The Role of Directed Information in Network Capacity

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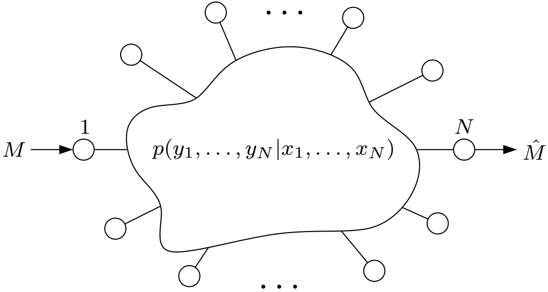
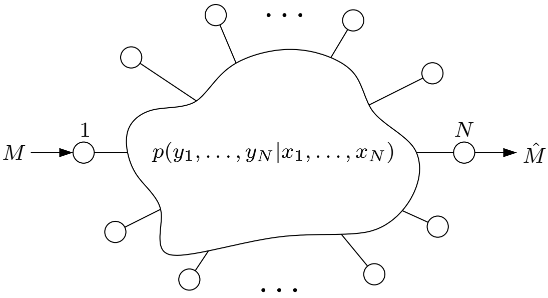
Communication over a general network



- Nodes: $(X_1, Y_1), \ldots, (X_N, Y_N)$
- Network: $p(y^N | x^N)$ captures noise, interference, broadcast, multiple access, ...

Communication over a general network



- Nodes: $(X_1, Y_1), \dots, (X_N, Y_N)$
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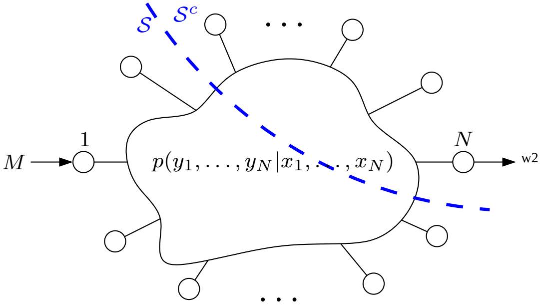
Network information flow questions

- Network capacity
- Optimal coding schemes

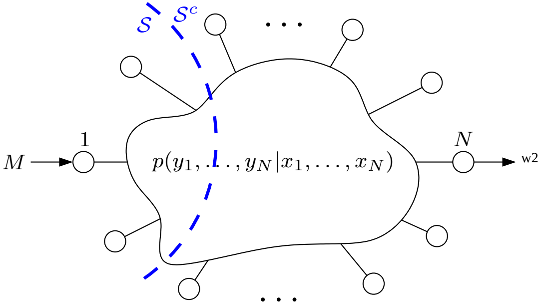
Cutset bound



Cutset bound (El Gamal 1981)

$$I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

Cutset bound



Cutset bound (El Gamal 1981)

$$\min_{\mathcal{S}} \; I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

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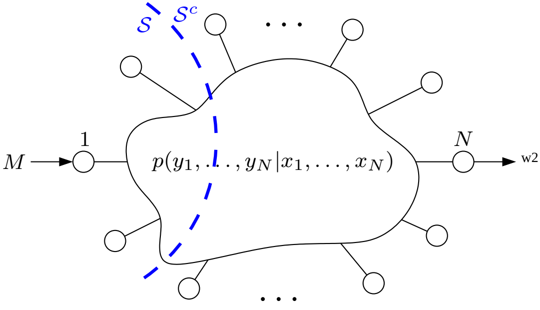

Cutset bound (El Gamal 1981)

$$C \leq \max_{p(x^N)} \min_{\mathcal{S}} \; I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

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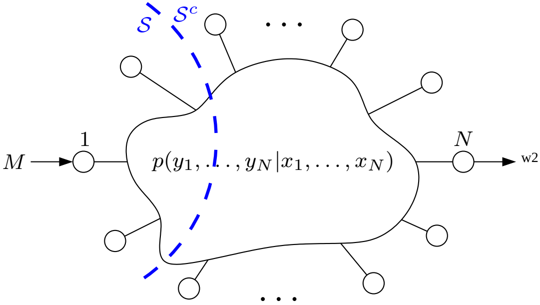

