Causality Through Directed Information

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Related publications

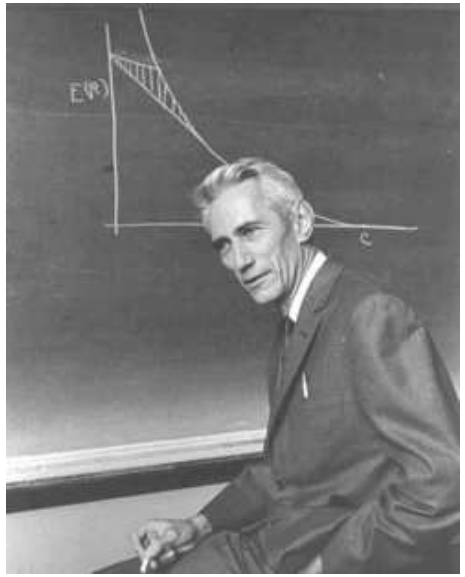
- Haim H. Permuter, Young-Han Kim, and Tsachy Weissman, "Interpretations of directed information in portfolio theory, data compression, and hypothesis testing," *IEEE Transactions on Information Theory,* vol. 57, no. 6, pp. 3248–3259, June 2011.

- Tsachy Weissman, Young-Han Kim, and Haim H. Permuter, "Directed information, causal estimation, and communication in continuous time," *IEEE Transactions on Information Theory,* vol. 59, no. 3, pp. 1271–1287, March 2013.

- Jiantao Jiao, Lei Zhao, Haim H. Permuter, Young-Han Kim, and Tsachy Weissman, "Universal estimation of directed information," to appear in *IEEE Transactions on Information Theory*, 2013.

- For more information, visit http://circuit.ucsd.edu/~yhk

Shannon's information measures (1948)



- Entropy: "uncertainty in a random variable X"

$$H(X) = \sum_x p(x) \log \frac{1}{p(x)}$$

- Mutual information: "information about X provided by Y"

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

- Relative entropy (Kullback–Leibler 1951): "distinction between p and q"

$$D(p\|q) = \sum_x p(x) \log \frac{p(x)}{q(x)}$$

Where do they come from?

- Mathematical communication theory (Shannon 1948)
  - Fundamental limits on communication and compression
  - Probability theory and statistics

- Axiomatic definitions (Aczél–Daróczy 1975)
  - "Reasonable" properties for information measures
  - Functional equations: $f(p \times q) = f(p) + f(q) \;\Rightarrow\; f \cong H$

- How about finance and economics?

Gambling in horse races

- Horses: $1, 2, \ldots, m$
- Odds: $o(1), o(2), \ldots, o(m)$ (say, $o(x) \equiv m$)
- Win probabilities: $p(1), p(2), \ldots, p(m)$

Optimal gambling

- Bets: $b(1), b(2), \ldots, b(m)$
  - No short: $b(x) \geq 0, x = 1, 2, \ldots, m$
  - No margin: $\sum_x b(x) = 1$
  - In other words, $b(x)$ lies in the probability simplex

- Payoff: If horse x wins (with probability $p(x)$), then \$1 turns into $\$b(x)o(x)$

Question

How should we choose our portfolio $b(x)$?

Kelly gambling and log-optimal portfolio

- Kelly (1956), "A new interpretation of information rate":

$$b^*(x) = p(x)$$

- Maximize $\mathsf{E}[\log(b(X)o(X))]$
  - Logarithmic utility
  - Growth rate optimality
  - Competitive optimality (Bell–Cover 1980)
  - Other properties (MacLean–Thorp–Ziemba 2011)

- Optimal growth rate:

$$W^*(X) = \max_{b(x)} \mathsf{E}[\log(b(X)o(X))] = \mathsf{E}(\log o(X)) - H(X)$$

With $o(x) \equiv m$,

$$W^*(X) = \log m - H(X)$$

Entropy

$$H(X) = \sum_x p(x) \log \frac{1}{p(x)} = \mathsf{E}\left[\log \frac{1}{p(X)}\right]$$

- Amount of randomness (information, uncertainty) in X
- Fundamental limit on lossless compression (Shannon 1948)
- Can be generalized to measures other than the counting measure

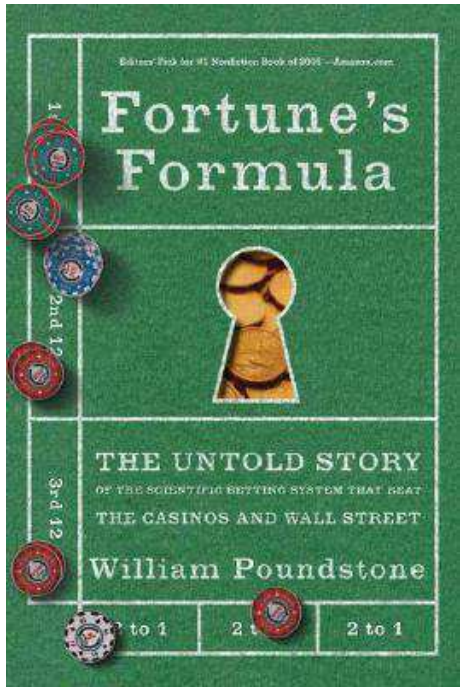
- Conditional entropy:

$$H(X|Y) = \sum_{x,y} p(x,y) \log \frac{1}{p(x|y)} = \mathsf{E}\left[\log \frac{1}{p(X|Y)}\right]$$

Gambling with side information

- Side information Y about the horse race outcome X
- Bets: $b(x|y), x = 1, 2, \ldots, m$
- Kelly gambling: $b^*(x|y) = p(x|y)$
- Optimal growth rate:

$$W^*(X|Y) = \max_{b(x|y)} \mathsf{E}[\log(b(X|Y)o(X))]$$
$$= \mathsf{E}(\log o(X)) - H(X|Y)$$



