# Hybrid Coding

Young-Han Kim, UCSD

Living Information Theory Workshop

Happy <70.42>$^{\text{th}}$ birthday, Professor Berger!

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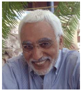
Sung Hoon Lim, Paolo Minero, and Abbas El Gamal



# Gratitude #1

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INFORMATION AND CONTROL **13**, 254–273 (1968)

## Rate Distortion Theory for Sources with Abstract Alphabets and Memory

TOBY BERGER†

*Advanced Development Laboratory, Raytheon Company*
*Wayland, Massachusetts 01778*

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## Rate Distortion Theory for Sources with Abstract Alphabets and Memory

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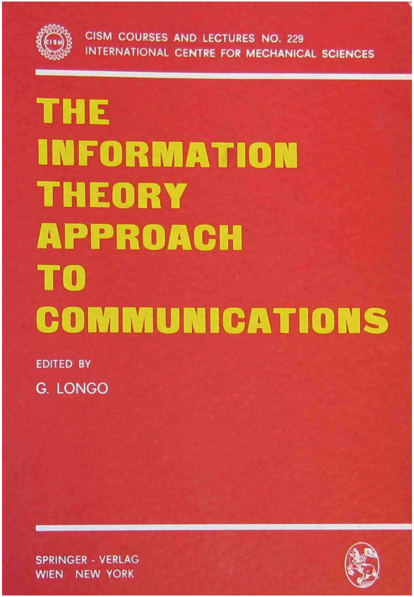
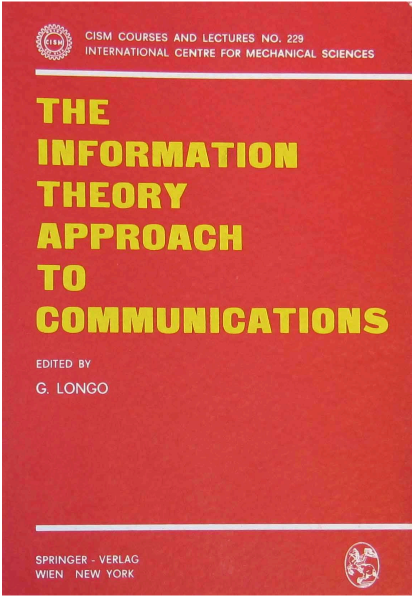
DEFINITION 1. A source $[X, \mu]$ is *block ergodic* if it is $\tau$-ergodic for every positive $\tau \in M$.

We show in the appendix that block ergodicity lies between ergodicity and weak mixing in restrictiveness. Note that $\tau$-ergodicity of $[X, \mu]$ and ergodicity of $[X, \mu]_\tau$ are equivalent.

# Gratitude #2


CISM COURSES AND LECTURES NO. 229
INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES
THE INFORMATION THEORY APPROACH TO COMMUNICATIONS
EDITED BY
G. LONGO
SPRINGER - VERLAG
WIEN NEW YORK

# Gratitude #2

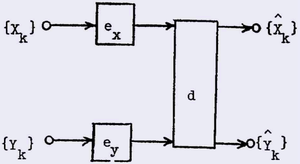



MULTITERMINAL SOURCE CODING

TOBY BERGER

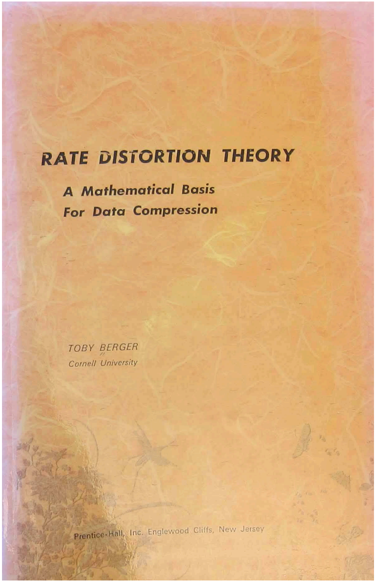
School of Electrical Engineering

Cornell University, Ithaca, New York

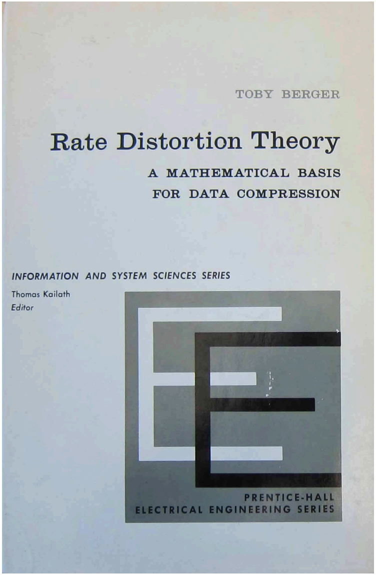



- Strong typicality
- (Extended) Markov lemma

# Gratitude #3


RATE DISTORTION THEORY
A Mathematical Basis
For Data Compression
TOBY BERGER
Cornell University
Prentice-Hall, Inc. Englewood Cliffs, New Jersey

