Index Coding and Fractional Graph Theory

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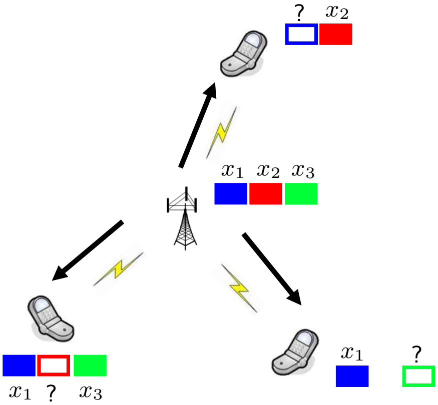
Between Shannon and Hamming
Network Information Theory and Combinatorics
March 3, 2015

Index coding (Birk–Kol 1998)



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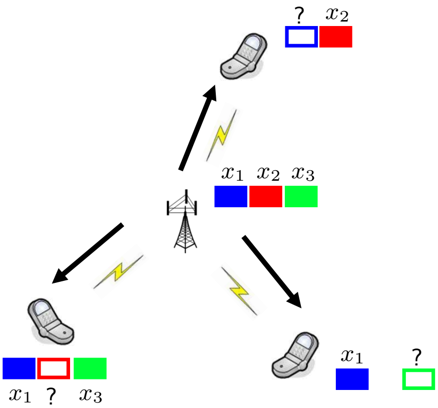

- Send $x_1 + x_2$ and x_3

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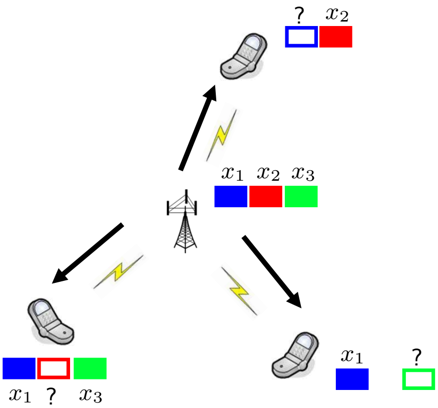

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- What is the fundamental limit on the number of transmissions?

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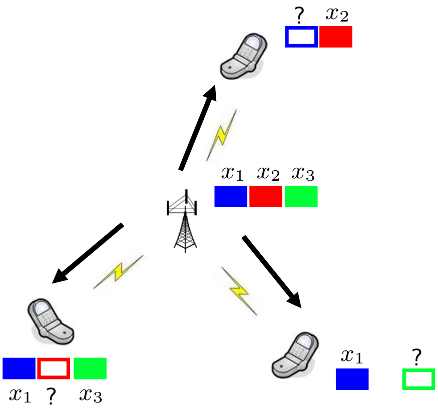
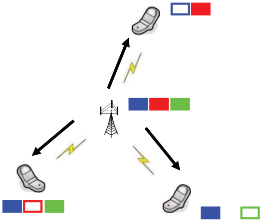


- Send $x_1 + x_2$ and x_3
- What is the fundamental limit on the number of transmissions?
- Which coding scheme achieves the limit?

Representations

- Side information

$$A_1 = \{2\},\ A_2 = \{1,3\},\ A_3 = \{1\}$$



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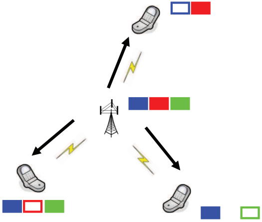

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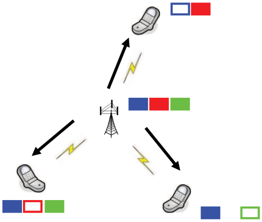

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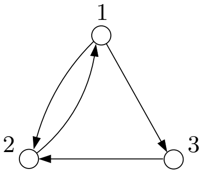

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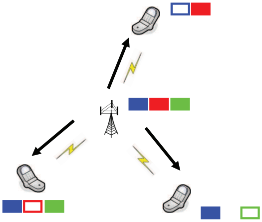

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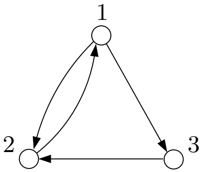