Value of side information (Kelly 1956)

$$\Delta W = W^*(X|Y) - W^*(X) = I(X;Y)$$

Mutual information

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

- Amount of information about X provided by Y (and vice versa)
  - For a general stock market (Barron–Cover 1988): $\Delta W \leq I(X;Y)$
- Fundamental limit on communication (Shannon 1948)
- Fundamental limit on lossy compression/quantization (Shannon 1959)
- Can be generalized to any pair of random objects
- Conditional mutual information:

$$I(X;Y|Z) = H(X|Z) + H(Y|Z) - H(X,Y|Z)$$

Repeated gambling in horse races with memory

- Win probabilities: $p(x_1), p(x_2|x_1), p(x_3|x_1, x_2), \ldots, p(x_n|x^{n-1})$
- Odds: $o(x_i) \equiv m$
- Bets: $b(x_1), b(x_2|x_1), b(x_3|x_1, x_2), \ldots, b(x_n|x^{n-1})$
- Kelly gambling: $b^*(x_i|x^{i-1}) = p(x_i|x^{i-1}), i = 1, 2, \ldots$
- Optimal growth rate:

$$W^*(X^n) = \log m - \frac{1}{n} H(X^n) = \log m - \frac{1}{n} \sum_{i=1}^{n} H(X_i|X^{i-1})$$

- If the horse race process $\{X_n\}$ is stationary ergodic, then
  - $(1/n)H(X^n) \to H^*(X)$
  - $W^*(X^n) \to W^*(X)$
  - wealth $\doteq 2^{nW^*}$ almost surely (Shannon 1948, McMillan 1953, Breiman 1957)

Gambling with causal side information

- Side information $Y_1, Y_2, \ldots$
- Bets: $b(x_i|x^{i-1}, y^i)$
- Kelly gambling: $b^*(x_i|x^{i-1}, y^i) = p(x_i|x^{i-1}, y^i), i = 1, 2, \ldots$
- Optimal growth rate:

$$W^*(X^n\|Y^n) = \log m - \frac{1}{n}\sum_{i=1}^{n} H(X_i|X^{i-1}, Y^i) = \log m - \frac{1}{n}H(X^n\|Y^n)$$

- If the $\{(X_n, Y_n)\}$ is stationary ergodic, then $(1/n)H(X^n\|Y^n) \to H^*(X\|Y)$

Value of causal side information (Permuter–K–Weissman 2011)

$$\Delta W = W^*(X^n\|Y^n) - W^*(X^n) = \frac{1}{n}(H(X^n) - H(X^n\|Y^n)) = \frac{1}{n}I(Y^n \to X^n)$$

Directed information

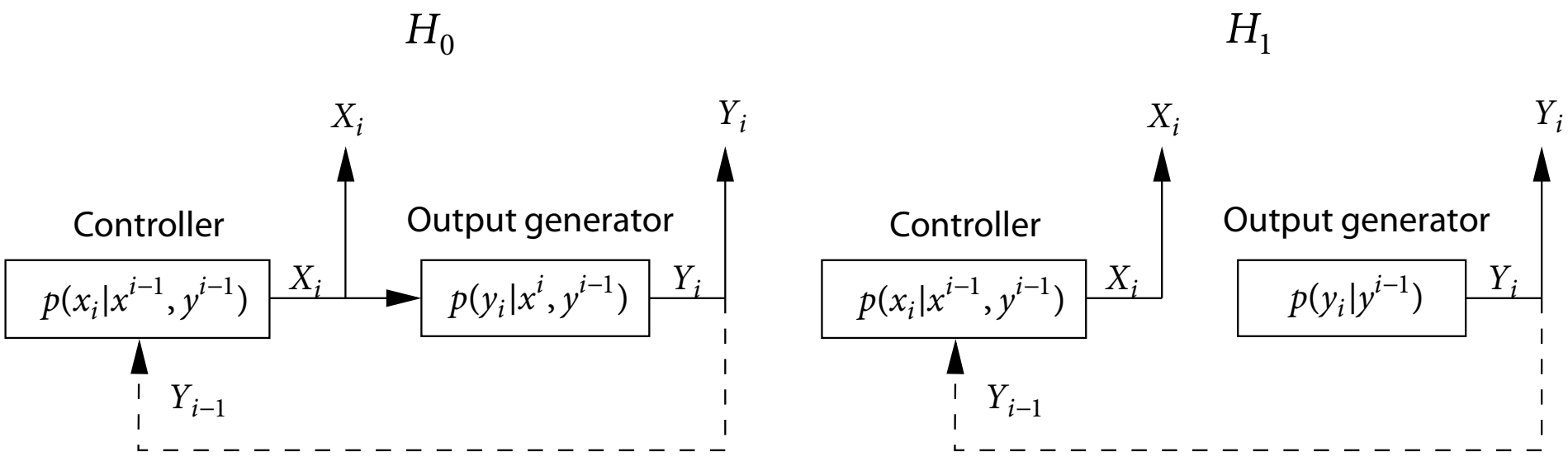
$$I(Y^n \to X^n) = H(X^n) - H(X^n \| Y^n) = \sum_{i=1}^{n} I(X_i; Y^i | X^{i-1})$$

- Amount of information about X causally provided by Y
  - For a general stock market: $\Delta W \leq (1/n) I(Y^n \to X^n)$
- Arrow of time: directed and asymmetric

$$I(Y^n \to X^n) \neq I(X^n \to Y^n)$$

- Fundamental limit on feedback communication (Tatikonda–Mitter 2009, K 2008, Permuter–Weissman–Goldsmith 2009)
- Can be generalized to continuous time (Weissman–K–Permuter 2013)

Test for causal dependence



- Type-I and type-II error probabilities: $\alpha = \mathrm{P}(\mathcal{A}^c|H_0)$, $\beta = \mathrm{P}(\mathcal{A}|H_1)$

Chernoff–Stein lemma for the causal dependence test

$$\beta^* = \min_{\mathcal{A} \subseteq \mathcal{X}^n \times \mathcal{Y}^n : \alpha < \epsilon} \beta \doteq 2^{-I(X^n \to Y^n)}$$