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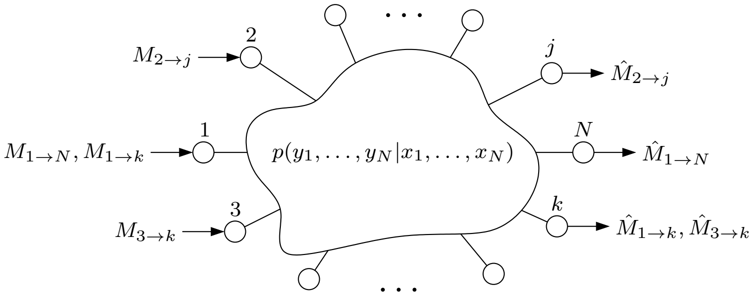
$$C \le \max_{p(x^N)} \min_{\mathcal{S}} \; I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

"amount of information nodes in $\mathcal{S}$ can transmit to nodes in $\mathcal{S}^c$"

Networks with multiple flows (Cover–Thomas 2006)



- Multiple information flows: $M_{1\to k}$, $M_{1\to N}$, $M_{2\to j}$, $M_{3\to k}$, $\ldots$

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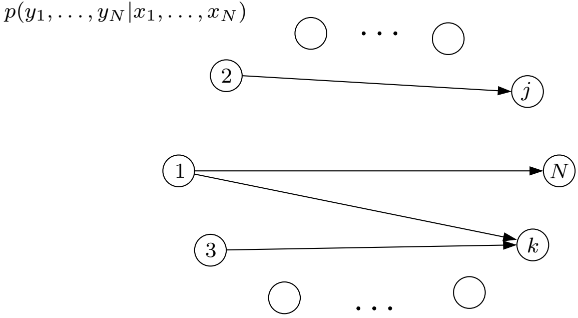
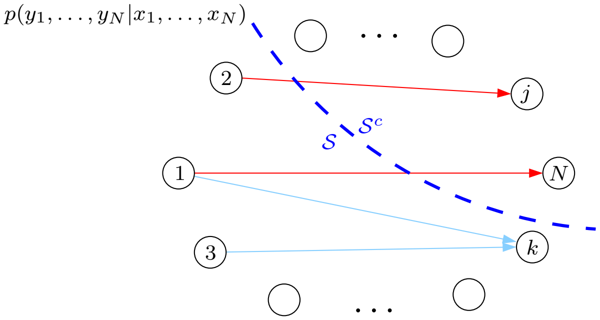


- Multiple information flows: $M_{1\to k}$, $M_{1\to N}$, $M_{2\to j}$, $M_{3\to k}$, $\ldots$
- Circles: nodes
- Edges: unicast flows (can be generalized to subset-to-subset multicast)
- Background: physical network $p(y_1, \ldots, y_N | x_1, \ldots, x_N)$

Cutset bound for multiple flows



Cutset bound

$$\sum_{j \in \mathcal{S}, k \in \mathcal{S}^c} R_{j \to k} \leq I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

for all $\mathcal{S}$ for some $p(x^N)$

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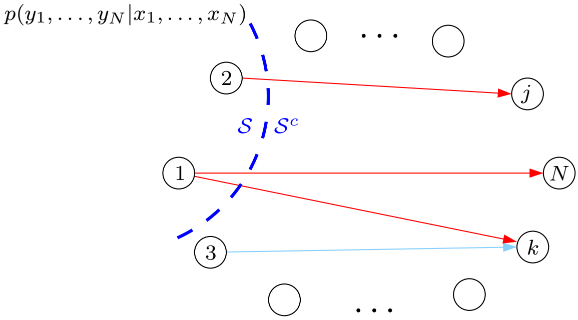

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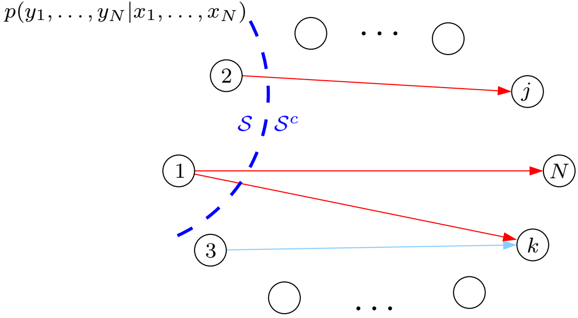