# Gratitude #3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Rate Distortion Theory
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Thomas Kailath
Editor
PRENTICE-HALL
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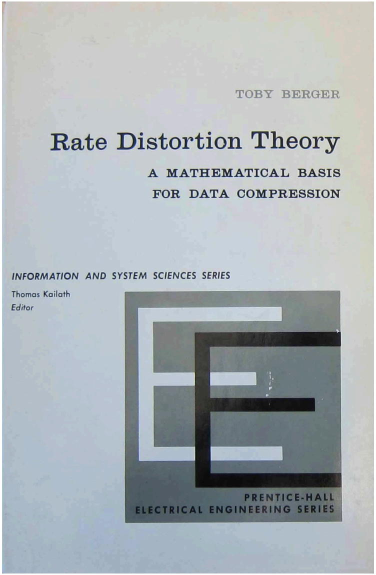
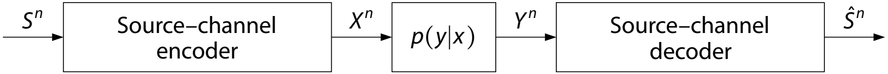



But life is short and information endless: nobody has time for everything. In practice we are generally forced to choose between an unduly brief exposition and no exposition at all. Abbreviation is a necessary evil and the abbreviator's business is to make the best of a job which, though intrinsically bad, is still better than nothing. He must learn to simplify, but not to the point of falsification. He must learn to concentrate upon the essentials of a situation, but without ignoring too many of reality's qualifying side issues.

—Aldous Huxley
*Brave New World Revisited*

# Point-to-Point Communication System




## Question

Find the sufficient and necessary condition to achieve

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(S_i, \hat{S}_i) \le D$$

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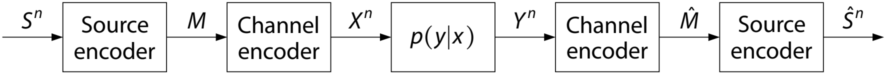


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## Answer: Source–channel separation (Shannon 1948, 1959)

- Sufficient and necessary condition:

$$R(D) = \min_{p(\hat{s}|s):\mathrm{E}(d(S,\hat{S}))\le D} I(S;\hat{S}) < \max_{p(x)} I(X;Y) = C$$

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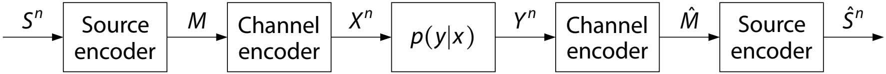


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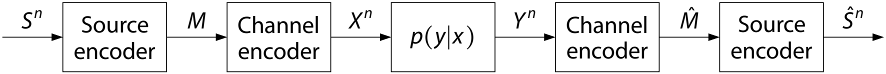


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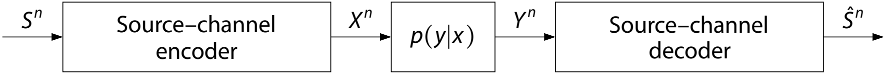
- Digital source–channel interface

# Uncoded Transmission



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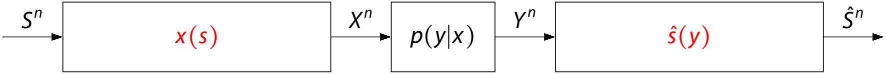
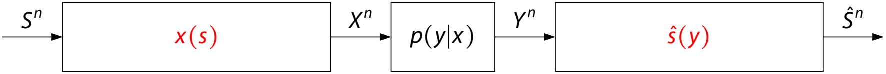


# Uncoded Transmission

$$S^n \longrightarrow \boxed{\quad x(s) \quad} \xrightarrow{X^n} \boxed{p(y|x)} \xrightarrow{Y^n} \boxed{\quad \hat{s}(y) \quad} \xrightarrow{\hat{S}^n}$$

- Optimal for quadratic Gaussian source coding over the Gaussian channel with average power constraint (Goblick 1965)

# Uncoded Transmission

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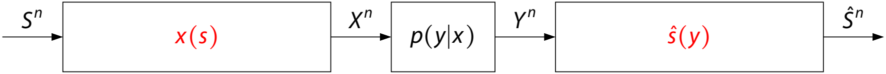


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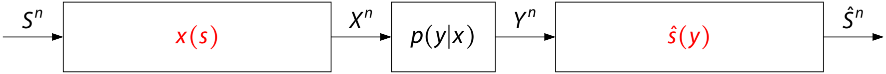


- Optimal for quadratic Gaussian source coding over the Gaussian channel with average power constraint (Goblick 1965)

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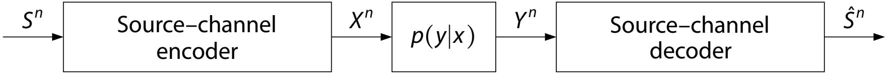
- "To code or not to code" (Gastpar, Rimoldi, and Vetterli 2003)
- Analog source–channel interface