Brief history

- Marko (1973), "The bidirectional communication theory: A generalization of information theory"
  - Direction of information flow for mutually coupled statistical systems
  - Cybernetics: Group behavior with monkeys
- Massey (1990), "Causality, feedback, and directed information"

Relationship to other notions for causality

- Granger causality (Granger 1969, Geweke 1982):

$$G(X^n \to Y^n) = \sum_{i=1}^{n} \log \frac{\mathrm{LMMSE}(Y_i | Y_{i-p}^{i-1})}{\mathrm{LMMSE}(Y_i | Y_{i-p}^{i-1}, X_{i-p}^{i})}$$

  - The higher $G(X^n \to Y^n)$ is, the more X influences Y
  - If $\{(X_n, Y_n)\}$ is Gauss–Markov of order p, then

$$I(X^n \to Y^n) \equiv G(X^n \to Y^n)$$

- Transfer entropy (Schreiber 2000):

$$T_i(X \to Y) = I(X^{i-1}; Y_i | Y^{i-1})$$

  - The higher $T_i(X \to Y)$ is, the more X influences Y (with one step delay)
  - If $\{(X_n, Y_n)\}$ is stationary, then

$$\frac{1}{n} I(X^{n-1} \to Y^n) \to T(X \to Y)$$

Causal conditioning

- Causally conditional probability (Kramer 1998):

$$p(y^n \| x^n) = \prod_{i=1}^{n} p(y_i | x^i, y^{i-1})$$

$$p(y^n \| x^{n-1}) = \prod_{i=1}^{n} p(y_i | x^{i-1}, y^{i-1})$$

- Causally conditional entropy:

$$H(Y^n \| X^n) = -\mathsf{E}[\log p(Y^n \| X^n)],$$

$$H(Y^n \| X^{n-1}) = -\mathsf{E}[\log p(Y^n \| X^{n-1})]$$

Chain rules

$$p(x^n, y^n) = p(x^n \| y^n) p(y^n \| x^{n-1}) = p(x^n \| y^{n-1}) p(y^n \| x^n),$$

$$H(X^n, Y^n) = H(X^n \| Y^n) + H(Y^n \| X^{n-1}) = H(X^n \| Y^{n-1}) + H(Y^n \| X^n)$$

Properties of directed information

$$I(X^n \to Y^n) = H(Y^n) - H(Y^n \| X^n),$$
$$I(X^{n-1} \to Y^n) = H(Y^n) - H(Y^n \| X^{n-1})$$

- $I(X^n \to Y^n) \leq I(X^n; Y^n)$
- $I(X^n \to Y^n) = I(X^n; Y^n)$ if $p(x^n \| y^{n-1}) = p(x^n)$
- $I(X^n \to Y^n) = I(X^n; Y^n) = nI(X; Y)$ if $\{(X_n, Y_n)\}$ is IID

Conservation law

$$I(X^n; Y^n) = I(X^n \to Y^n) + I(Y^{n-1} \to X^n) = I(X^{n-1} \to Y^n) + I(Y^n \to X^n)$$

- Measure of causal influence

Universal estimation of directed information

- In reality, the probability distribution may not be known or may not even exist

Something out of nothing

- Can we perform as if the distribution were known?
- Can we perform as well as the best estimator in a given class?

- Answer: Yes! (Jiao–Zhao–Permuter–K–Weissman 2013)

Universal probability assignments

- Probability assignment: $q(x^n)$
- Sequential probability assignment: $q(x_1), q(x_2|x_1), q(x_3|x_1, x_2), \ldots, q(x_n|x^{n-1})$
- Probability assignment q is universal if

$$\lim_{n\to\infty} \frac{1}{n} D(p(x^n)\|q(x^n)) = 0$$

for every stationary distribution p

- Probability assignment q is pointwise universal if

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{p(X^n)}{q(X^n)} \leq 0 \quad p\text{–a.s.}$$

for every stationary ergodic distribution p

- (Pointwise) universal probability assignments
  - Compression-based approaches: Ziv–Lempel (1978), Willems–Shtarkov–Tjalkens (1995)
  - Ergodic theoretic approaches: Ornstein (1978), Morvai–Yakowitz–Algoet (1997)

Algorithm 1

$$\hat{I}_1(X^n \to Y^n) = \hat{H}_1(Y^n) - \hat{H}_1(Y^n \| X^n)$$

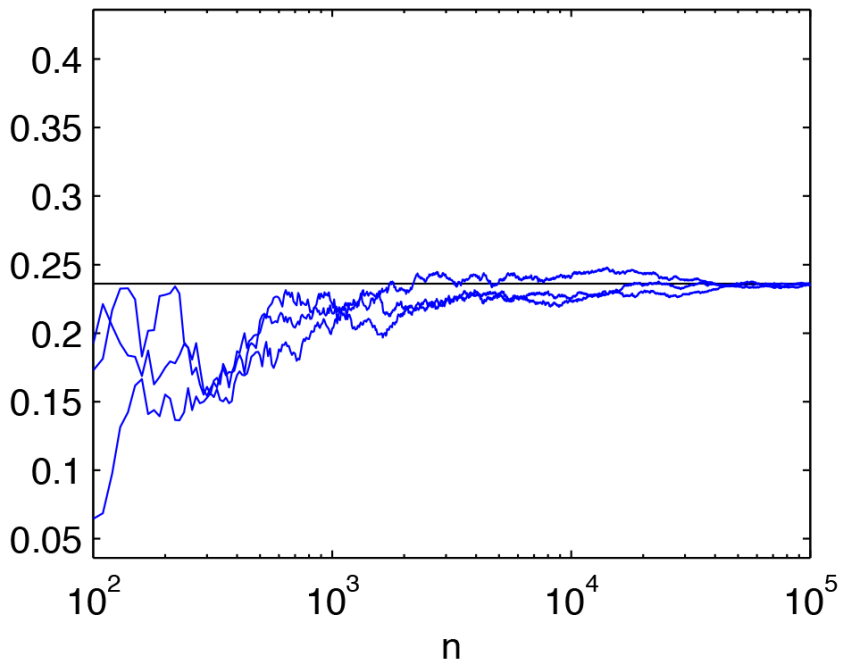
- $\hat{H}_1(Y^n) = -\frac{1}{n} \log q(Y^n)$ and $\hat{H}_1(Y^n \| X^n) = -\frac{1}{n} \log q(Y^n \| X^n)$