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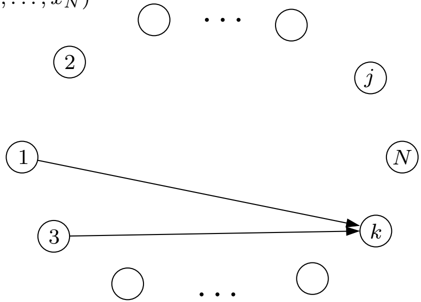

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- Good for single destination

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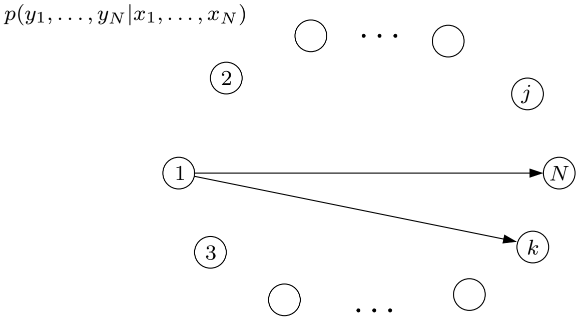

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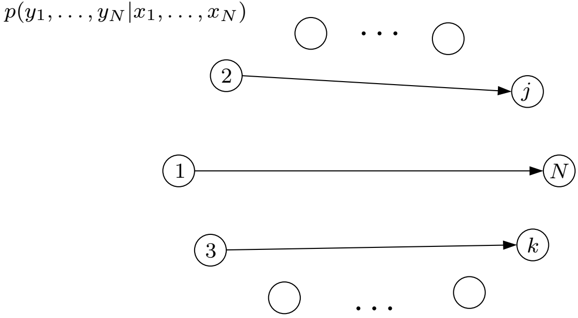

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- Good for single destination
- Fair for single source
- Not good for multiple unicast

How to improve the cutset bound

- Auxiliary random variables
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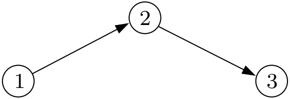
- What we are missing is a new bound
  - Simple, intuitive, general (just like cutset)
  - Yet better than cutset

Not-relay channel

$p(y_2, y_3|x_1, x_2)$



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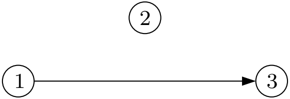

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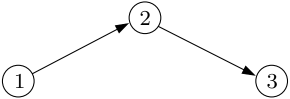

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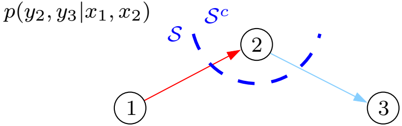

Cutset bound

$$R_{1\to 2} \le I(X_1; Y_2 | X_2)$$

for some $p(x_1, x_2)$

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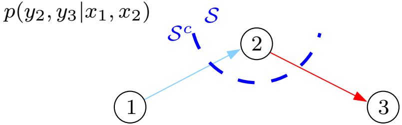

Cutset bound

$$R_{1\to 2} \le I(X_1; Y_2 | X_2)$$
$$R_{2\to 3} \le I(X_2; Y_3 | X_1)$$

for some $p(x_1, x_2)$

Not-relay channel

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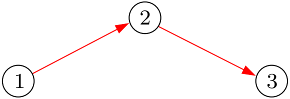

Improved bound (Kamath–Kim 2014)

$$R_{1\to 2} \le I(X_1; Y_2 | X_2)$$
$$R_{2\to 3} \le I(X_2; Y_3 | X_1)$$
$$R_{1\to 2} + R_{2\to 3} \le I(X_1; Y_2, Y_3 | X_2) + I(X_2; Y_3)$$

for some $p(x_1, x_2)$

The third inequality

$$R_{1\to 2} \le I(X_1; Y_2 | X_2) =: I_1$$
$$R_{2\to 3} \le I(X_2; Y_3 | X_1) =: I_2$$
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- Not too tight: $I_1 \le I_3$, $I_2 \le I_3$

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- The proof follows mechanically from Fano's inequality:

$$n(R_{1\to 2} + R_{2\to 3}) \le I(M_{1\to 2}; Y_2^n, Y_3^n | M_{2\to 3}) + I(M_{2\to 3}; Y_3^n) + n\epsilon_n$$

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- More questions than answers

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  - Is it really good?