# Hybrid Coding



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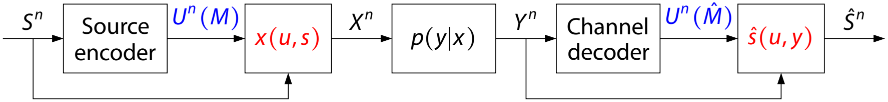

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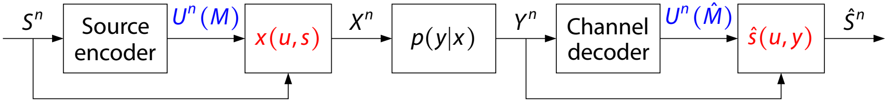


- Sufficient condition:

$$I(U;S) < I(U;Y)$$

for some $p(u|s)$, $x(u,s)$, $\hat{s}(u,y)$ such that $\mathrm{E}(d(S,\hat{S})) \le D$

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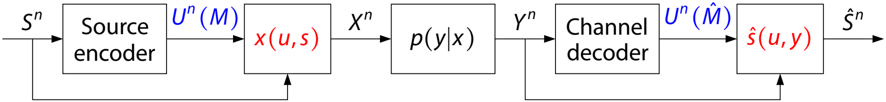


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- Analog/digital hybrid source–channel interface

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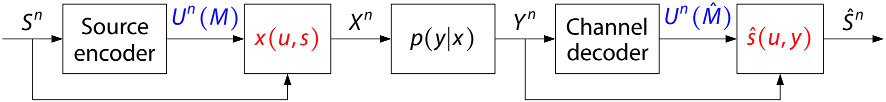


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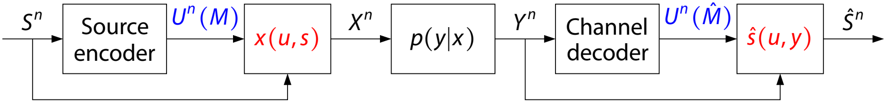


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  - $U = (\hat{S}, X) \sim p(\hat{s}|s)p(x)$: separate source and channel coding

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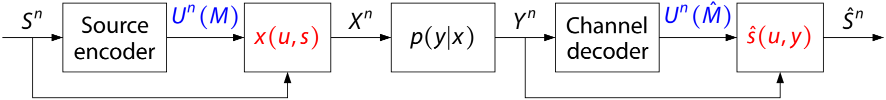


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  - $U = \emptyset$: uncoded transmission
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- "Beaten?" (Shamai, Verdú, and Zamir 1998, Mittal and Phamdo 2002, Tinguely and Lapidoth 2006, Caire and Narayanan 2007, Kochman, Khina, Erez, and Zamir 2009, Tian, Diggavi, and Shamai 2010)

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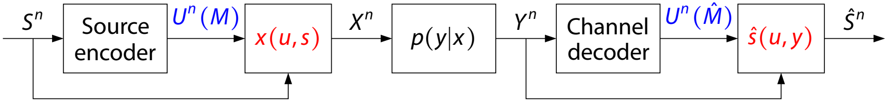


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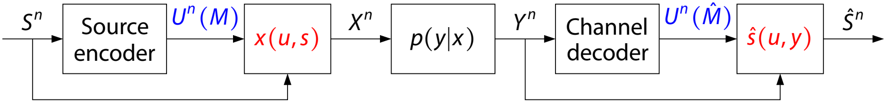
# Proof of Achievability



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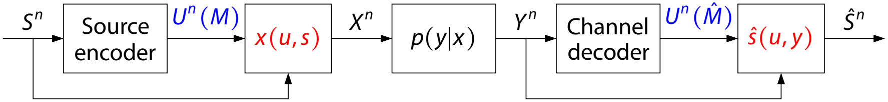
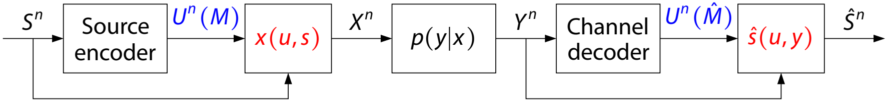



## Random codebook generation

$2^{nR}$ sequences $u^n(m) \sim \prod_{i=1}^{n} p_U(u_i)$ for $m \in [1:2^{nR}]$

# Proof of Achievability

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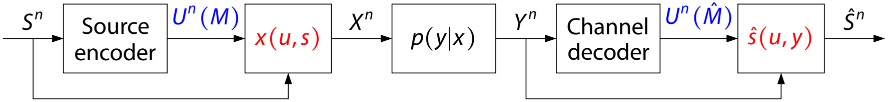
## Joint typicality encoding

Find an index $m$ such that $(u^n(m), s^n) \in \mathcal{T}_{\epsilon'}^{(n)}$ and transmit $x_i = x(u_i(m), s_i)$:

# Proof of Achievability

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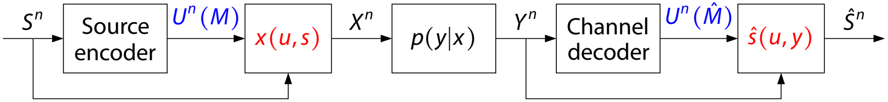


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## Joint typicality decoding

Find the unique index $\hat{m}$ such that $(u^n(\hat{m}), y^n) \in \mathcal{T}_{\epsilon}^{(n)}$ and reconstruct $\hat{s}_i = \hat{s}(u_i(\hat{m}), y_i)$:

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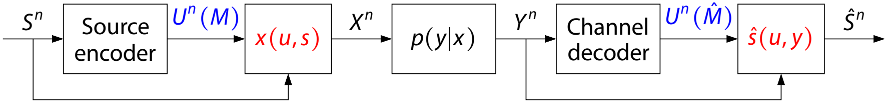


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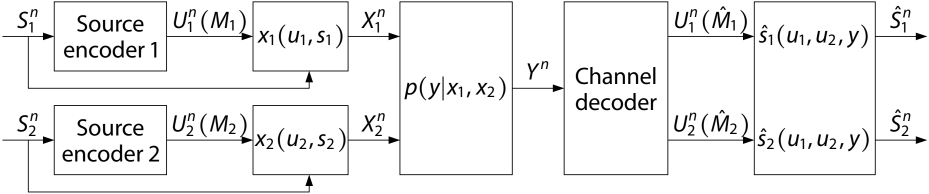
# Correlated Sources over a MAC




## Hybrid coding

- Berger–Tung source coding + MAC coding

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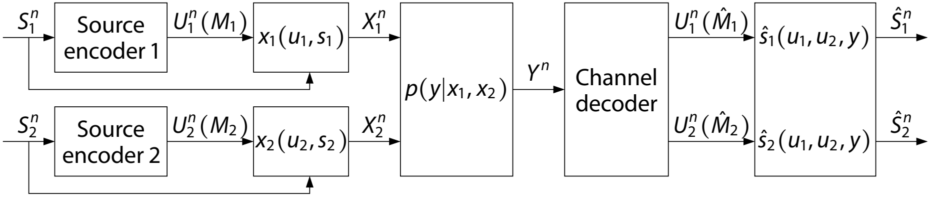


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$$R_1 > I(U_1; S_1),$$
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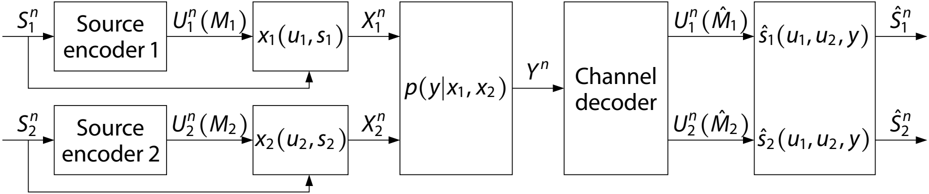


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$$R_1 > I(U_1; S_1),$$
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$$R_1 < I(U_1; Y, U_2),$$
$$R_2 < I(U_2; Y, U_1),$$
$$R_1 + R_2 < I(U_1, U_2; Y) + I(U_1; U_2)$$

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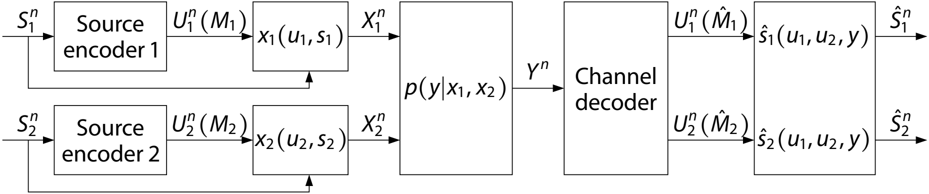


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$$
\begin{aligned}
I(U_1; S_1|Q) &< I(U_1; Y, U_2|Q), \\
I(U_2; S_2|Q) &< I(U_2; Y, U_1|Q), \\
I(U_1; S_1|Q) + I(U_2; S_2|Q) &< I(U_1, U_2; Y|Q) + I(U_1; U_2|Q)
\end{aligned}
$$

for some $p(q)p(u_1|s_1, q)p(u_2|s_2, q)$, $x_j(u_j, s_j, q)$, $\hat{s}_j(u_1, u_2, y, q)$ such that $\mathrm{E}(d_j(S_j, \hat{S}_j)) \le D_j$, $j = 1, 2$

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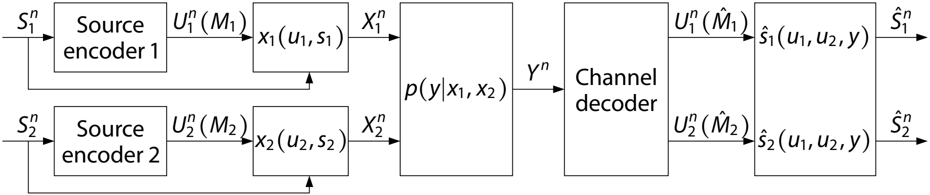