- # of n-message index coding problems = # of n-node directed graphs

$$1,\ 3,\ 16,\ 218,\ 9608,\ 1540944,\ 882033440,\ 1793359192848, \ldots$$

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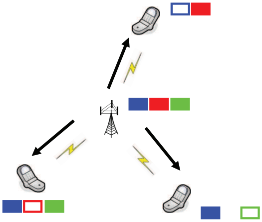

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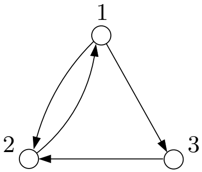
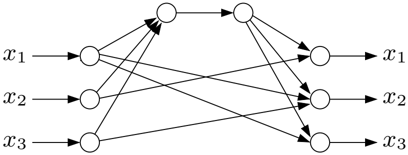


- Multiple unicast network coding



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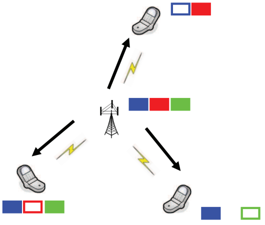

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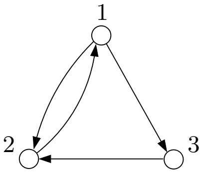

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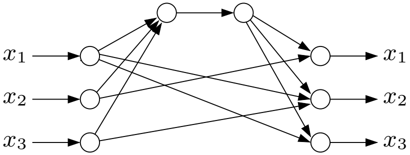

- # of n-message index coding problems = # of n-node directed graphs

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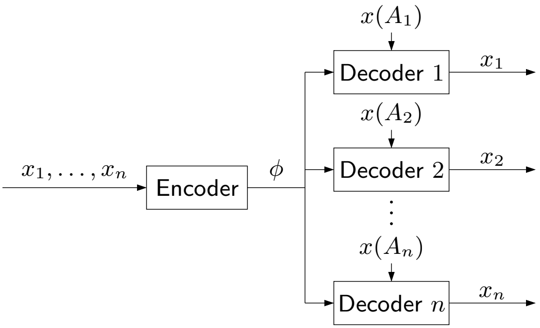
- "Equivalent" to network coding (Effros–El Rouayheb–Langberg 2013)

Index coding



- $(t_1, \ldots, t_n, r)$ index code
- Messages $x_j \in \{0,1\}^{t_j} \quad j \in [1:n]$,
- Codeword $\phi(x_1, \ldots, x_n) \in \{0,1\}^r$
- Side information $x(A_j) := (x_i : i \in A_j)$, $A_j \subseteq [1:n] \setminus \{j\}$

Achievability, Capacity region

- $(R_1, \ldots, R_n)$ is achievable iff $\exists$ $(t_1, \ldots, t_n, r)$ index code such that

$$R_j \leq \frac{t_j}{r}, \quad j \in [1:n]$$

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- The capacity region $\mathscr{C}$ is the closure of the set of all achievable rate tuples
- Broadcast rate $\beta = \min\{b\colon (1/b, \ldots, 1/b) \in \mathscr{C}\}$
  - $x_j \in \{0,1\}^t, \quad j \in [1:n]$
  - $1 \leq \beta \leq n$

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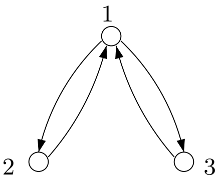
- No computable characterization of $\mathscr{C}$ or β in general
- No approximation within a factor of $O(n^{1-\epsilon})$ for any $\epsilon > 0$

Outline

- Existing results
  - Lower bounds on β (outer bounds on $\mathscr{C}$)
  - Coding schemes (upper bounds on β / inner bounds on $\mathscr{C}$)
- Structural properties of the capacity region
  - Graph products
  - Confusion graph
  - Characterization of the capacity region
  - Structural properties

Cycle-free bound



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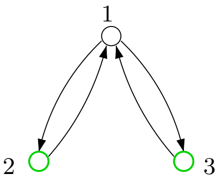

$$2 = \alpha(G) \leq \beta$$

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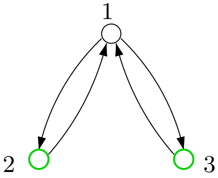

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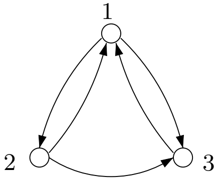