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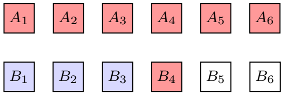
- More questions than answers
  - Is it really good?
  - Can we generalize and simplify?
  - How should we interpret?

Directed information

- Mutual information

$$I(A_1, \ldots, A_N; B_1, \ldots, B_N) = \sum_{j=1}^{N} I(A^N; B_j | B^{j-1})$$



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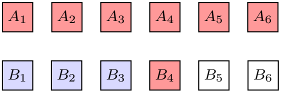
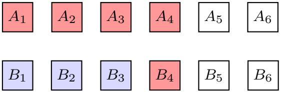


- Directed information (Marko 1973, Massey 1990)

$$I(A_1, \ldots, A_N \to B_1, \ldots, B_N) = \sum_{j=1}^{N} I(\color{red}{A^j}; B_j \mid B^{j-1})$$



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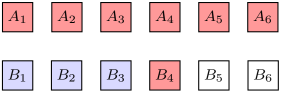

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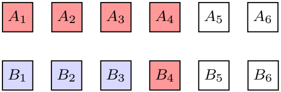
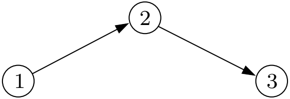


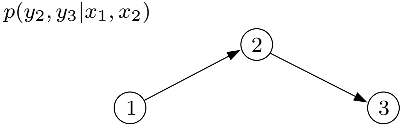
- Amount of information A^N causally provides about B^N (Permuter et al. 2011)

Directed cutset bound for the not-relay channel

$p(y_2, y_3|x_1, x_2)$



Directed cutset bound for the not-relay channel



Directed cutset bound (Kamath–Kim 2014, 2015)

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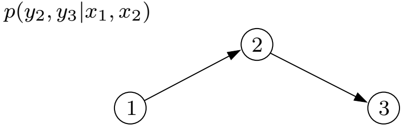
for some $p(x_1, x_2)$

Directed cutset bound for the not-relay channel



Directed cutset bound (Kamath–Kim 2014, 2015)

$$
\begin{aligned}
R_{1\to 2} &\le I(Y_2 \to X_1 | X_2) \\
R_{2\to 3} &\le I(Y_3 \to X_2 | X_1) \\
R_{1\to 2} + R_{2\to 3} &\le I(Y_3, Y_2 \to X_2, X_1)
\end{aligned}
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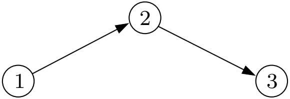
for some $p(x_1, x_2)$

Directed cutset bound for the not-relay channel

$p(y_2, y_3 | x_1, x_2)$



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for some $p(x_1, x_2)$

- Amount of information Y_3, Y_2 spatiocausally[TM] provide about X_2, X_1

Directed cutset bound for the not-relay channel

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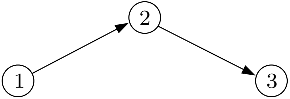

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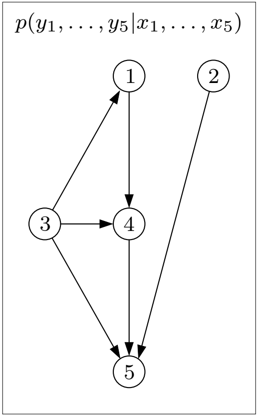
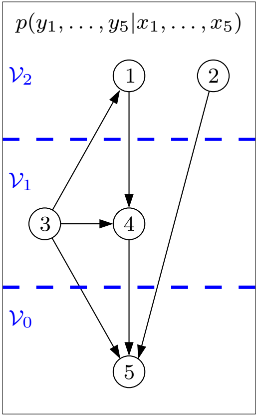
for some $p(x_1, x_2)$

- Amount of information Y_3, Y_2 spatiocausally™ provide about X_2, X_1
- Fano's inequality: $n(R_{1\to 2} + R_{2\to 3}) \le I(Y_3^n, Y_2^n \to M_{2\to 3}, M_{1\to 2}) + n\epsilon_n$

