unbounded gap from the cutset bound. To see these, we first establish the following.

Denote $\bar R_{\rm AF}(P_1,P_2,P_3)$ as the achievable rate of AF when the power constraints are P_1, P_2, and P_3. And denote $\bar R_{\rm AF}^{\rm (conc)}(P_1,P_2,P_3)$ as the concave envelope of $\bar R_{\rm AF}(P_1,P_2,P_3)$. We now have the following result.

Proposition 3. *Given any channel gains* g_{21}, g_{31}, g_{42}, *and* g_{43}, *there exists* $P_0(g_{21},g_{31},g_{42},g_{43})$ *such that*

$$C_{\rm L}(P)\le \bar R_{\rm AF}^{(conc)}(4P)$$

for all $P\ge P_0(g_{21},g_{31},g_{42},g_{43})$.

Thus we actually have

$$\bar R_{\rm AF}(P)\le R_{\rm BAF}(P)\le C_{\rm L}(P)\le \bar R_{\rm AF}^{\rm (conc)}(4P)$$

asymptotically in P, which means that AF performs as well as the bursty AF and also optimal linear relaying scheme (up to 2 bits asymptotically).

Proof of Proposition 3: We first note that each term in the summation of (5) satisfies

$$\bar R_{\rm AF}^{\rm (conc)}(P_{1k},P_{2k},P_{3k})\le \bar R_{\rm AF}^{\rm (conc)}(P/\alpha_k,P/\alpha_k,P/\alpha_k)$$

for each k. In addition, for P sufficiently large, $\bar R_{\rm AF}^{\rm (conc)\prime\prime}(P)<0$. Hence, for $P>P_0(g_{21},g_{31},g_{42},g_{43})$,

$$\begin{aligned}
C_{\rm L}(P)&=\max\sum_{k=1}^{4}\alpha_kR_{\rm AF}(P_{1k},P_{2k},P_{3k})\\
&\le\max\sum_{k=1}^{4}\alpha_k\bar R_{\rm AF}^{\rm (conc)}(P_{1k},P_{2k},P_{3k})\\
&\le\max\sum_{k=1}^{4}\alpha_k\bar R_{\rm AF}^{\rm (conc)}(P/\alpha_k,P/\alpha_k,P/\alpha_k)\\
&\le\max\bar R_{\rm AF}^{\rm (conc)}\Big(\sum_{k=1}^{4}\alpha_kP_k\Big)\\
&=\bar R_{\rm AF}^{\rm (conc)}(4P).
\end{aligned}$$

This completes the proof of Proposition 3. ■

For a general Gaussian diamond network, AF performs rather poorly, and $\bar R_{\rm AF}^{\rm (conc)}(P)$ (and consequently $C_{\rm L}(P)$) has an unbounded gap from $R_{\rm CS}(P)$. To see this, suppose that $g_{21}^2=g_{43}^2=g$ and $g_{31}^2=g_{42}^2=g^2$. On one hand, the cutset upper bound in (2) behaves as

$$R_{\rm CS}(P)\sim\frac12\log(g^2P)$$

as $g\to\infty$. On the other hand, the amplify–forward lower bound reduces to

$$R_{\rm AF}(P)=\mathsf{C}\left(\frac{\left(\sqrt{\frac{gg^2P^2}{gP+1}}+\sqrt{\frac{g^2gP^2}{g^2P+1}}\right)^2}{1+\frac{g^2P}{gP+1}+\frac{gP}{g^2P+1}}\right),$$

and hence $\bar R_{\rm AF}^{\rm (conc)}(4P)\sim\frac12\log(gP)$ as $g\to\infty$.

Combining this observation with Proposition 3 establishes the following.

Proposition 4.

$$\sup_{P,g}\big[R_{\rm CS}(P;\sqrt g,g,g,\sqrt g)-C_{\rm L}(P;\sqrt g,g,g,\sqrt g)\big]=\infty.$$

III. Proof of Theorem 1

We prove Theorem 1 in four steps.