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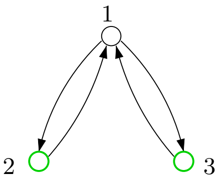

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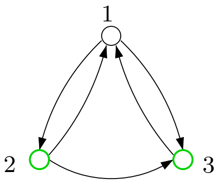

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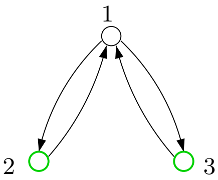

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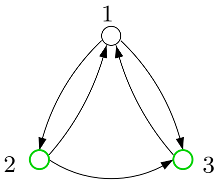

$2 \leq \beta$

Cycle-free lower bound (Neely–Tehrani–Zhang 2013)

$$\max_{S \subseteq [1:n]:\ G|_S \text{ is cycle-free}} |S| \leq \beta$$

Polymatroidal outer bound

Polymatroidal bound (Dougherty et al. 2011, Blasiak et al. 2011)

If $(R_1, \ldots, R_n)$ is achievable for index coding problem $(j \mid A_j), j \in [1:n]$, then

$$R_j \leq f(B_j \cup \{j\}) - f(B_j), \quad j \in [1:n]$$

for some set function f such that

- $f(\emptyset) = 0$
- $f([1:n]) = 1$
- $f(J) \leq f(K), \quad J \subseteq K \subseteq [1:n]$
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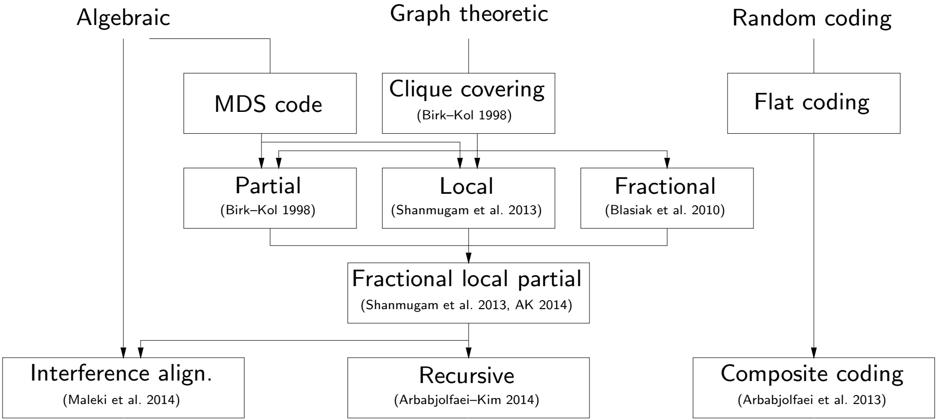
- Tighter than the cycle-free bound
- Not tight (Sun–Jafar 2013)

Overview of coding schemes



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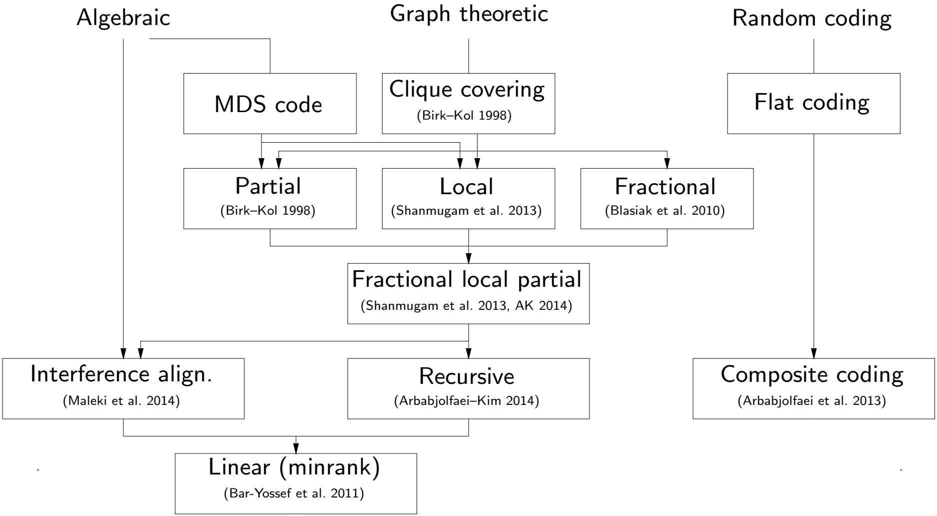