☺ Consistency (almost sure and L_1 convergence)

☺ Essentially optimal convergence rate

☹ Erratic for small n

☹ Unbounded support



Algorithm 2

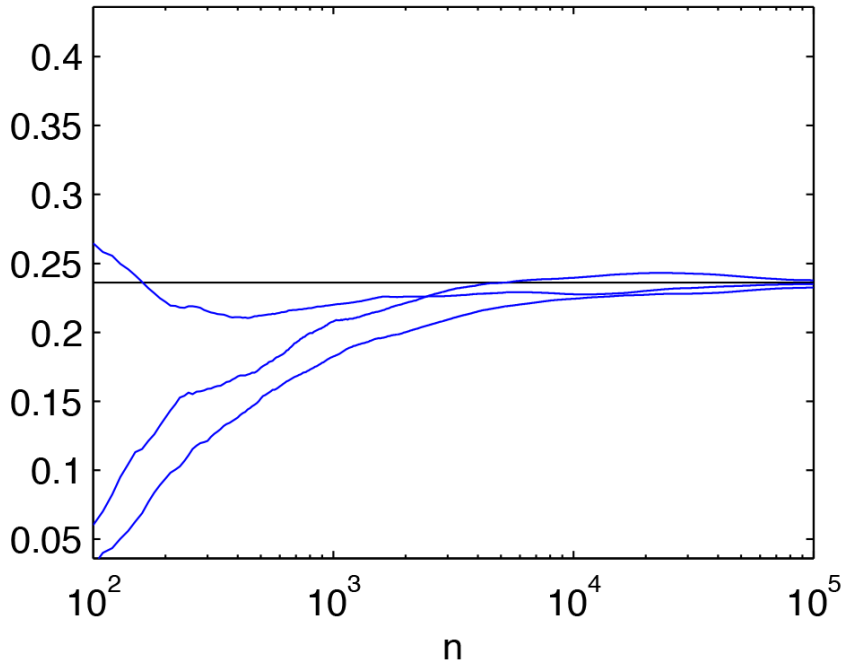
$$\hat{I}_2(X^n \to Y^n) = \hat{H}_2(Y^n) - \hat{H}_2(Y^n \| X^n)$$

- $\hat{H}_2(Y^n) = \frac{1}{n}\sum_{i=1}^{n} H(q(y_i|Y^{i-1}))$ and $\hat{H}_2(Y^n\|X^n) = \frac{1}{n}\sum_{i=1}^{n} H(q(y_i|x_i, X^{i-1}, Y^{i-1}))$

☺ Similar convergence rate as $\hat{I}_1$

☺ Smooth and bounded support

☹ Can be negative

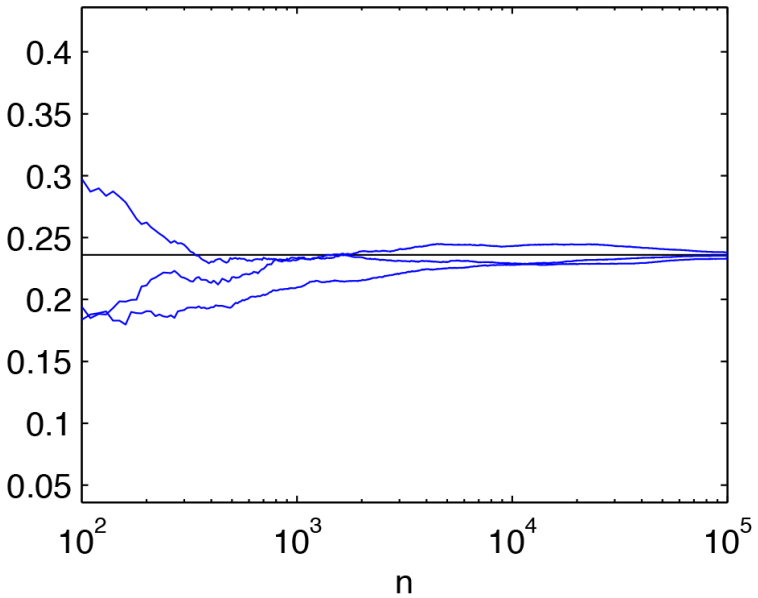


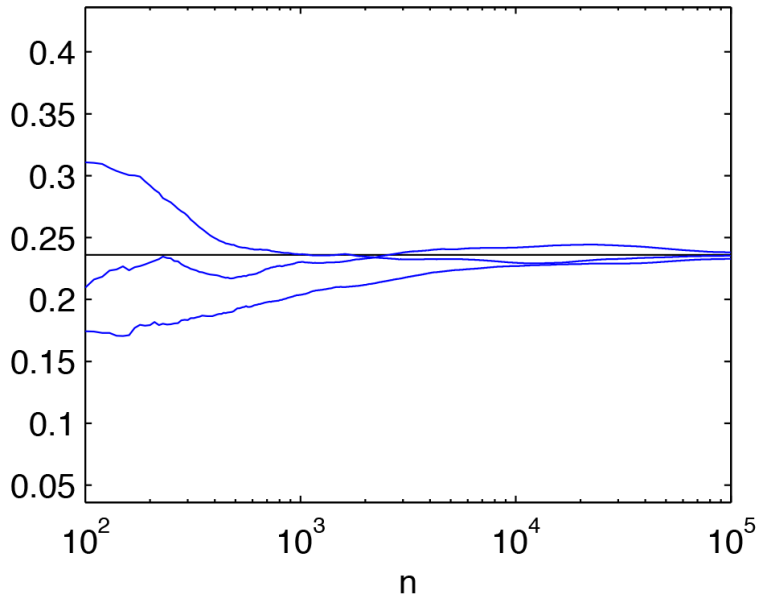
Algorithms 3 and 4

$$\hat{I}_3(X^n \to Y^n) = \frac{1}{n}\sum_{i=1}^{n} D(q(y_i|X^i, Y^{i-1})\|q(y_i|Y^{i-1})),$$