Directed cutset bound for the general network



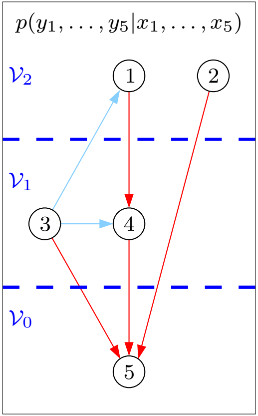
Directed cutset bound for the general network



- Partition the network into

$$\mathcal{V}_0 \uplus \mathcal{V}_1 \uplus \cdots \uplus \mathcal{V}_L$$

Directed cutset bound for the general network

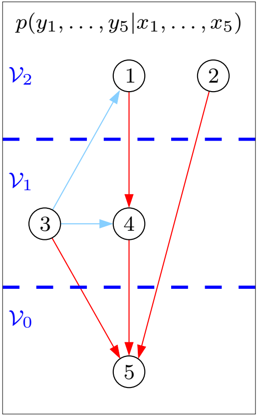


- Partition the network into

$$\mathcal{V}_0 \uplus \mathcal{V}_1 \uplus \cdots \uplus \mathcal{V}_L$$

- Consider the flows cut by the partition
  "information flows from top to bottom"

Directed cutset bound for the general network



- Partition the network into

$$\mathcal{V}_0 \uplus \mathcal{V}_1 \uplus \cdots \uplus \mathcal{V}_L$$

- Consider the flows cut by the partition
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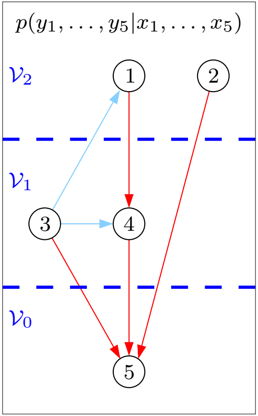
- Bound the sum rate by

$$I(Y(\mathcal{V}_0), \ldots, Y(\mathcal{V}_{L-1}) \to X(\mathcal{V}_1), \ldots, X(\mathcal{V}_L) | X(\mathcal{V}_0))$$

"amount of information nodes in $\mathcal{V}_0, \ldots, \mathcal{V}_{L-1}$ can receive spatiocausally from nodes in $\mathcal{V}_1, \ldots, \mathcal{V}_L$"

Directed cutset bound for the general network



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"amount of information nodes in $\mathcal{V}_0, \ldots, \mathcal{V}_{L-1}$ can receive spatiocausally from nodes in $\mathcal{V}_1, \ldots, \mathcal{V}_L$"

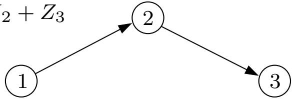
$$\sum_{f \text{ cut by } \mathcal{V}_0^L} R_f \leq I(Y(\mathcal{V}_0), \ldots, Y(\mathcal{V}_{L-1}) \to X(\mathcal{V}_1), \ldots, X(\mathcal{V}_L) | X(\mathcal{V}_0))$$

for every $(L+1)$ partition $\mathcal{V}_0^L$ and every $L = 1, \ldots, N$ for some $p(x_1, \ldots, x_N)$

Feature 1: High SNR performance

$Y_2 = X_1 + Z_2$

$Y_3 = X_1 + X_2 + Z_3$



- Cutset:

$$R_{1\to 2} \leq I(X_1; Y_2 | X_2) = \mathsf{C}(P)$$
$$R_{2\to 3} \leq I(X_2; Y_3 | X_1) = \mathsf{C}(P)$$

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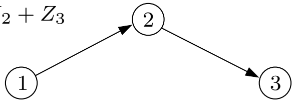

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- Time division with power control:

$$R_{1\to 2} + R_{2\to 3} \ge \mathsf{C}(2P)$$

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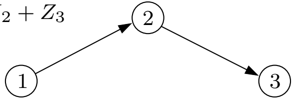

- Cutset:

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- Time division with power control:

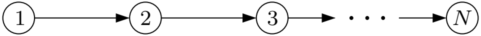
$$R_{1\to 2} + R_{2\to 3} \geq \mathsf{C}(2P)$$