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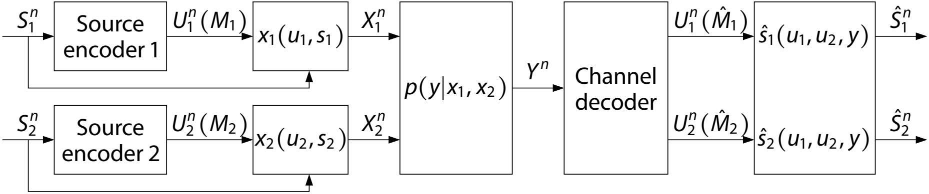


## Hybrid coding

- Berger–Tung source coding + MAC coding
- Special cases
  - $(\hat{S}_1, \hat{S}_2) = (S_1, S_2)$: lossless coding (Cover, El Gamal, and Salehi 1980)

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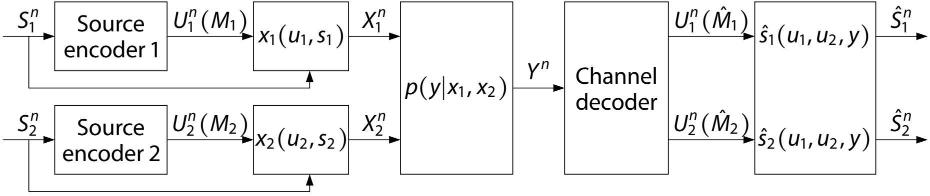


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  - $Y = (X_1, X_2)$: noiseless channel (Berger 1978, Tung 1978)

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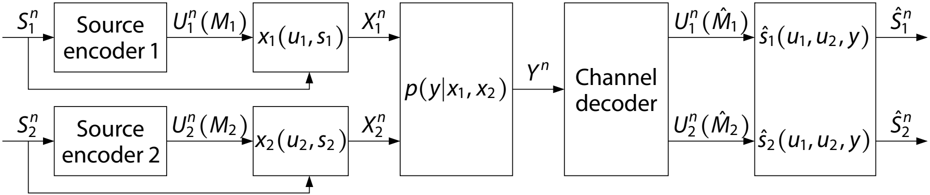


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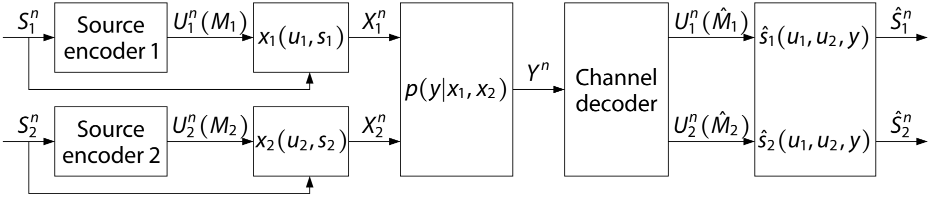


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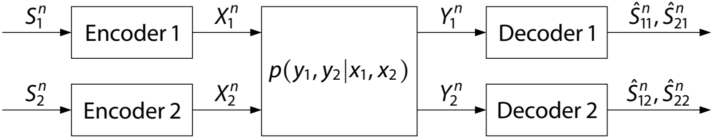
  - Gaussian $(S_1, S_2)$ and Gaussian MAC (Tinguely and Lapidoth 2006)
- Common part: Kaspi–Berger coding + Slepian–Wolf MAC coding

# 2-Sender 2-Receiver Channel



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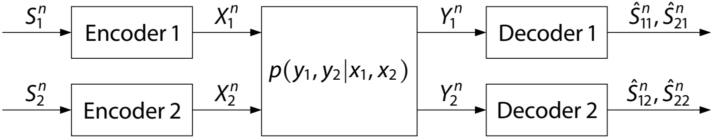


## Hybrid coding

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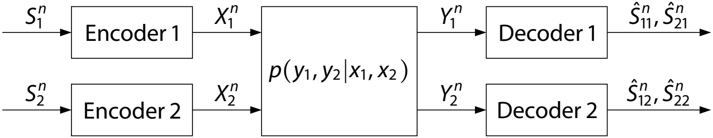


## Hybrid coding

- Conceptually simple yet performs very well

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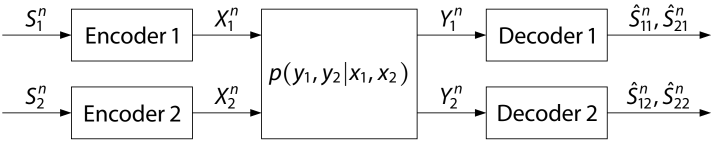
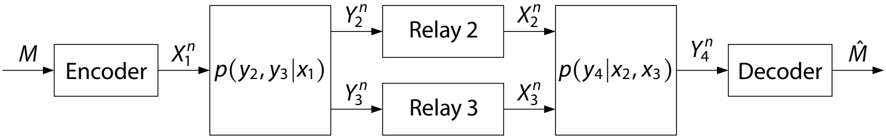



