Feedback Capacity of Stationary Gaussian Channels

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Abstract— The feedback capacity of additive stationary Gaussian noise channels is characterized as the solution to a variational problem. Toward this end, it is proved that the optimal feedback coding scheme is stationary. When specialized to the first-order autoregressive moving-average noise spectrum, this variational characterization yields a closed-form expression for the feedback capacity. In particular, this result shows that the celebrated Schalkwijk–Kailath coding scheme achieves the feedback capacity for the first-order autoregressive moving-average Gaussian channel, positively answering a long-standing open problem studied by Butman, Schalkwijk–Tiernan, Wolfowitz, Ozarow, Ordentlich, Yang–Kavčić–Tatikonda, and others. More generally, it is shown that a k-dimensional extension of the Schalkwijk–Kailath coding scheme achieves the feedback capacity for any autoregressive moving-average noise spectrum of order k. Simply put, the optimal transmitter iteratively refines the receiver's knowledge of the intended message.

I. INTRODUCTION

We consider a communication scenario in which one wishes to communicate a message $W \in \{1, \ldots, 2^{nR}\}$ over the additive Gaussian noise channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, where the additive Gaussian noise process $\{Z_i\}_{i=1}^{\infty}$ is stationary with $Z^n = (Z_1, \ldots, Z_n) \sim N_n(0, K_{Z,n})$ for each $n = 1, 2, \ldots$. For block length n, we specify a $(2^{nR}, n)$ feedback code with codewords $X^n(W, Y^{n-1}) = (X_1(W), X_2(W, Y_1), \ldots, X_n(W, Y^{n-1}))$, $W = 1, \ldots, 2^{nR}$, satisfying the average power constraint

$$\frac{1}{n} \sum_{i=1}^{n} EX_i^2(W, Y^{i-1}) \leq P$$

and decoding function $\hat{W}_n : \mathbb{R}^n \to \{1, \ldots, 2^{nR}\}$. The probability of error $P_e^{(n)}$ is defined as

$$P_e^{(n)} := \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \Pr\{\hat{W}_n(Y^n) \neq w | X^n = X^n(w, Y^{n-1})\}$$
$$= \Pr\{\hat{W}_n(Y^n) \neq W\}$$

where the message W is uniformly distributed over $\{1, 2, \ldots, 2^{nR}\}$ and is independent of Z^n. We say that the rate R is achievable if there exists a sequence of $(2^{nR}, n)$ codes with $P_e^{(n)} \to 0$ as $n \to \infty$. The feedback capacity C_{FB} is defined as the supremum of all achievable rates. We also consider the case in which there is no feedback, corresponding to the codewords $X^n(W) = (X_1(W), \ldots, X_n(W))$ independent of the previous channel outputs. We define the

nonfeedback capacity C, or the capacity in short, in a manner similar to the feedback case.

It is well known that the nonfeedback capacity is characterized by water-filling on the noise spectrum, which is arguably one of the most beautiful results in information theory. More specifically, the capacity C of the additive Gaussian noise channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P, is given by

$$C = \int_{-\pi}^{\pi} \frac{1}{2} \log \frac{\max\{S_Z(e^{i\theta}), \lambda\}}{S_Z(e^{i\theta})} \frac{d\theta}{2\pi} \tag{1}$$

where $S_Z(e^{i\theta})$ is the power spectral density of the stationary noise process $\{Z_i\}_{i=1}^{\infty}$, i.e., the Radon-Nikodym derivative of the spectral distribution of $\{Z_i\}_{i=1}^{\infty}$ (with respect to Lebesgue measure), and the water-level λ is chosen to satisfy

$$P = \int_{-\pi}^{\pi} \max\{0, \lambda - S_Z(e^{i\theta})\} \frac{d\theta}{2\pi}. \tag{2}$$

Although (1) and (2) give only a parametric characterization of the capacity $C(\lambda)$ under the power constraint $P(\lambda)$ for each parameter $\lambda \geq 0$, this solution is considered to be simple and elegant enough to be called *closed-form*. Just like many other fundamental developments in information theory, the idea of water-filling comes from Shannon [7].