Step 1. Multiletter Characterization

The linear relaying capacity can be characterized as

$$\begin{aligned}
C_{\rm L}&=\lim_{n\to\infty}R_{\rm L}^{(n)} &(8)\\
&:=\lim_{n\to\infty}\sup\frac1n I(X_1^n;Y_4^n), &(9)
\end{aligned}$$

where the supremum is over all $F(x_1^n)$ and strictly lower triangular matrices B_2 and B_3 such that $\sum_{i=1}^n\mathsf{E}(X_{ji}^2)\le nP$,

$j \in [1:3]$. Here the output sequence at node 4 is

$$\begin{aligned}
&Y_4^n \\
&= g_{42}X_2^n + g_{43}X_3^n + Z_4^n \\
&= g_{42}B_2(g_{21}X_1^n + Z_2^n) + g_{43}B_3(g_{31}X_1^n + Z_3^n) + Z_4^n \\
&= (g_{42}g_{21}B_2 + g_{43}g_{31}B_3)X_1^n + g_{42}B_2Z_2^n + g_{43}B_3Z_3^n + Z_4^n.
\end{aligned}$$

It can be easily shown that the supremum in (9) is attained by Gaussian distributions. Denoting the covariance matrix of X_1^n by K_1, we can simplify (9) as (11), where the maximum is over all $K_1 \succeq 0$ and strictly lower triangular B_2 and B_3 such that $\mathrm{tr}(K_1) \le nP$ and $\mathrm{tr}(B_j(g_{j1}^2K_1+1)B_j^T) \le nP$, $j = 2, 3$.

This is a nonconvex optimization problem. One potential approach to solving this problem is to adapt the technique by El Gamal, Mohseni, and Zahedi [4] for the linear relaying capacity of the receiver-frequency-division (RFD) Gaussian relay channel. Here we take an alternative route and study a variational version of the problem as in [11].

Step 2. Variational Characterization

We can establish the following.

Lemma 2. *C_{L} can be expressed as* (12)*, where the supremum is over all power spectral densities $K_1(e^{i\theta})$ and strictly causal filters $B_j(e^{i\theta}) = \sum_{k=1}^{\infty} b_k^{(j)} e^{ki\theta}$, $j = 2, 3$, satisfying*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_1(e^{i\theta})\,d\theta \le P,$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |B_j(e^{i\theta})|^2(g_{j1}^2K_1(e^{i\theta})+1)\,d\theta \le P, \quad j = 2, 3.$$

The proof of the lemma, which is quite technical and follows similar lines as in [11], is omitted.

Step 3. Time-Shared Amplify–Forward

By the triangle inequality, we have

$$\begin{aligned}
&\left|g_{42}g_{21}B_2(e^{i\theta}) + g_{43}g_{31}B_3(e^{i\theta})\right| \\
&\qquad \le g_{42}g_{21}\left|B_2(e^{i\theta})\right| + g_{43}g_{31}\left|B_3(e^{i\theta})\right|
\end{aligned}$$

and hence the upper bounded in (13). Denoting

$$K_j(e^{i\theta}) := |B_j(e^{i\theta})|^2(g_{j1}^2K_1(e^{i\theta})+1), \quad j = 2, 3,$$

relaxing the causality constraint on $B_2(e^{i\theta})$ and $B_3(e^{i\theta})$, and substituting $K_2(e^{i\theta})$ and $K_3(e^{i\theta})$ in (13), we can further upper bound C_{L} as (14), where the supremum is over all power spectral densities $K_1(e^{i\theta})$, $K_2(e^{i\theta})$, and $K_3(e^{i\theta})$ satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_j(e^{i\theta})\,d\theta \le P, \quad j = 1, 2, 3.$$

Noting the similarity to (3), we can express this upper bound equivalently as

$$\begin{aligned}
&C_{\mathrm{L}} \le R_{\mathrm{L}}^* \\
&\quad := \sup_{K_1,K_2,K_3} \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{\mathrm{AF}}(K_1(e^{i\theta}), K_2(e^{i\theta}), K_3(e^{i\theta}))\,d\theta.
\end{aligned} \tag{15}$$

Now this bound, although expressed in the frequency domain, matches exactly the rate achieved when we use AF at frequency $\theta \in [-\pi, \pi]$ with power constraints on $K_1(e^{i\theta})$, $K_2(e^{i\theta})$, and $K_3(e^{i\theta})$. Thus, the $C_{\mathrm{L}} = R_{\mathrm{L}}^*$.