## Hybrid coding

- Conceptually simple yet performs very well
- Paolo Minero's talk on Monday 10:05 am (right before David Tse)

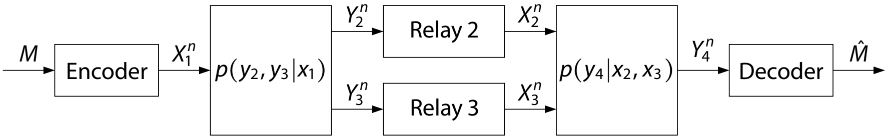
# Diamond Network




## Existing coding schemes

- Decode–forward (digital-to-digital interface)

# Diamond Network




## Existing coding schemes

- Decode–forward (digital-to-digital interface)
- Amplify–forward (analog-to-analog interface)

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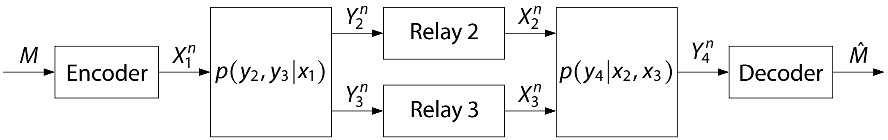


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- Noisy network coding (analog-to-digital interface)

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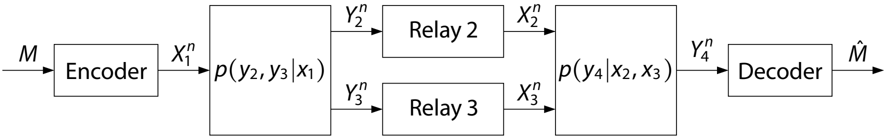


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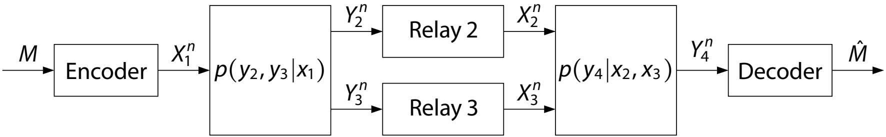
## Question

Can we do better than these coding schemes?

# Diamond Network




## Answer

- Hybrid coding (analog-to-[analog/digital] interface)

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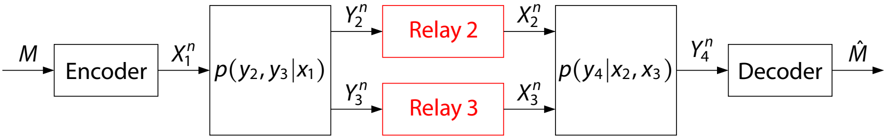


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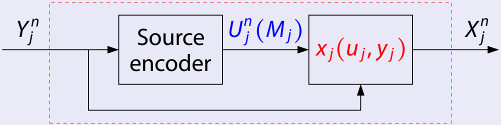

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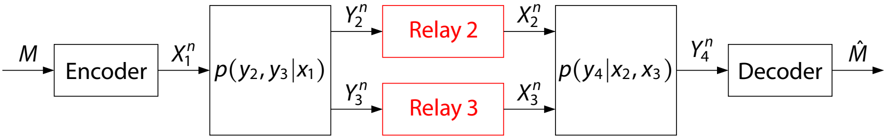


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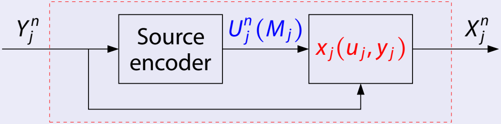
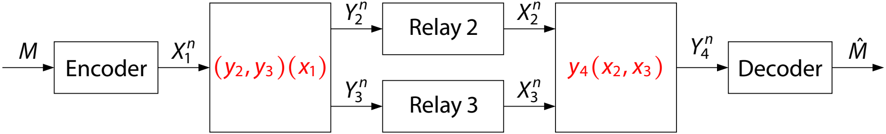



- Naturally combines noisy network coding and amplify–forward

# Example 1: Deterministic Diamond Network



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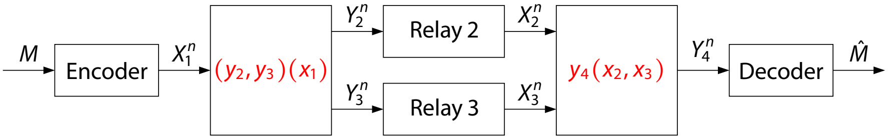


## Cutset bound

$$C \leq \max_{p(x_1,x_2,x_3)} \min\{H(Y_2, Y_3),\ H(Y_2)+H(Y_4|X_3),\ H(Y_3)+H(Y_4|X_2),\ H(Y_4)\}$$

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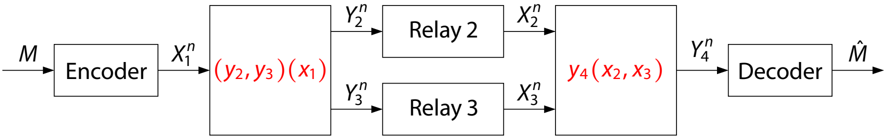