For the case of feedback, no such elegant solution exists. Most notably, Cover and Pombra [3] characterized the n-block feedback capacity $C_{\text{FB},n}$ for arbitrary time-varying Gaussian channels via the asymptotic equipartition property (AEP) for arbitrary nonstationary nonergodic Gaussian processes as

$$C_{\text{FB},n} = \max_{K_{V,n}, B_n} \frac{1}{2} \log \frac{\det(K_{V,n} + (B_n + I) K_{Z,n} (B_n + I)')^{1/n}}{\det(K_{Z,n})^{1/n}} \tag{3}$$

where the maximum is taken over all positive semidefinite matrices $K_{V,n}$ and all strictly lower triangular B_n of sizes $n \times n$ satisfying $\text{tr}(K_{V,n} + B_n K_{Z,n}(B_n)') \leq nP$. Note that we can also recover the nonfeedback case by taking $B_n \equiv 0$. When specialized to the stationary noise processes, the Cover–Pombra characterization gives the feedback capacity as a limiting expression

$$C_{\text{FB}} = \lim_{n \to \infty} C_{\text{FB},n}. \tag{4}$$

Despite its generality, the Cover–Pombra formulation of the feedback capacity falls short of what we can call a closed-form solution. It is very difficult, if not impossible, to obtain an

analytic expression for the optimal $(K^\star_{V,n}, B^\star_n)$ in (3) for each n. Furthermore, the sequence of optimal $\{K^\star_{V,n}, B^\star_n\}^\infty_{n=1}$ is not necessarily consistent, that is, $(K^\star_{V,n}, B^\star_n)$ is not necessarily a subblock of $(K^\star_{V,n+1}, B^\star_{n+1})$. Hence the characterization (3) in itself does not give much hint on the structure of optimal $\{K^\star_{V,n}, B^\star_n\}^\infty_{n=1}$ achieving $C_{\text{FB},n}$, or more importantly, its limiting behavior.

In this paper, we make one step forward by first showing that the feedback capacity C_{FB} of the additive Gaussian noise channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, is given by

$$C_{\text{FB}} = \sup_{S_V, B} \int^\pi_{-\pi} \frac{1}{2} \log \frac{S_V(e^{i\theta}) + |1 + B(e^{i\theta})|^2 S_Z(e^{i\theta})}{S_Z(e^{i\theta})} \frac{d\theta}{2\pi} \quad (5)$$

where $S_Z(e^{i\theta})$ is the power spectral density of the noise process $\{Z_i\}^\infty_{i=1}$ and the supremum is taken over all power spectral densities $S_V(e^{i\theta}) \geq 0$ and strictly causal filters $B(e^{i\theta}) = \sum^\infty_{k=1} b_k e^{ik\theta}$ satisfying the power constraint $\int^\pi_{-\pi}(S_V(e^{i\theta}) + |B(e^{i\theta})|^2 S_Z(e^{i\theta})) \frac{d\theta}{2\pi} \leq P$. Roughly speaking, this characterization shows the asymptotic optimality of the stationary solution $(K^*_{V,n}, B^*_n)$ in (3) and hence it can be viewed as the justification for interchange of the order of limit and maximum in (4).

Since our characterization is in a variational form, we will subsequently obtain necessary and sufficient conditions for the optimal $(S^\star_V(e^{i\theta}), B^\star(e^{i\theta}))$ from Lagrange duality theory and additional information theoretic arguments. This result, when specialized to the first-order autoregressive (AR) noise spectrum $S_Z(e^{i\theta}) = |1 + \beta e^{i\theta}|^{-2}, -1 < \beta < 1$, yields the closed-form solution for feedback capacity as

$$C_{\text{FB}} = -\log x_0$$

where x_0 is the unique positive root of the fourth-order polynomial

$$P x^2 = \frac{(1 - x^2)}{(1 + |\beta|x)^2},$$

establishing the long-standing conjecture by Butman [1], [2], Tiernan–Schalkwijk [8], [9], and Wolfowitz [10]. In fact, we will obtain an explicit feedback capacity formula for the first-order autoregressive moving average (ARMA) noise spectrum, generalizing the result in [4] and confirming a recent conjecture by Yang, Kavčić, and Tatikonda [11]. As we will see later, our result shows that the celebrated Schalkwijk–Kailath coding scheme [5], [6] achieves the feedback capacity. More generally, we will show that a k-dimensional generalization of the Schalkwijk–Kailath coding scheme achieves the feedback capacity for any autoregressive moving-average noise spectrum of order k.