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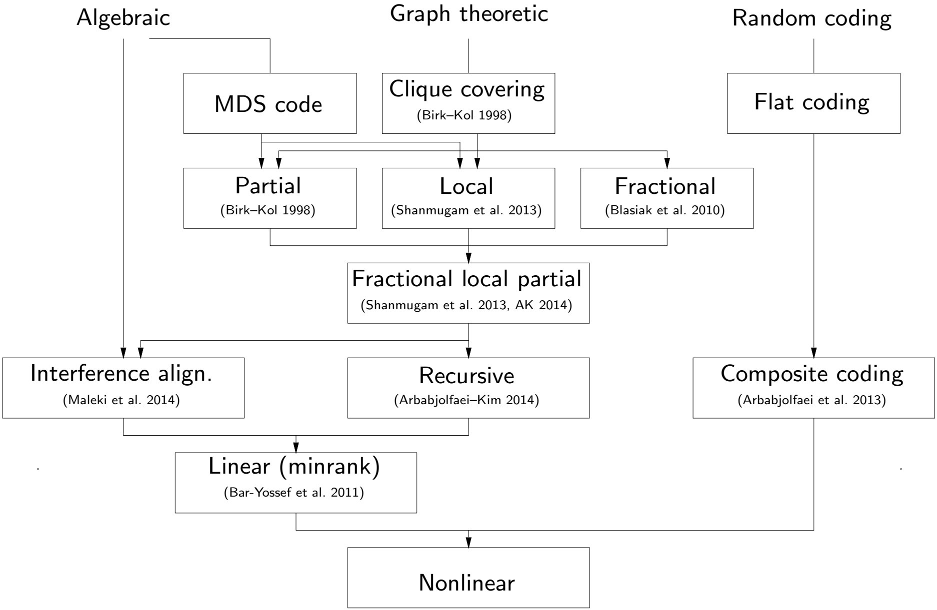

MDS code

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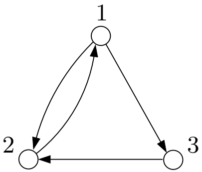

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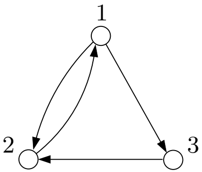

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- Systematic $(5,3)$ MDS code

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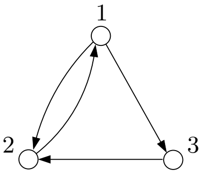

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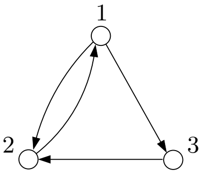

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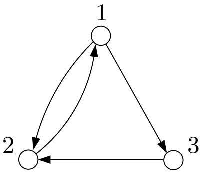

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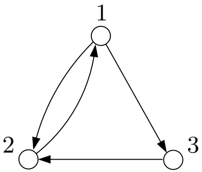

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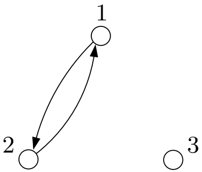
- $\beta = 1$ if the graph is a clique

Clique covering (Birk–Kol 1998)

$(1|2)$, $(2|1)$, (3)



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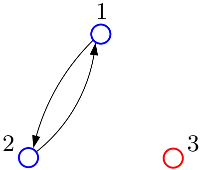

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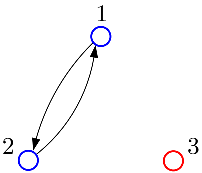

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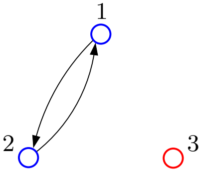

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- Integer programming characterization

$$
\begin{aligned}
\text{minimize} \quad & \sum_{S \in \mathcal{K}} \rho_S \\
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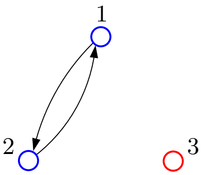

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- Not comparable with MDS code