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## General lower bound (Avestimehr, Diggavi, and Tse 2007)

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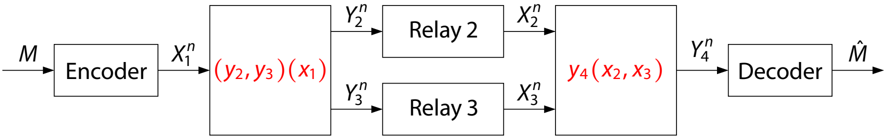


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## Hybrid coding lower bound

$$C \ge \max_{p(x_1)p(x_2|y_2)p(x_3|y_3)} \min\{H(Y_2, Y_3),\ H(Y_2)+H(Y_4|X_3),\ H(Y_3)+H(Y_4|X_2),\ H(Y_4)\}$$

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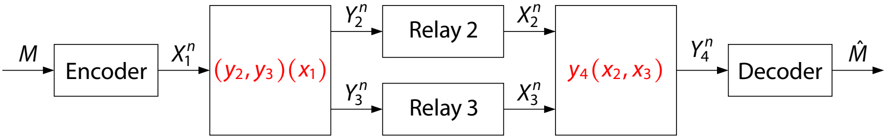
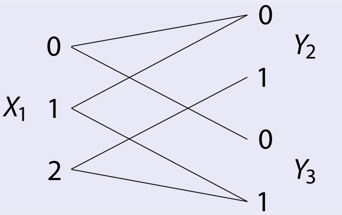



### Blackwell BC $(y_2(x_1), y_3(x_1))$




### Binary erasure MAC $y_4(x_2, x_3)$

$$Y_4 = X_2 + X_3$$

$$X_2, X_3 \in \{0, 1\},$$

$$Y_4 \in \{0, 1, 2\}$$

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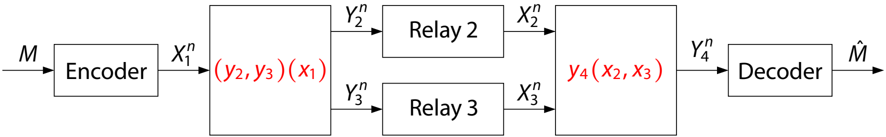


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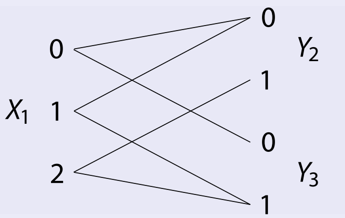


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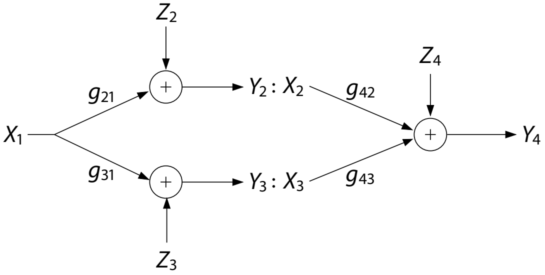
General lower bound = 1.5 < log 3 = Hybrid coding lower bound

# Example 2: Gaussian Diamond Network



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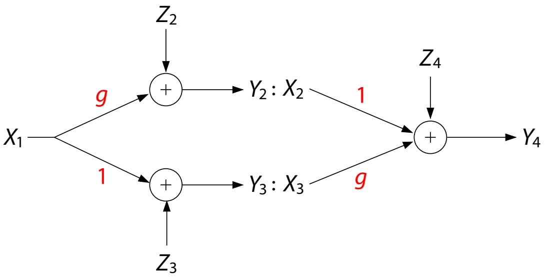
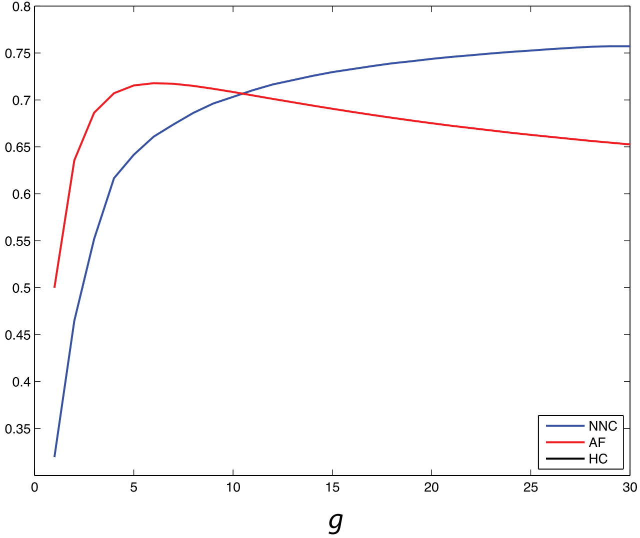