The rest of the paper is organized as follows. We present a variational characterization of Gaussian feedback capacity in the next section. We find properties of the optimal solution to our variational problem in Section III, and apply these results to the first-order ARMA noise spectrum (Section IV) and the general finite-order ARMA noise spectrum (Section V).

II. VARIATIONAL CHARACTERIZATION OF THE FEEDBACK CAPACITY

We start from the Cover-Pombra formulation of the n-block feedback capacity $C_{\text{FB},n}$ in (3). Tracing the development of Cover and Pombra [3] backwards, we express $C_{\text{FB},n}$ as

$$C_{\text{FB},n} = \max_{V^n + B_n Z^n} \frac{h(Y^n) - h(Z^n)}{n} = \max_{V^n + B_n Z^n} \frac{I(V^n; Y^n)}{n}$$

where the maximization is over all X^n of the form $X^n = V^n + B_n Z^n$, resulting in $Y^n = V^n + (I + B_n)Z^n$, with strictly lower-triangular B_n and multivariate Gaussian V^n, independent of Z^n, satisfying the power constraint $E \sum^n_{i=1} X^2_i \leq nP$.

Define

$$\tilde{C}_{\text{FB}} = \sup_{S_V, B} \int^\pi_{-\pi} \frac{1}{2} \log \frac{S_V(e^{i\theta}) + |1 + B(e^{i\theta})|^2 S_Z(e^{i\theta})}{S_Z(e^{i\theta})} \frac{d\theta}{2\pi}$$

where $S_Z(e^{i\theta})$ is the power spectral density of the noise process $\{Z_i\}^\infty_{i=1}$ and the supremum is taken over all power spectral densities $S_V(e^{i\theta}) \geq 0$ and strictly causal filters $B(e^{i\theta}) = \sum^\infty_{k=1} b_k e^{ik\theta}$ satisfying the power constraint $\int^\pi_{-\pi}(S_V(e^{i\theta}) + |B(e^{i\theta})|^2 S_Z(e^{i\theta})) \frac{d\theta}{2\pi} \leq P$. In the light of Szegő–Kolmogorov–Krein theorem, we can express $\tilde{C}_{\text{FB}}$ also as

$$\tilde{C}_{\text{FB}} = \sup_{\{X_i\}} h(\mathcal{Y}) - h(\mathcal{Z})$$

where the supremum is taken over all stationary Gaussian processes $\{X_i\}^\infty_{i=-\infty}$ of the form $X_i = V_i + \sum^\infty_{k=1} b_k Z_{i-k}$ where $\{V_i\}^\infty_{i=-\infty}$ is stationary and independent of $\{Z_i\}^\infty_{i=-\infty}$ such that $EX^2_0 \leq P$.

Fix n and let $(K^\star_{V,n}, B^\star_n)$ achieve $C_{\text{FB},n}$. Then, we can construct a stationary Gaussian process by first considering a blockwise stationary process $\{X_i, Y_i\}$ generated from $(K^\star_{V,n}, B^\star_n)$, randomizing the starting point T to get a stationary (non-Gaussian) process $\{X_i(T), Y_i(T)\}$, and then taking a Gaussian process $\{\tilde{X}_i, \tilde{Y}_i\}$ with the same second-order statistics. It is easy to see that the derived processes $\{X_i(T), Y_i(T)\}$ and $\{\tilde{X}_i, \tilde{Y}_i\}$ satisfy the input power constraint and the causal relationship with the noise process. From this observation (and from the usual information theoretic inequalities), we can show that