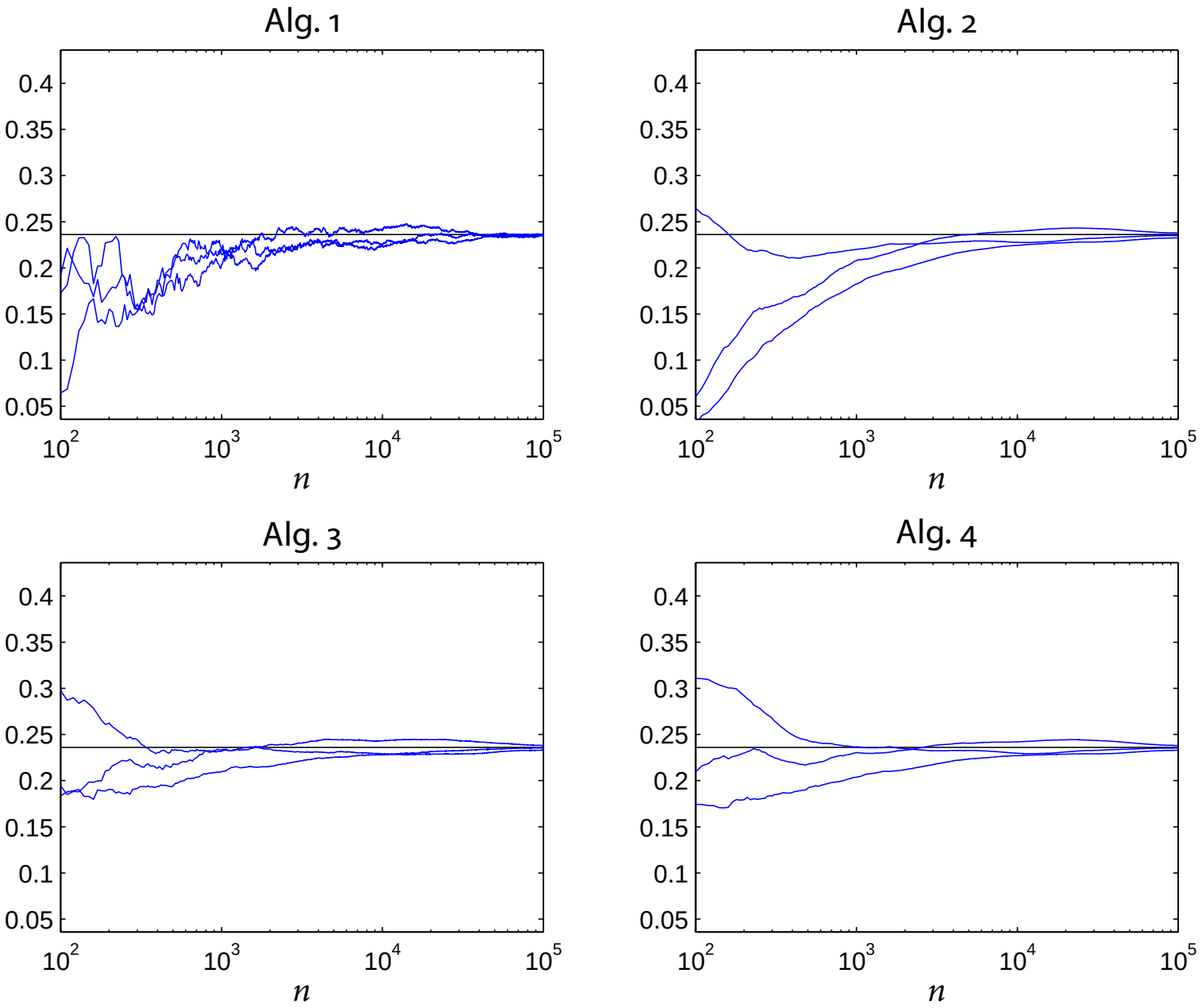
$$\hat{I}_4(X^n \to Y^n) = \frac{1}{n}\sum_{i=1}^{n} D(q(x_i, y_i|X^i, Y^{i-1})\|q(y_i|Y^{i-1})q(x_i|X^i, Y^i))$$

- ☺ Smooth, nonnegative, and bounded support
- ☹ Weaker performance guarantee than $\hat{I}_1$ and $\hat{I}_2$