Step 4. Cardinality Reduction

To further simplify (15), we observe that the point $(P, P, P, R_{\mathrm{L}}^*)$ lies on the convex hull of the four-dimensional region $\mathscr{R}$ with the boundary characterized by $(P_1, P_2, P_3, R_{\mathrm{AF}}(P_1, P_2, P_3))$. Thus, by the Fenchel–Eggleston–Carathéodory theorem [12], the point can be represented as a convex combination of at most four points on the boundary. Thus, the upper bound R_{L}^* can be written as (5), which completes the proof of Theorem 1.

IV. Discussion

Our results can be extended in a straightforward manner to the N-relay Gaussian diamond network, in which nodes 2 through $N+1$ relay communication between nodes 1 and $N+2$. Let $R_{\mathrm{AF}}^{(N)}(P_1, P_2, ..., P_{N+1})$ be the rate achieved by AF with powers $P_1, \ldots, P_{N+1}$ at nodes $1, \ldots, N+1$. We characterize the linear relaying capacity as follows.

$$R_{\mathrm{L}}^{(n)} = \max \frac{1}{2n}\left(\log \frac{\left|I + g_{42}^2B_2B_2^T + g_{43}^2B_3B_3^T + (g_{42}g_{21}B_2 + g_{43}g_{31}B_3)K_1(g_{42}g_{21}B_2 + g_{43}g_{31}B_3)^T\right|}{\left|I + g_{42}^2B_2B_2^T + g_{43}^2B_3B_3^T\right|}\right) \tag{11}$$

$$C_{\mathrm{L}} = \sup \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathsf{C}\left(\frac{\left|g_{42}g_{21}B_2(e^{i\theta}) + g_{43}g_{31}B_3(e^{i\theta})\right|^2 K_1(e^{i\theta})}{1 + g_{42}^2|B_2(e^{i\theta})|^2 + g_{43}^2|B_3(e^{i\theta})|^2}\right) d\theta \tag{12}$$

$$\le \sup \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathsf{C}\left(\frac{\left(g_{42}g_{21}\left|B_2(e^{i\theta})\right| + g_{43}g_{31}\left|B_3(e^{i\theta})\right|\right)^2 K_1(e^{i\theta})}{1 + g_{42}^2|B_2(e^{i\theta})|^2 + g_{43}^2|B_3(e^{i\theta})|^2}\right) d\theta \tag{13}$$

$$\le \sup \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathsf{C}\left(\frac{\left(\sqrt{\frac{g_{42}^2K_2(e^{i\theta})\cdot g_{21}^2K_1(e^{i\theta})}{g_{21}^2K_1(e^{i\theta})+1}} + \sqrt{\frac{g_{43}^2K_3(e^{i\theta})\cdot g_{31}^2K_1(e^{i\theta})}{g_{31}^2K_1(e^{i\theta})+1}}\right)^2}{1 + \frac{g_{42}^2K_2(e^{i\theta})}{g_{21}^2K_1(e^{i\theta})+1} + \frac{g_{43}^2K_3(e^{i\theta})}{g_{31}^2K_1(e^{i\theta})+1}}\right) d\theta \tag{14}$$

Theorem 2. *The linear relaying capacity of the N-relay Gaussian diamond network with power constraint P on each node is*

$$C_{\mathrm{L}}^{(N)} = \max \sum_{k=1}^{N+2} \alpha_k R_{\mathrm{AF}}^{(N)}(P_{1k}, P_{2k}, ..., P_{(N+1)k}),$$

where the maximum is over all $\alpha_k \geq 0$ and $P_{jk} \geq 0$, $j \in [1:N+1]$, $k \in [1:N+2]$ such that $\sum_{k=1}^{N+2} \alpha_k = 1$ and $\sum_{k=1}^{N+2} \alpha_k P_{jk} \leq P$, $j \in [1:N+1]$.

When the channel is symmetric, i.e., $g_{j1} \equiv g$ and $g_{N+2,j} \equiv h$ for all j, bursty AF achieves the linear relaying capacity.

Proposition 5.

$$C_{\mathrm{L}}^{(N)}(P; g, \dots, g, h, \dots, h) = \max_{0<\alpha\leq 1} \alpha\, \mathsf{C}\left(1 + \frac{(NghP/\alpha)^2}{1+(N+1)h^2P/\alpha}\right).$$

In general, linear relaying does not improve much upon AF.