$$C_{\text{FB},n} \leq \frac{1}{m}\big(h(Y^m_1(T)|T) - h(Z^m_1)\big) + \epsilon_m$$

$$\leq \frac{1}{m}\big(h(Y^m_1(T)) - h(Z^m_1)\big) + \epsilon_m$$

$$\leq \frac{1}{m}\big(h(\tilde{Y}^m_1) - h(Z^m_1)\big) + \epsilon_m,$$

where ϵ_m vanishes as $m \to \infty$. Thus, from the definition of $\tilde{C}_{\text{FB}}$, we have

$$C_{\text{FB},n} \leq h(\tilde{\mathcal{Y}}) - h(\mathcal{Z}) \leq \tilde{C}_{\text{FB}}. \quad (6)$$

We can also show the other direction of inequality from a relatively simple functional analysis argument, which proves the variational characterization of the Gaussian feedback capacity as follows:

Theorem 2.1: The feedback capacity C_{FB} of the Gaussian channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P, is given by (5).

III. OPTIMAL FEEDBACK CODING SCHEME

In this section, we explore many features of the variational characterization of the Gaussian feedback capacity we established in Theorem 2.1. We begin with presenting the necessary condition for the optimal solution $(S_V^\star, B^\star)$ to our variational problem (5) and a few applications of this development.

Proposition 3.1: If $(S_V^\star(e^{i\theta}), B^\star(e^{i\theta}))$ achieves the feedback capacity, it must satisfy all of the following:

(7) Power: $\int_{-\pi}^{\pi} S_V^\star(e^{i\theta}) + |B^\star(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} = P$.

(8) Water-filling: $S_V^\star(e^{i\theta})$ water-fills the modified noise spectrum $|1 + B^\star(e^{i\theta})|^2 S_Z(e^{i\theta})$, that is,

$$S_V^\star(e^{i\theta})(S_Y^\star(e^{i\theta}) - \lambda^\star) = 0 \quad \text{a.e.}$$

where $S_Y^\star(e^{i\theta}) = S_V^\star(e^{i\theta}) + |1 + B^\star(e^{i\theta})|^2 S_Z(e^{i\theta})$ and $\lambda^\star = \operatorname{ess\,inf}_{\theta \in [-\pi, \pi)} S_Y^\star(e^{i\theta})$.

(9) Orthogonality: The current input X_0 is independent of the past output $\{Y_i\}_{-\infty}^{-1}$. Equivalently, $S_V^\star(e^{i\theta}) + B^\star(e^{i\theta})S_Z(e^{i\theta})(1 + B^\star(e^{-i\theta}))$ is anticausal.

The necessity of (7) and (8) is obvious. The orthogonality condition (9) follows roughly from the fact that the transmitter can save its power by not sending the information the receiver already has.

Corollary 3.2: Feedback does not increase the capacity if and only if the noise spectrum is white, i.e., $S_Z(e^{i\theta})$ is constant.

Corollary 3.3: Suppose $(S_V^\star(e^{i\theta}), B^\star(e^{i\theta}))$ achieves the Gaussian feedback capacity. Then, there exists $B^{\star\star}(e^{i\theta})$ such that

$$S_Y^\star(e^{i\theta}) = |1 + B^{\star\star}(e^{i\theta})|^2 S_Z(e^{i\theta})$$

and

$$\int_{-\pi}^{\pi} S_V^\star(e^{i\theta}) + |B^\star(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi}$$
$$= \int_{-\pi}^{\pi} |B^{\star\star}(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi}.$$

In particular, $(0, B^{\star\star}(e^{i\theta}))$ achieves the feedback capacity.

The essential content of Corollary 3.3 is that we can restrict our attention to the solutions of the form $(0, B(e^{i\theta}))$. In other words, the question becomes finding the optimal feedback filter, which yields a simpler characterization of the feedback capacity.