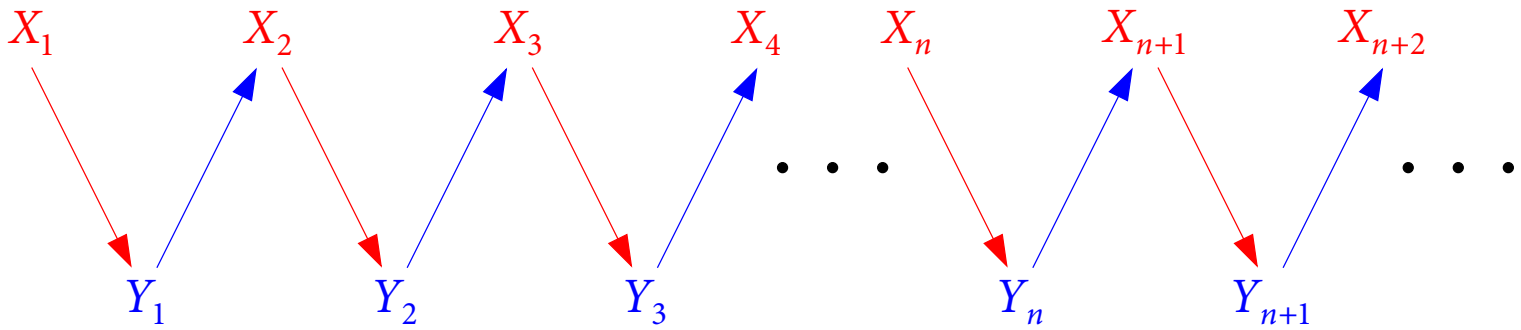




Performance comparison



Causal influence

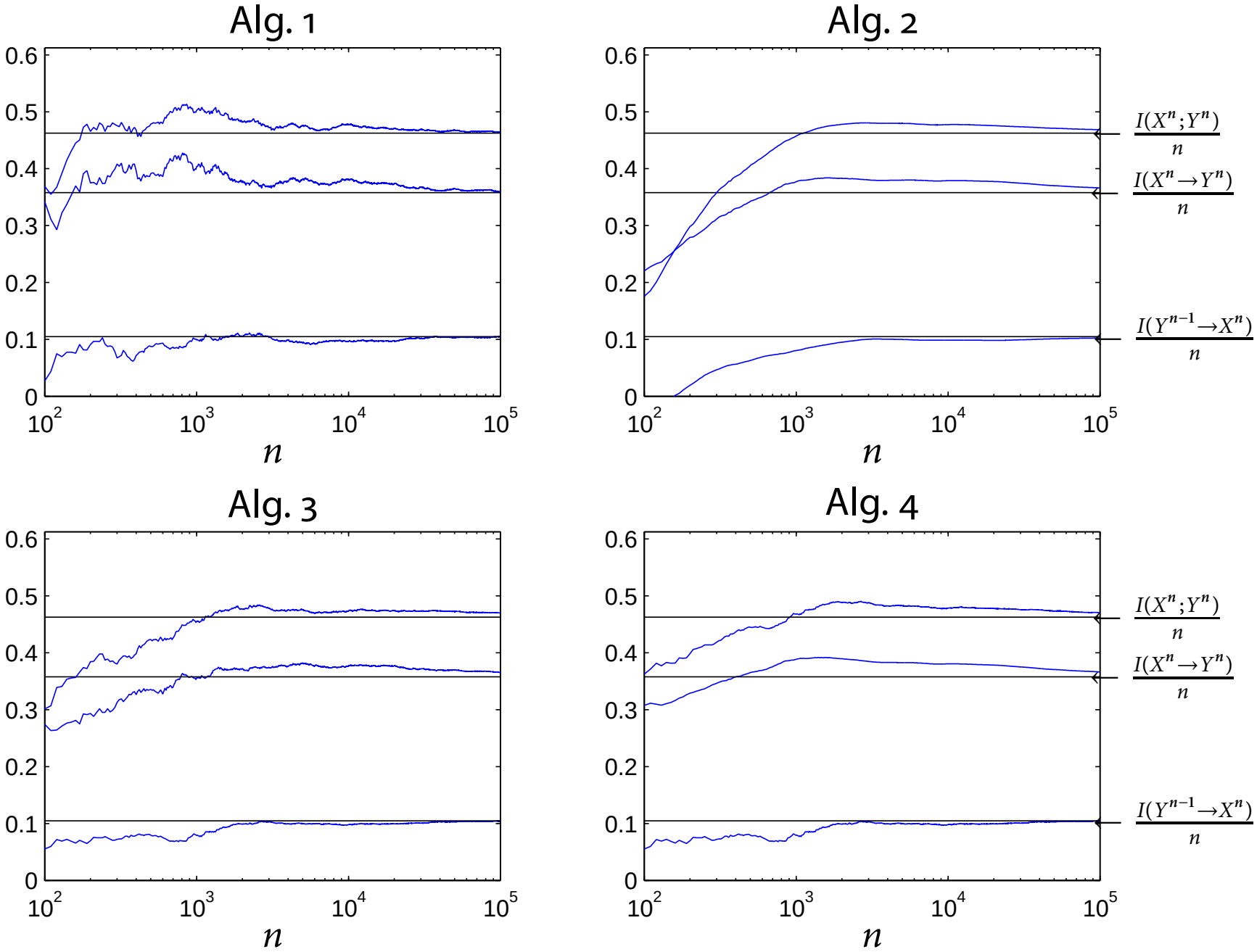


Question

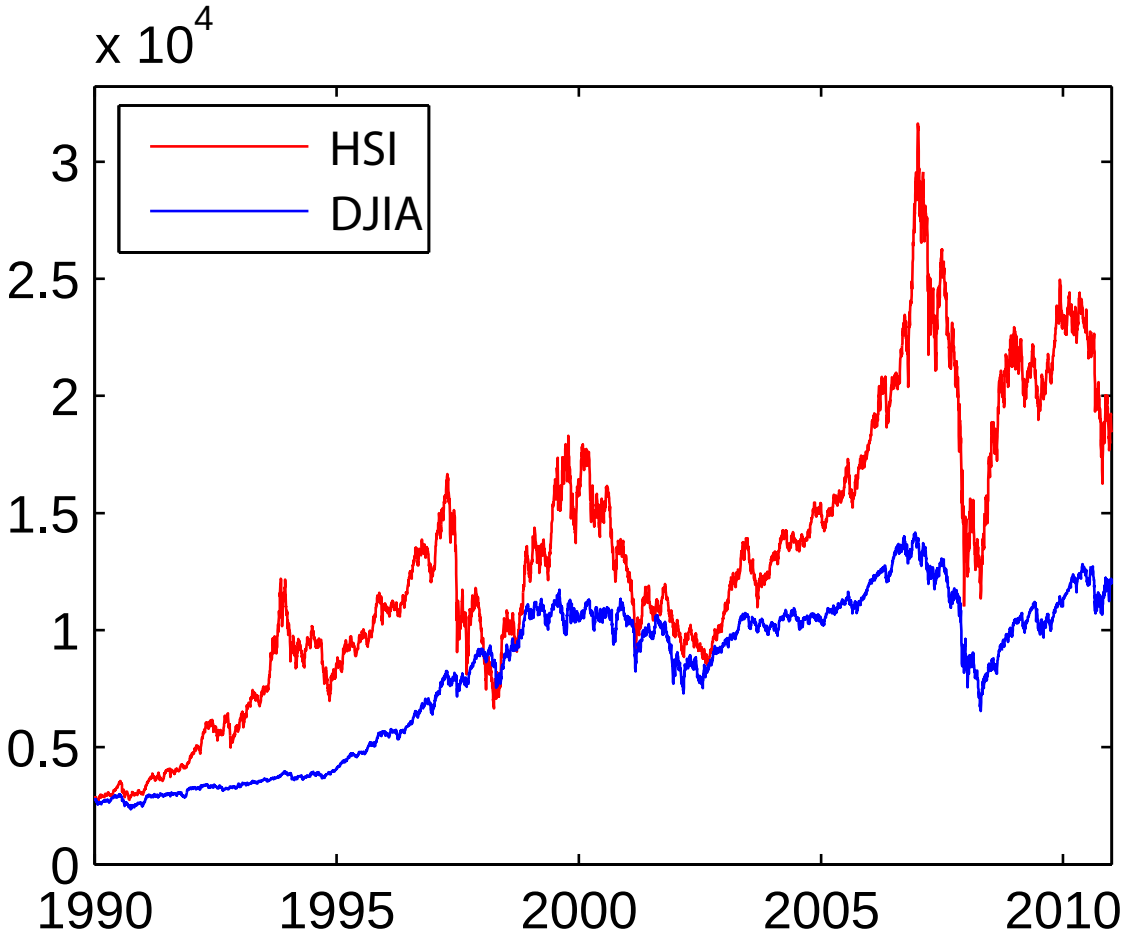
Which process influences the other?

- Conversation law: $I(X^n; Y^n) = I(X^n \to Y^n) + I(Y^{n-1} \to X^n)$
- If $I(X^n \to Y^n) \gg I(Y^{n-1} \to X^n)$, then X causes Y
- If $I(X^n \to Y^n) \ll I(Y^{n-1} \to X^n)$, then Y causes X