Partial clique covering (Birk–Kol 1998)

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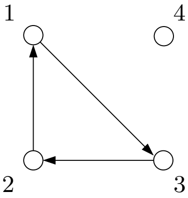

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Partial clique covering (Birk–Kol 1998)

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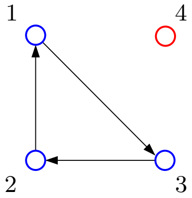

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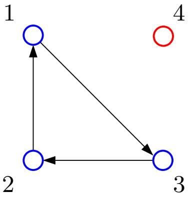

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- Integer programming characterization

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\begin{aligned}
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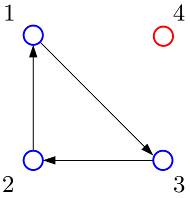

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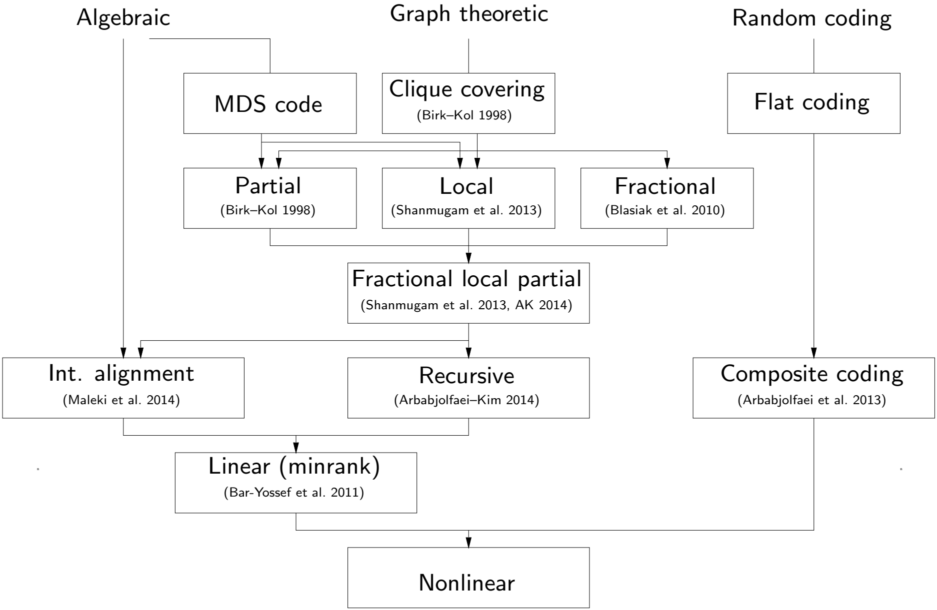
- Linear programming relaxation: fractional partial clique covering

Overview of coding schemes



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- Do we need to study them all?
  - Irreducibility: Cannot be decomposed into subproblems
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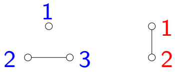
Outline

- Overview of previous work
  - Lower bounds
  - Coding schemes
- Structural properties of the capacity region
  - Graph products
  - Confusion graph
  - Characterization of the capacity region
  - Structural properties

Graph products

- Γ_1, Γ_2: undirected graphs



Graph products

- Γ_1, Γ_2: undirected graphs
- Disjunctive product: $\Gamma = \Gamma_1 * \Gamma_2$
  - $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2)$
  - $(u_1, u_2) \sim (v_1, v_2)$ iff

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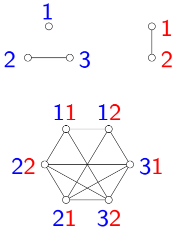

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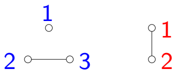

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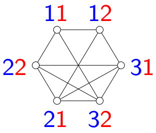

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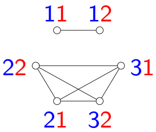

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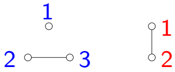

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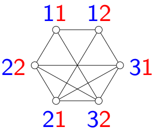

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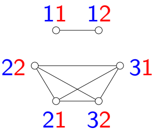

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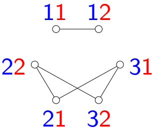