Theorem 3.4: Suppose the noise process $\{Z_i\}_{i=1}^{\infty}$ has the absolutely continuous power spectral distribution $d\mu_Z(\theta) = S_Z(e^{i\theta})d\theta$. Then, the feedback capacity C_{FB} of the Gaussian channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P, is given by

$$C_{\text{FB}} = \sup_{B(e^{i\theta})} \int_{-\pi}^{\pi} \frac{1}{2} \log \frac{|1 + B(e^{i\theta})|^2 S_Z(e^{i\theta})}{S_Z(e^{i\theta})} \frac{d\theta}{2\pi}$$
$$= \sup_{B(e^{i\theta})} \int_{-\pi}^{\pi} \frac{1}{2} \log |1 + B(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

where the supremum is taken over all strictly causal filters $B(e^{i\theta}) = \sum_{k=1}^{\infty} b_k e^{ik\theta}$ satisfying the power constraint $\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} \leq P$.

Although the necessary condition in Proposition 3.1 reveals the structure of the capacity-achieving feedback filter, it is still short of characterizing the capacity-achieving feedback filter itself. We remedy the situation by deriving a universal upper bound on the feedback capacity and finding the condition in which this upper bound is tight.

Take any $\nu > 0$, $\phi, \psi_1 \in L_\infty$, and $\psi_2, \psi_3 \in L_1$ such that $\phi > 0$, $\log \phi \in L_1$, $\psi_1 = \nu - \phi \geq 0$, $A := \psi_2 + \nu S_Z$ is anticausal, and

$$\begin{bmatrix} \psi_1(e^{i\theta}) & \psi_2(e^{i\theta}) \\ \hline \overline{\psi_2(e^{i\theta})} & \psi_3(e^{i\theta}) \end{bmatrix} \succeq 0.$$

Now that any feasible $B(e^{i\theta})$ and $S_Y(e^{i\theta})$ satisfy

$$\begin{bmatrix} S_Y(e^{i\theta}) & 1 + B(e^{i\theta}) \\ 1 + \overline{B(e^{i\theta})} & S_Z^{-1}(e^{i\theta}) \end{bmatrix} \succeq 0,$$

we have

$$\operatorname{tr}\left(\begin{bmatrix} S_Y & 1 + B \\ 1 + \overline{B} & S_Z^{-1} \end{bmatrix} \begin{bmatrix} \psi_1 & \psi_2 \\ \overline{\psi_2} & \psi_3 \end{bmatrix} \right)$$
$$= \phi_1 S_Y + \psi_2(1 + \overline{B}) + \overline{\psi_2}(1 + B) + \psi_3 S_Z^{-1} \geq 0.$$

Invoking the inequality $\log x \leq x - 1$ with $x = \phi S_Y$, we get

$$\log S_Y \leq -\log \phi + \phi S_Y - 1 = -\log \phi + \nu S_Y - \psi_1 S_Y - 1$$
$$\leq -\log \phi + \nu S_Y + \psi_2(1 + \overline{B}) + \overline{\psi_2}(1 + B) + \psi_3 S_Z^{-1} - 1. \quad (10)$$

Further, since $A = \psi_2 + \nu S_Z$ is anticausal and B is strictly causal, $A\overline{B}$ is strictly anticausal, so that

$$\int_{-\pi}^{\pi} A(e^{i\theta})\overline{B(e^{i\theta})} \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \overline{A(e^{i\theta})}B(e^{i\theta}) \frac{d\theta}{2\pi} = 0. \quad (11)$$

By integrating both sides of (10), we get

$$\int_{-\pi}^{\pi} \log S_Y$$
$$\leq \int_{-\pi}^{\pi} -\log \phi + \nu S_Y + \psi_2(1 + \overline{B}) + \overline{\psi_2}(1 + B) + \psi_3 S_Z^{-1} - 1$$
$$\leq \int_{-\pi}^{\pi} -\log \phi + \nu\big((B + \overline{B} + 1)S_Z + P\big)$$
$$\qquad + \psi_2(1 + \overline{B}) + \overline{\psi_2}(1 + B) + \psi_3 S_Z^{-1} - 1$$
$$= \int_{-\pi}^{\pi} -\log \phi + \psi_2 + \overline{\psi_2} + \psi_3 S_Z^{-1} + \nu(S_Z + P) - 1 \quad (12)$$

where the second inequality follows from the power constraint and the last equality follows from (11).