Proposition 6. *There exists $P_0 = P_0(g_{21}, \dots, g_{N+2,N+1})$ such that*

$$C_{\mathrm{L}}^{(N)}(P) \leq R_{\mathrm{AF}}^{(N)}((N+2)P)$$

for all $P \geq P_0$.

The techniques developed in this paper are applicable to other classes of relay networks with layered structure in showing that the optimal linear relaying scheme simplifies as time-shared AF. For instance, consider the RFD Gaussian relay channel studied by El Gamal, Mohseni, and Zahedi [4]. Following the same four steps in the previous section, we can establish the linear relaying capacity of RFD Gaussian relay channel as

$$C_{\mathrm{L}}^{\mathrm{RFD}} = \max \sum_{k=1}^{3} \alpha_k R_{\mathrm{AF}}^{\mathrm{RFD}}(P_{1k}, P_{2k}), \tag{16}$$

where $\alpha_k \geq 0$, $P_{jk} \geq 0$, $j \in [1:2]$, $k \in [1:3]$ such that $\sum_{k=1}^{3} \alpha_k = 1$ and $\sum_{k=1}^{3} \alpha_k P_{jk} = P$, $j \in [1:2]$, and $R_{\mathrm{AF}}^{\mathrm{RFD}}(P_1, P_2)$ is the achievable rate of AF with powers P_1 and P_2 at the nodes 1 and 2 of the channel. Incidentally, this result slightly simplifies the characterization of C_{L} in [4].

Appendix

Let

$$r(\alpha, P_1, P_2) := \alpha R_{\mathrm{AF}}(P_1, P_2)$$

where $R_{\mathrm{AF}}(P_1, P_2) := R_{\mathrm{AF}}(P_1, P_2, P_2)$. It is the rate of AF with power $P_1 > 0$ at node 1 and $P_2 > 0$ at nodes 2 and 3 over a fraction $\alpha \in (0, 1]$ of time.

If we slightly increase the power P_1 and P_2 to $P_1 + \mathrm{d}P_1$ and $P_2 + \mathrm{d}P_2$ respectively, and, in order to keep the amount of power used unchanged, slightly decrease the duty period α to $\alpha - \mathrm{d}\alpha$ with

$$\mathrm{d}\alpha = \frac{\alpha}{P_1}\mathrm{d}P_1 = \frac{\alpha}{P_2}\mathrm{d}P_2 \tag{17}$$

we can then characterize the increment of the rate by

$$\mathrm{d}r(\alpha, P_1, P_2) := \alpha \frac{\partial R_{\mathrm{AF}}}{\partial P_1}(P_1, P_2)\mathrm{d}P_1 + \alpha \frac{\partial R_{\mathrm{AF}}}{\partial P_2}(P_1, P_2)\mathrm{d}P_2 - R_{\mathrm{AF}}(P_1, P_2)\mathrm{d}\alpha$$

with (17) satisfied.

Now let $P_{jk}^\star$ and $\alpha_k^\star$, $j \in [1:2], k \in [1:4]$ be a optimal solution.

Let index k_0 be such that $P_{1k_0}^\star = P_{2k_0}^\star = 0$ and $\alpha_{k_0}^\star > 0$, and without loss of generality we assume that $P_{1k}^\star > 0$, $P_{2k}^\star > 0$ and $\alpha_k^\star > 0$ for all $k \neq k_0$.

If such k_0 exists, then we have

$$\mathrm{d}r(\alpha_k^\star, P_{1k}^\star, P_{2k}^\star) = 0$$

for all $k \neq k_0$.

Otherwise

$$\mathrm{d}r(\alpha_k^\star, P_{1k}^\star, P_{2k}^\star) < 0$$

for all $k \in [1:4]$.

Combining both cases, we actually have

$$\mathrm{d}r(\alpha_k^\star, P_{1k}^\star, P_{2k}^\star) \leq 0$$

which, by (17) plugged in, leads to

$$\frac{\partial R_{\mathrm{AF}}}{\partial P_1}(P_{1k}^\star, P_{2k}^\star)P_{1k} + \frac{\partial R_{\mathrm{AF}}}{\partial P_2}(P_{1k}^\star, P_{2k}^\star)P_{2k} \leq R_{\mathrm{AF}}(P_{1k}^\star, P_{2k}^\star)$$

for all $k \neq k_0$.