Causal influence



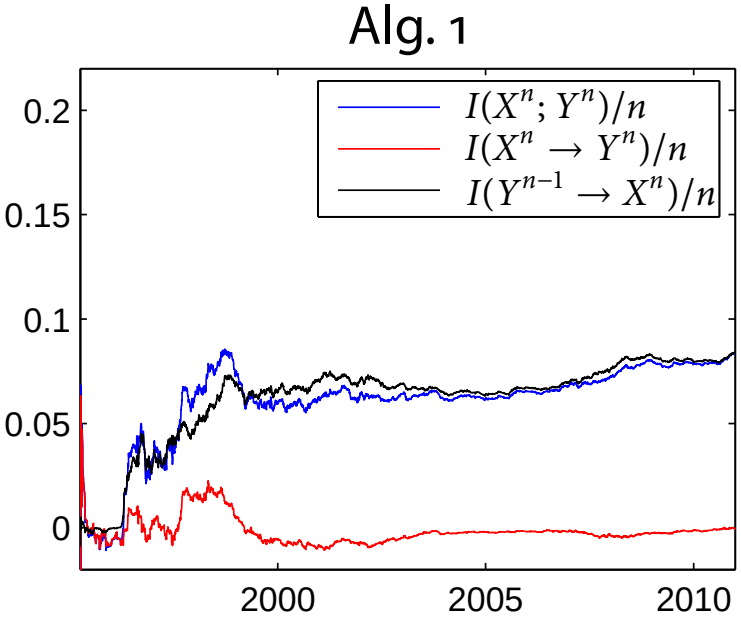
HSI versus DJIA

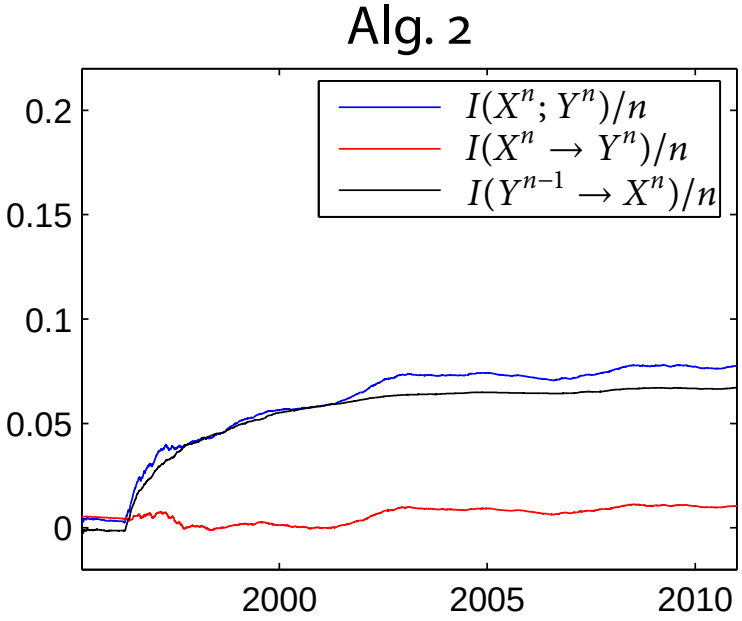


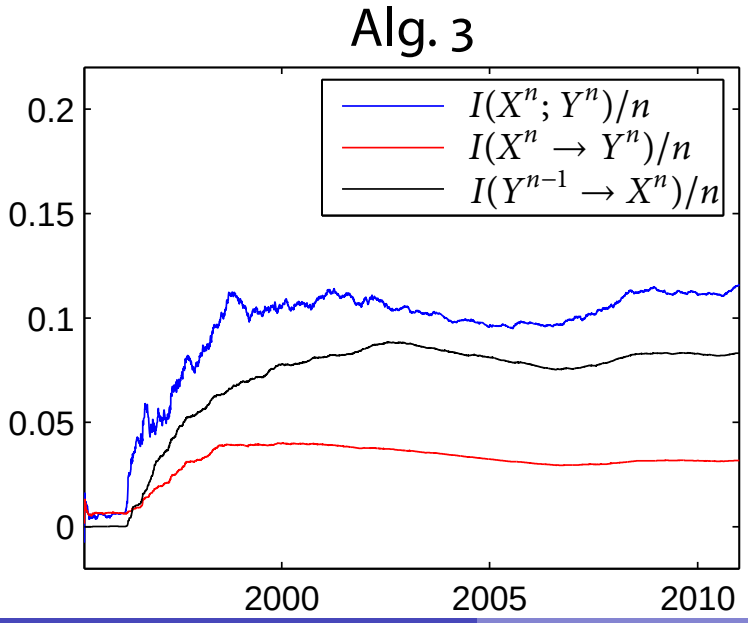
Questions

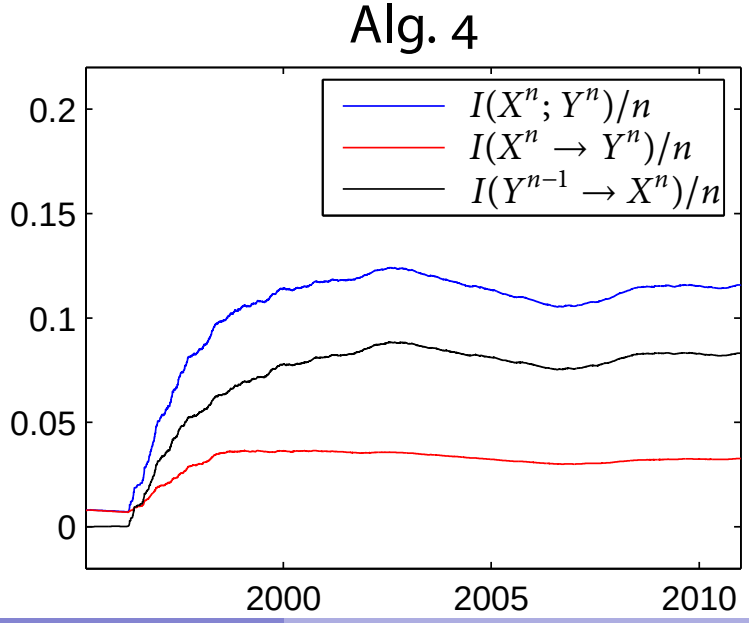
- Are these markets correlated?
- Which index leads the other?

HSI versus DJIA

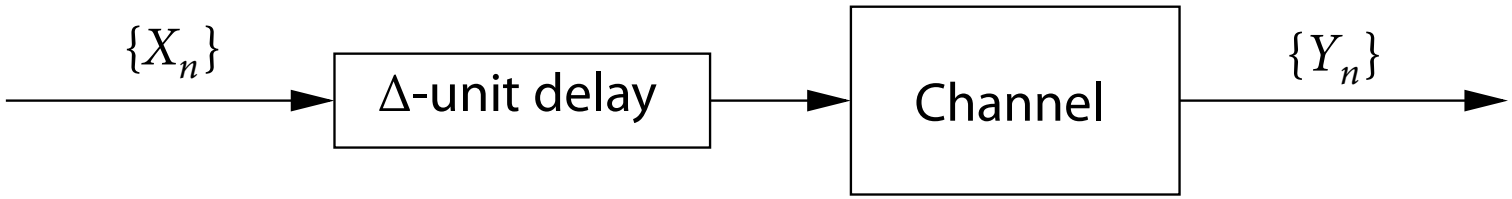








Delay estimation



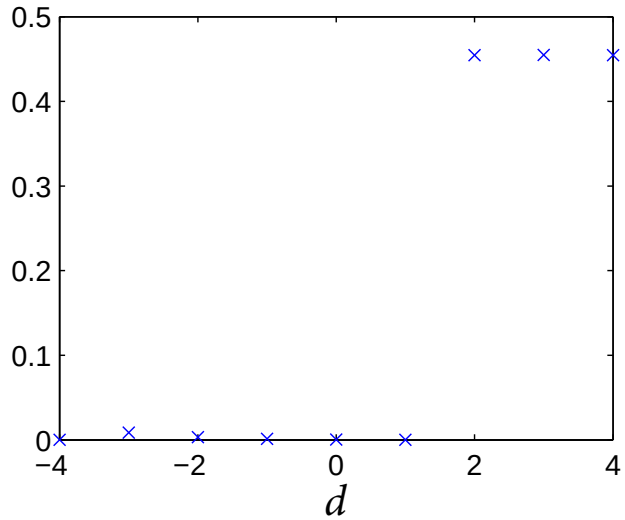
Question

Can we estimate Δ efficiently?

- Shifted directed information

$$I(Y_{d+1}^n \to X^{n-d}) = \sum_{i=1}^{n-d} H(X_i|X^{i-1}) - H(X_i|X^{i-1}, Y_{d+1}^{d+i})$$

- If $d < \Delta$, then $I(Y_{d+1}^n \to X^{n-d}) = 0$
- If $d \geq \Delta$, then $I(Y_{d+1}^n \to X^{n-d}) \gg 0$



Concluding remarks

- Directed information $I(X^n \to Y^n)$
  - Arrow of time + Shannon's mutual information
  - A natural generalization of Granger causality
  - Causation beyond correlation
  - Universal estimation algorithms (MATLAB codes available)

- Looking forward
  - Large alphabets
  - More than a pair of random processes
  - Piecewise stationary processes
  - More applications (economics, biology, climate change, …)

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