Tracing the equality conditions in (12), we have the following sufficient condition for the optimality of a specific $B(e^{i\theta})$.

Proposition 3.5: Suppose $S_Z(e^{i\theta})$ is bounded away from zero. Suppose $B(z)$ is strictly causal (i.e., B(0) = 0) with

$$\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} = P. \quad (13)$$

If there exists $\lambda > 0$ such that

(14) $\lambda \leq \operatorname{ess\,inf}_{\theta \in [-\pi, \pi)} |1 + B(e^{i\theta})|^2 S_Z(e^{i\theta})$

and that

(15) $\quad \dfrac{\lambda}{1 + B(e^{-i\theta})} - B(e^{i\theta})S_Z(e^{i\theta}) \in L_1$ is anticausal,

then $B(e^{i\theta})$ achieves the feedback capacity.

The structural properties of the optimal feedback filter we discovered so far is still abstract in nature and one may question their applicability. In the next two section we narrow our focus to rational noise spectra and show how the results in this section can be used to yield a concrete solution.

IV. FIRST-ORDER ARMA NOISE SPECTRUM

The first-order autoregressive moving average noise spectrum is defined as

$$S_Z(e^{i\theta}) = \left| \frac{1 + \alpha e^{i\theta}}{1 + \beta e^{i\theta}} \right|^2 \qquad (16)$$

for $\alpha \in [-1, 1]$ and $\beta \in (-1, 1)$. This spectral density corresponds to the stationary noise process given by

$$Z_i + \beta Z_{i-1} = U_i + \alpha U_{i-1}, \qquad i \in \mathbb{Z}$$

where $\{U_i\}_{i=-\infty}^{\infty}$ is a white Gaussian process with zero mean and unit variance.

Theorem 4.1: Suppose the noise process $\{Z_i\}_{i=1}^{\infty}$ has the power spectral density $S_Z(z)$ defined in (16). Then, the feedback capacity C_{FB} of the Gaussian channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P is given by

$$C_{\text{FB}} = -\log x_0$$

where x_0 is the unique positive root of the fourth-order polynomial

$$P x^2 = \frac{(1 - x^2)(1 + \sigma\alpha x)^2}{(1 + \sigma\beta x)^2} \qquad (17)$$

and $\sigma = \text{sgn}(\beta - \alpha)$.

Proof: (Sketch.) Without loss of generality, we assume $|\alpha| < 1$; for the case $|\alpha| = 1$, we can perturb the noise spectrum with small power to transform it into another ARMA(1) spectrum with $|\alpha| \neq 1$. Under the assumption $|\alpha| < 1$, $S_Z(e^{i\theta})$ is bounded away from zero, so we can apply Proposition 3.5.

Here is the bare-bone summary of the proof: We will take the feedback filter of the form

$$B(z) = \frac{1 + \beta z}{1 + \alpha z} \cdot \frac{yz}{1 - \sigma x z} \qquad (18)$$

where $x \in (0, 1)$ is an arbitrary parameter corresponding to each power constraint $P \in (0, \infty)$ under the choice of

$$y = \frac{x^2 - 1}{\sigma x} \cdot \frac{1 + \alpha\sigma x}{1 + \beta\sigma x} = -P\sigma x \left(\frac{1 + \beta\sigma x}{1 + \alpha\sigma x} \right). \qquad (19)$$

Then, we can show that $B(z)$ satisfies the sufficient condition in Proposition 3.5 under the power constraint

$$P = \int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \frac{y^2}{|1 - xe^{i\theta}|^2} \frac{d\theta}{2\pi} = \frac{y^2}{1 - x^2}.$$

The corresponding information rate is given by

$$\int_{-\pi}^{\pi} \frac{1}{2} \log |1 + B(e^{i\theta})|^2 \frac{d\theta}{2\pi} = -\frac{1}{2} \log x^2.$$

The rest of the proof is the actual implementation of this idea that checks the sufficient condition directly. ∎

Now we interpret the optimal feedback filter $B^\star(z)$ we found in (18). In particular, we show that the celebrated Schalkwijk–Kailath signaling scheme is asymptotically equivalent to our feedback filter $B^\star$, establishing the optimality of the Schalkwijk–Kailath coding scheme for the ARMA(1) noise spectrum.