# AF vs. NNC vs. Hybrid Coding



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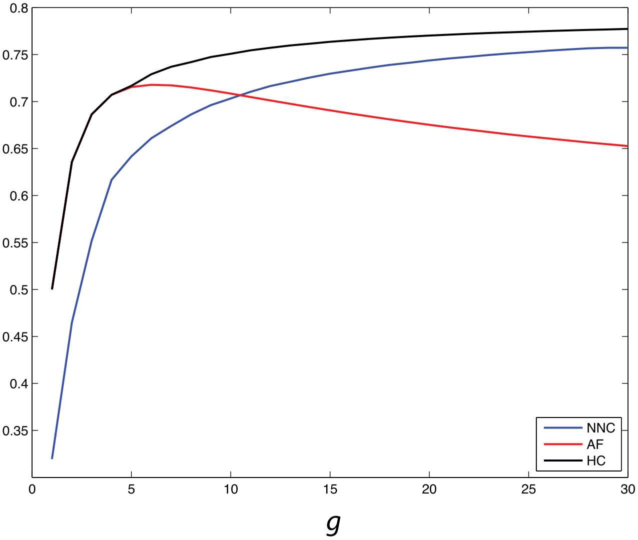
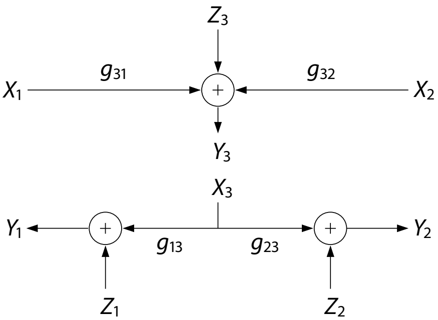


# Example 3: Gaussian Two-Way Relay Channel



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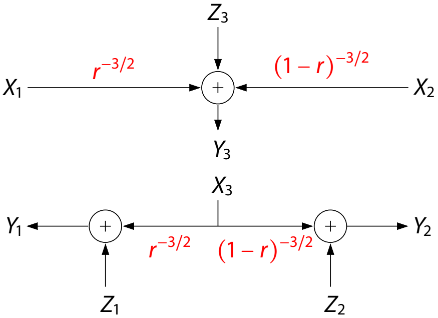
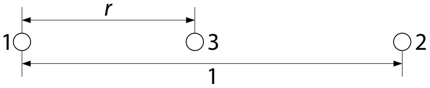
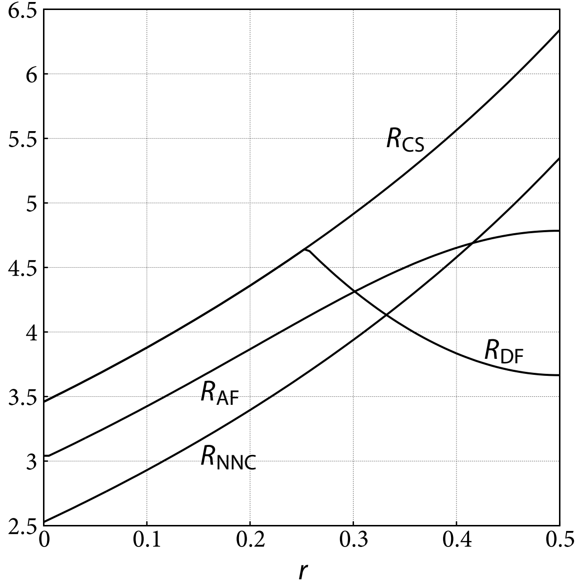


# Hybrid Coding vs. Other Coding Schemes



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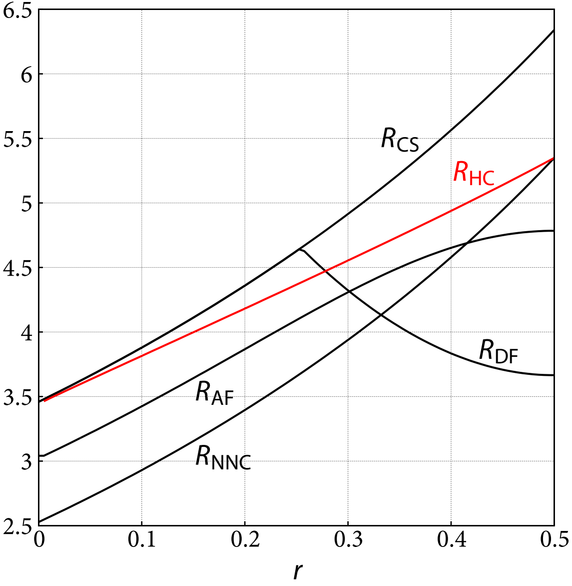

# Concluding Remarks

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## Back to reality

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- Barron and Oppenheim (2002): US Patent 6,441,764

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- Meir Feder: Amimon high-definition wireless audio-video modem

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## Back to reality

- Barron and Oppenheim (2002): US Patent 6,441,764
- Meir Feder: Amimon high-definition wireless audio-video modem
- Any good code for both (source) encoding and (channel) decoding?