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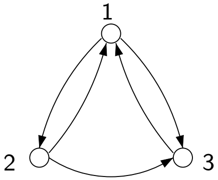
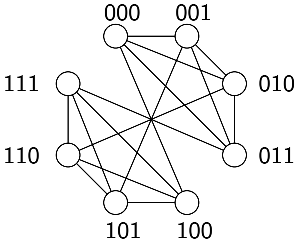
- Properties
  - Undirected
  - Vertex transitive

Confusion graph example





$$(t_1, t_2, t_3) = (1, 1, 1)$$

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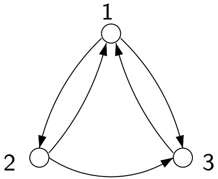

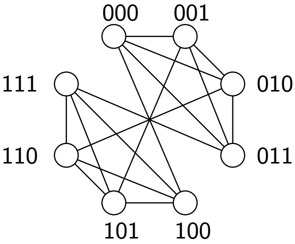

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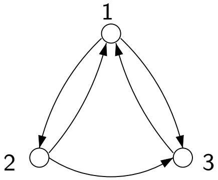

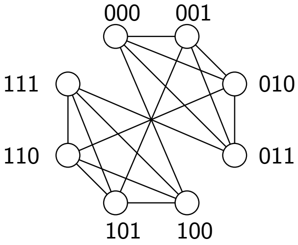

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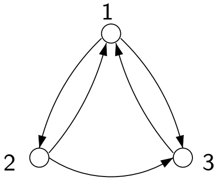

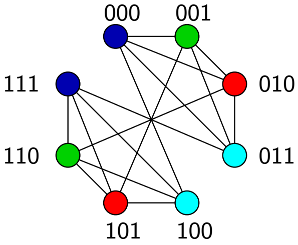

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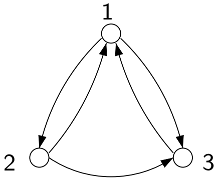

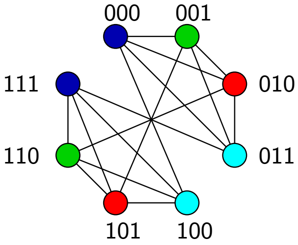

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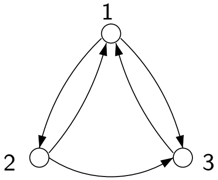

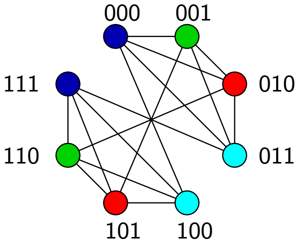

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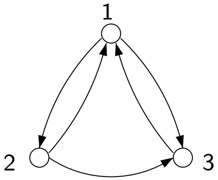

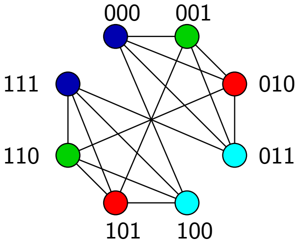

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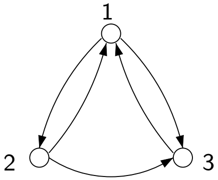
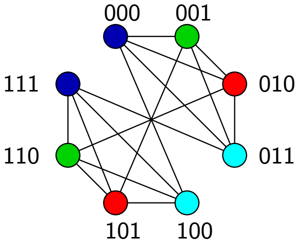
Confusion graph example





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Proposition 1 (Alon et al. 2008)

$(R_1, \ldots, R_n)$ is achievable iff

$$R_j \leq \frac{t_j}{\lceil \log(\chi(\Gamma_{\mathbf{t}}(G))) \rceil}, \quad j \in [1:n]$$

for some integer tuple $\mathbf{t} = (t_1, \ldots, t_n)$

Index coding capacity region via the confusion graph

Proposition 2

The capacity region $\mathscr{C}$ is the closure of all rate tuples $(R_1, \ldots, R_n)$ such that

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- Lemma (McEliece–Posner 1971, Berge–Simonovits 1974):

$$\chi_f(\Gamma) = \lim_{k\to\infty} \sqrt[k]{\chi(\Gamma^k)} = \inf_k \sqrt[k]{\chi(\Gamma^k)}.$$