- Directed cutset:

$$R_{1\to 2} + R_{2\to 3} \leq I(Y_3, Y_2 \to X_2, X_1) \leq \mathsf{C}(8P)$$

Feature 2: Scaling law

$$Y_2 = X_1 + Z_2 \qquad \cdots \qquad Y_N = X_1 + \cdots + X_{N-1} + Z_N$$



- Cutset:

$$R_{\text{sum}} = O(N)$$

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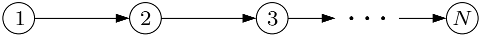

- Cutset:

$$R_{\text{sum}} = O(N)$$

- Time division with power control:

$$R_{\text{sum}} = \Omega(\log N)$$

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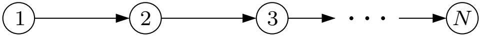


- Cutset:

$$R_{\text{sum}} = O(N)$$

- Time division with power control:

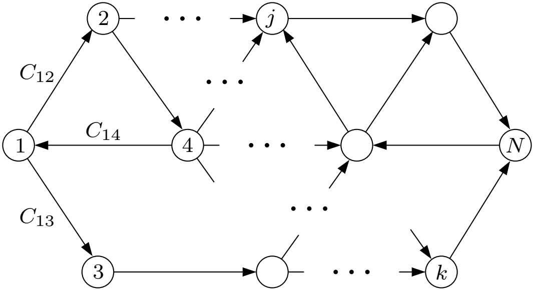
$$R_{\text{sum}} = \Omega(\log N)$$

- Directed cutset:

$$R_{\text{sum}} = O(\log N)$$

Feature 3: Graphical networks

$$X_k = (X_{kl} : (k,l) \in \mathcal{E}), \quad Y_k = (X_{jk} : (j,k) \in \mathcal{E}), \quad X_{jk} \in [1 : 2^{C_{jk}}]$$



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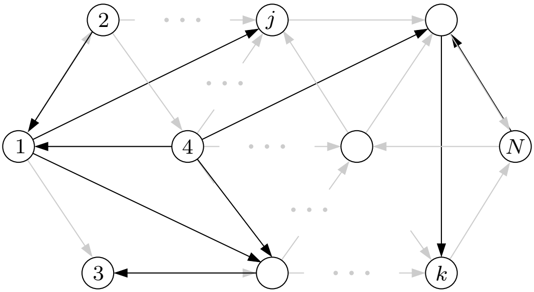

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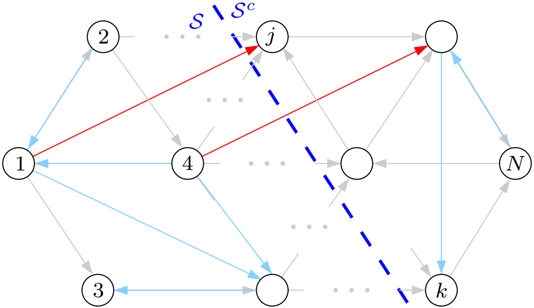

- Cutset (node-cut, not edge-cut):

$$\sum_f R_f \le H(Y(\mathcal{S}^c)|X(\mathcal{S}^c)) \le C(\mathcal{S}, \mathcal{S}^c)$$

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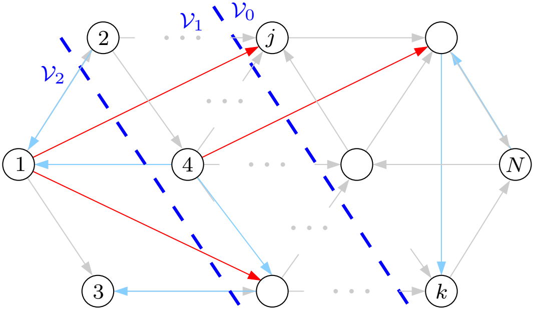

- Cutset (node-cut, not edge-cut):

$$\sum_f R_f \le H(Y(\mathcal{S}^c)|X(\mathcal{S}^c)) \le C(\mathcal{S}, \mathcal{S}^c)$$

- Directed cutset (still node-cut):

$$\sum_f R_f \le \sum_l H(Y(\mathcal{V}_l)|X(\mathcal{V}_0^l), Y(\mathcal{V}_0^{l-1})) \le \sum_l C(\mathcal{V}(l+1), \mathcal{V}(l))$$

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- Cutset (node-cut, not edge-cut):

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