Consider the following coding scheme. Let $V \sim N(0, 1)$. Over the channel $Y_i = X_i + Z_i$, the transmitter initially sends $X_1 = V$ and subsequently

$$X_n = (\sigma x)^{-(n-1)}(V - \hat{V}_{n-1}), \qquad n = 2, 3, \ldots, \qquad (20)$$

where $\sigma = \text{sgn}(\beta - \alpha)$, x is the unique positive root of the fourth-order polynomial (17), and $\hat{V}_n = E(V|Y^n)$ is the minimum mean-squared error estimate of V given the channel output signals $Y^n = (Y_1, \ldots, Y_n)$ up to time n. For all $m < n$, we have

$$X_n = (\sigma x)^{m-n}(X_m - E(X_m|Y^{n-1})) \qquad (21)$$

$$= -(\sigma x)^{m-n}(Z_m - E(Z_m|Y^{n-1})). \qquad (22)$$

Furthermore, from the structure of noise $Z_n = -\beta Z_{n-1} + U_n + \alpha U_{n-1}$, we can show that

$$Z_n - E(Z_n|Y^{n-1}) \approx \left(\frac{\beta - \alpha}{\alpha + (\sigma x)^{-1}} \right) X_n + U_n \qquad (23)$$

for large n, or equivalently,

$$X_n \approx \frac{\alpha + (\sigma x)^{-1}}{\beta - \alpha} (E(Z_n|Z^{n-1}) - E(Z_n|Y^{n-1})). \qquad (24)$$

Now from (21) with $m = n - 1$ and the orthogonality of X_{n-1} and Y^{n-2},

$$X_n = (\sigma x)^{-1} (X_{n-1} - E(X_{n-1}|\tilde{Y}_{n-1})) \qquad (25)$$

where $\tilde{Y}_{n-1} := Y_{n-1} - E(Y_{n-1}|Y^{n-2})$ is the innovation of the output process at time $n - 1$. Also from (23) and the orthogonality of X_{n-1} and Y^{n-2}, we can check that

$$\tilde{Y}_{n-1} \approx c X_{n-1} + U_{n-1}$$

where

$$c = 1 + \frac{\beta - \alpha}{\alpha + (\sigma x)^{-1}} = \frac{1 + \beta\sigma x}{1 + \alpha\sigma x}.$$

Finally, returning to (25), we can show that

$$X_n \approx \frac{(\sigma x)^{-1}}{c^2 P + 1} (X_{n-1} - cPU_{n-1})$$

$$= \sigma x X_{n-1} - yU_{n-1},$$

where x and y are the constants given by (17) and (19). Therefore, the feedback coding scheme given by (20) is asymptotically equivalent to filtering the noise through the feedback filter

$$B(z) = \frac{1 + \beta z}{1 + \alpha z} \cdot \frac{yz}{1 - \sigma x z},$$

which is exactly equal to the optimal feedback filter (18) we found in the proof of Theorem 4.1.

This coding scheme uses the minimum mean-square error decoding of the message V, but it is fairly straightforward to transform the Gaussian coding scheme to the original Schalkwijk–Kailath coding scheme with an equally spaced signal constellation and maximum likelihood decoding.

Finally note that (21), (22), and (24) give interesting alternative interpretations of the Schalkwijk–Kailath coding scheme; the optimal transmitter refines the receiver's knowledge of any past input (21), or equivalently, any past noise (22). Also asymptotically, the optimal transmitter sends the difference between what he knows about the noise and what the receiver knows about the noise (24).