Structural properties of capacity region

- Assumptions:
  - G_1 and G_2 are two vertex-induced subgraphs of G
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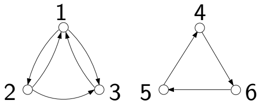

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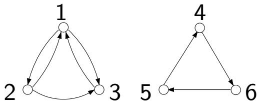
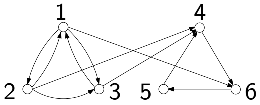


One-way interaction



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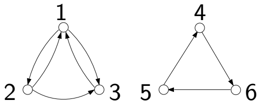

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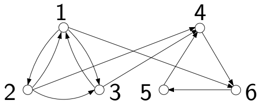
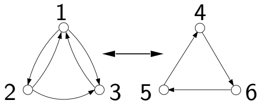
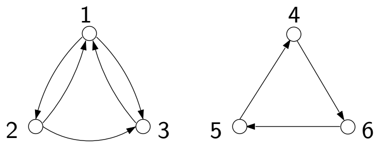
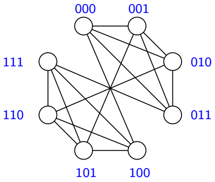


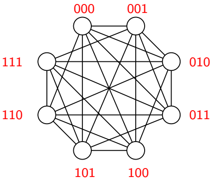
Complete interaction



Confusion graph in case of no interaction







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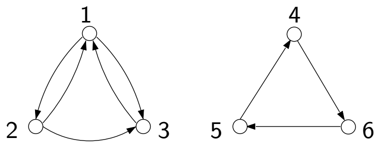

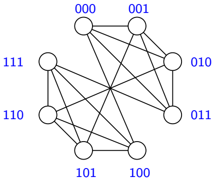

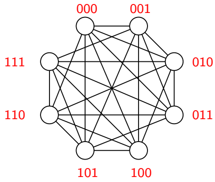

Message pair	Confusion?
000000 011100	Yes Yes Yes
000000 110100	No Yes Yes
000000 011111	Yes No Yes
000000 110000	No No No

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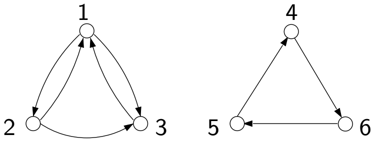

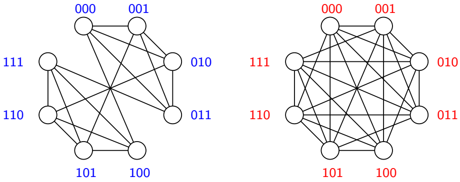

Disjunctive product: $\Gamma = \Gamma_1 * \Gamma_2$

$(u_1, u_2) \sim (v_1, v_2)$ iff

$u_1 \sim v_1$ or $u_2 \sim v_2$

Message pair	Confusion?
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011100	Yes
000000	No Yes
110100	Yes
000000	Yes No
011111	Yes
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$$\Gamma_{\mathbf{t}}(G) = \Gamma_{\mathbf{t}_1}(G_1) * \Gamma_{\mathbf{t}_2}(G_2)$$

Capacity region in case of no interaction

Proposition 3

If G has no edge between G_1 and G_2, then

$$\mathscr{C} = \bigcup_{\alpha \in [0,1]} \left\{ (\alpha \mathbf{R}_1, (1-\alpha)\mathbf{R}_2) \colon \mathbf{R}_1 \in \mathscr{C}_1, \mathbf{R}_2 \in \mathscr{C}_2 \right\}.$$

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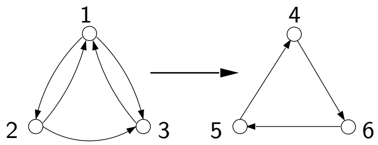
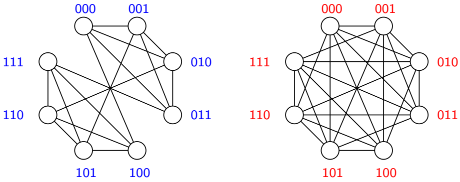
Confusion graph in case of one-way interaction





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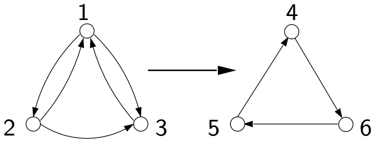