Assume $g = h = 1$ for simplicity. By plugging in the expressions of $\frac{\partial R_{\mathrm{AF}}}{\partial P_1}$, $\frac{\partial R_{\mathrm{AF}}}{\partial P_2}$ and R_{AF}, and after some simplifications along with the inequality $\log(1+x) \leq x$ for $x \geq 0$, we can obtain

$$4P_{1k}^\star P_{2k}^\star > 1 \tag{18}$$

for all $k \neq k_0$, which is a necessary condition of the optimality of $P_{1k}^\star$ and $P_{2k}^\star$.

Now define $A^\star$ as the convex hull of points $(P_{1k}^\star, P_{2k}^\star)$, $k \neq k_0$. It is easy to show that $4uv > 1$ for $\forall\ (u, v) \in A^\star$ (since $\{(x, y) : 4xy > 1, x > 0, y > 0\}$ is a convex set).

Now we show the concavity of function $R_{\mathrm{AF}}(\cdot, \cdot)$ in $A^\star$.

We have

$$\frac{\partial^2 R_{\mathrm{AF}}(x,y)}{\partial x^2} = -\frac{8y(2y+1)(x+4y+4xy+4y^2+1)}{(x+2y+1)^2(x+2y+4xy+1)^2} < 0,$$

$$\frac{\partial^2 R_{\mathrm{AF}}(x,y)}{\partial y^2} = -\frac{16x(x+1)(2x+2y+4xy+x^2+1)}{(x+2y+1)^2(x+2y+4xy+1)^2} < 0,$$

and

$$\frac{\partial^2 R_{\mathrm{AF}}(x,y)}{\partial x \partial y} = \frac{2}{(x+2y+1)^2} + \frac{2}{(x+2y+4xy+1)^2}$$

thus

$$\begin{aligned}
&\frac{\partial^2 R_{\mathrm{AF}}(x,y)}{\partial x^2}\frac{\partial^2 R_{\mathrm{AF}}(x,y)}{\partial y^2} - \left(\frac{\partial^2 R_{\mathrm{AF}}(x,y)}{\partial x \partial y}\right)^2 \\
=&\frac{64xy(4xy+4y+2x+1) - 32y - 16x - 16}{(x+2y+1)^2(x+2y+4xy+1)^3} \\
=&\frac{64xy(4xy+2y+x) + 16(2y+x+1)(4xy-1)}{(x+2y+1)^2(x+2y+4xy+1)^3} \\
>&0
\end{aligned}$$

if $4xy > 1$.

Therefore $R_{\rm AF}(\cdot,\cdot)$ is concave in $A^\star$. Consequently, we have

$$\begin{aligned}&\sum_{k\neq k_0}\alpha_k^\star R_{\rm AF}(P_{1k}^\star, P_{2k}^\star)\\ \leq&(\sum_{k\neq k_0}\alpha_k^\star)R_{\rm AF}\left(\frac{\sum_{k\neq k_0}\alpha_k^\star P_{1k}^\star}{\sum_{k\neq k_0}\alpha_k^\star}, \frac{\sum_{k\neq k_0}\alpha_k^\star P_{2k}^\star}{\sum_{k\neq k_0}\alpha_k^\star}\right)\\ =&\alpha^\star R_{\rm AF}\left(\frac{P}{\alpha^\star}, \frac{P}{\alpha^\star}\right)\end{aligned}$$

where $\alpha^\star := \sum_{k\neq k_0}\alpha_k^\star$. Note that $\alpha^\star = 1$ means that there does not exist $k_0 \in [1:4]$ such that $P_{1k_0}^\star = P_{2k_0}^\star = 0$ and $\alpha_{k_0}^\star > 0$.

It implies that the pair $(P_{1k}^\star, P_{2k}^\star)$ is either (0,0) or $(\frac{P}{\alpha^\star}, \frac{P}{\alpha^\star})$ for some $\alpha^\star$ optimally chosen, which completes the proof of the Lemma.

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