V. General Finite-Order ARMA Noise Spectrum

Now we turn to the general autoregressive moving-average noise spectrum with finite order, say, k. We assume that the noise power spectral density $S_Z(e^{i\theta})$ has the canonical spectral factorization $S_Z(e^{i\theta}) = H_Z(e^{i\theta})H_Z(e^{-i\theta})$ where

$$H_Z(z) = \frac{P(z)}{Q(z)} = \frac{1 + \sum_{n=1}^k p_n z^n}{1 + \sum_{n=1}^k q_n z^n} \qquad (26)$$

such that at least one of the monic co-prime polynomials $P(z)$ and $Q(z)$ has degree k and all zeros of $P(z)$ and $Q(z)$ lie strictly outside the unit circle (i.e, both $P(z)$ and $Q(z)$ are stable). In particular, $S_Z(e^{i\theta})$ is bounded away from zero.

Equivalently, we can represent the ARMA(k) noise process via a state-space model:

$$\begin{aligned} S_{n+1} &= FS_n + GU_n \\ Z_n &= HS_n + U_n \end{aligned} \qquad (27)$$

where real matrices F, G, and H of sizes $k \times k$, $k \times 1$, and $1 \times k$ are given by

$$F = \begin{bmatrix} -q_1 & -q_2 & \cdots & & -q_k \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}'$$

$$H = \begin{bmatrix} -(p_1 + q_1) & \cdots & -(p_k + q_k) \end{bmatrix},$$

$\{U_n\}_{n=-\infty}^{\infty}$ are independent and identically distributed normal random variables with zero mean and unit variance, and the state S_n and the innovation U_n are independent of each other.

Theorem 5.1: Suppose the noise process has the state-space representation (27). Then, the feedback capacity C_{FB} of the Gaussian channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P, is given by

$$C_{\text{FB}} = \max_X \frac{1}{2} \log(1 + (X + H)\Sigma_+(X)(X + H)'),$$

where the maximum is taken over all $X \in \mathbb{R}^{1 \times k}$ such that $F - G(X + H)$ has no unit-circle zero and $X\Sigma_+(X)X' \leq P$,

and $\Sigma_+(X)$ is the maximal solution to the discrete algebraic Riccati equation

$$\Sigma = F\Sigma F' + GG' - \frac{(F\Sigma(X+H)' + G)(F\Sigma(X+H)' + G)'}{1 + (X + H)\Sigma(X + H)'}.$$

The proof of Theorem 5.1 follows from the following structural result, whose proof, in turn, is a rather nontrivial consequence of Proposition 3.1. (We skip the proof.)

Lemma 5.2: Suppose the ARMA(k) noise process has the state-space representation (27). Then the feedback capacity is achieved by an input process $\{X_n\}_{n=-\infty}^{\infty}$ of the form

$$X_n = X\big(S_n - E(S_n|Y_{-\infty}^{n-1})\big)$$

for some $X \in \mathbb{R}^{1 \times k}$ such that $F - G(X + H)$ has no unit-circle eigenvalue.

From an argument similar to the one at the end of the previous section, we can show that a variant of the Schalkwijk–Kailath coding scheme achieves the feedback capacity — this time, with k-dimensional messages. More specifically, for initial k transmissions, the transmitter sends k orthogonal message symbols. Subsequently, it sends

$$X_n = X^\star(E(S_n|Z^{n-1}) - E(S_n|Y^n)), \qquad n \geq k + 1,$$

which results in the exponential decay of the minimum mean-squared error matrix of the message vector at rate $2^{C_{\text{FB}}}$. Here we can view $X^\star$ as the optimal direction for projecting the receiver's estimation error. We can also express the optimal coding scheme as the refinement of the receiver's estimate of the intended message, which proves the optimality of the k-dimensional Schalkwijk–Kailath feedback coding scheme.

Finding the optimal direction $X^\star$ is, in general, a very difficult optimization problem. Here we only comment that Proposition 3.5 can lead to a more refined structure on $X^\star$.

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