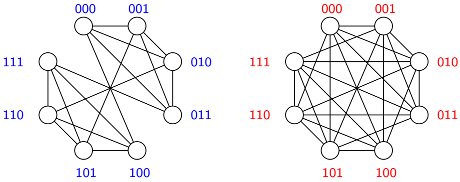

Message pair	Confusion?
000000	Yes No
011111	Yes
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110100	No
000000	Same Yes
000100	Yes
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110111	No

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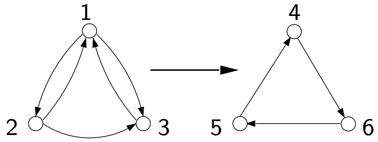

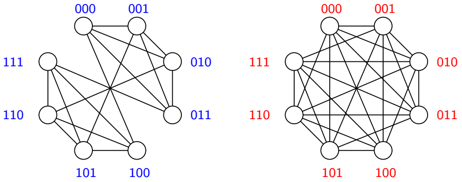

Lexicographic product: $\Gamma = \Gamma_1 \bullet \Gamma_2$

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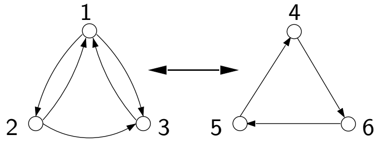
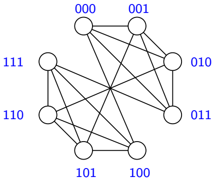
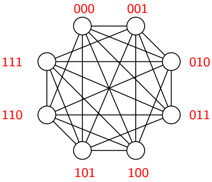
- Can remove edges not on directed cycles

Confusion graph in case of complete interaction







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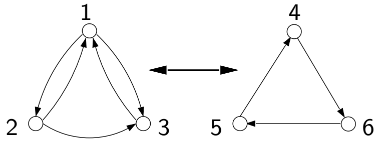

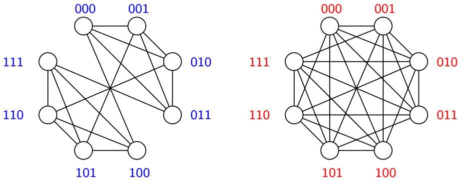

Message pair	Confusion?
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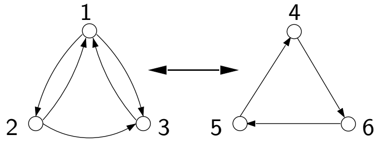

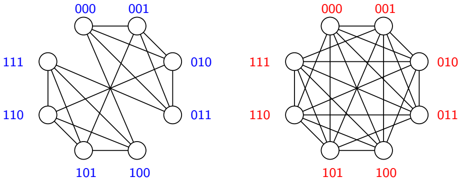

Cartesian product: $\Gamma = \Gamma_1 \wedge \Gamma_2$

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or

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Message pair	Confusion?
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$$\Gamma_{\mathbf{t}}(G) = \Gamma_{\mathbf{t}_1}(G_1) \wedge \Gamma_{\mathbf{t}_2}(G_2)$$

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Theorem 3 (Arbabjolfaei–Kim 2015)

If there are edges from every node in G_1 to every node in G_2 and vice versa, then

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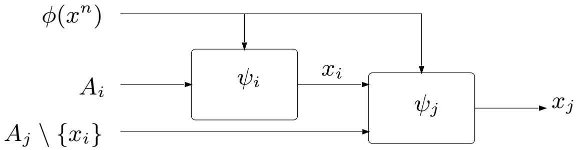
- Lemma (Sabidussi 1957): $\chi(\Gamma_1 \wedge \Gamma_2) = \max\{\chi(\Gamma_1), \chi(\Gamma_2)\}$
- Complete interaction in G $\leftrightarrow$ disconnected complement $\bar{G}$

Conclusion

n	$\#G$	"Nontrivial"
3	16	1
4	218	20
5	9608	1009
6	1,540,944	174,161

- "Nontrivial" $=$ irreducible $\wedge$ critical
  - Strongly connected (two-way interaction)
  - Connected complement (but not complete interaction)
  - Nondegraded side information sets



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