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Correlated Sources over Broadcast Channels

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***Abstract*—An alternative characterization is given for the coding theorem by Han and Costa, which finds a set of admissible source pairs that can be transmitted reliably over the broadcast channel. The associated coding technique is conceptually simpler and the resulting admissible source region can be shown to include the Gray–Wyner distributed source coding region in a straightforward manner. Incidentally, this new characterization illustrates how the *common part* between the two random sources does not play any role in broadcasting correlated sources, unlike transmission of correlated sources over the multiple access channel.**

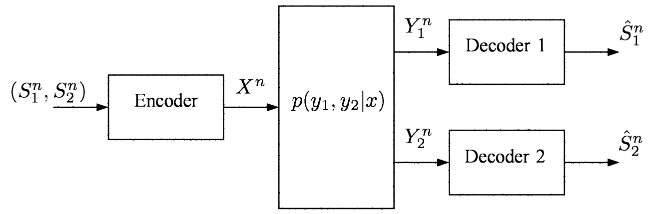

Fig. 1. The two-user broadcast channel with arbitrarily correlated sources.

I. Introduction

The problem of broadcasting two arbitrarily correlated sources over a general two-user discrete memoryless broadcast channel (BC) is depicted in Figure 1. A memoryless stationary source produces n independent copies of the random variables (S_1, S_2), which are distributed according to a joint distribution $p(s_1, s_2)$, generating the sequences $(S_1^n, S_2^n) := [(S_{1,1}, S_{2,1}), \cdots, (S_{1,n}, S_{2,n})]$. The encoder maps the source sequences into a random codeword X^n of n random variables with values in finite sets, and broadcasts X^n to two separate receivers. The communication channel is modeled via a transition probability matrix $p(y_1, y_2|x)$, which maps each channel input x into a pair of finite valued output symbols (y_1, y_2). The channel is supposed to be memoryless. Decoder 1 maps the channel output sequence Y_1^n into a source sequence $\hat{S}_1^n$, which represents an estimate of the transmitted sequence S_1^n. Similarly, decoder 2 maps Y_2^n into a sequence $\hat{S}_2^n$. For a given coding/decoding scheme, the average probability of decoding errors at the two decoders as a function of sequence length n are defined as

$$P_{e_i}^n = \Pr\{\hat{S}_i^n \neq S_i^n\}, \quad i = 1, 2. \tag{1}$$

The source (S_1, S_2) is said to be *admissible* for the BC $p(y_1, y_2|x)$ if there exists a sequence of encoders and decoders such that $P_{e_i}^n$ vanishes as n grows to infinity, $i = 1, 2$. The (closure of the) set of all admissible sources (S_1, S_2) for the given BC is called the *admissible source region.*

Han and Costa [12] provided a sufficient condition for (S_1, S_2) to be admissible for a general discrete memoryless BC. An outer bound on the admissible source region for an arbitrarily varying BC was derived by Gohari and Anantharam [10]. Unfortunately, necessary and sufficient conditions do not coincide, so the admissible source region for a general BC is still an open problem.

Up to present, conclusive results have been established only for special classes of sources and channels. An important subclass of sources is given by $S_1 = (W_0, W_1)$ and $S_2 = (W_0, W_2)$, where W_0, W_1, and W_2 are three mutually independent random variables uniformly distributed over sets with cardinalities 2^{nR_0}, 2^{nR_1}, and 2^{nR_2}, respectively, for some nonnegative triple (R_0, R_1, R_2). The set of admissible rate triples (R_0, R_1, R_2) is called the *capacity region* of the BC. Inner bounds on the capacity region of a general BC are developed in [1], [3], [9], [14], [16], [19]. Specializing to the case of independent sources, the coding scheme by Han and Costa includes all these results as special cases. Outer bounds on the capacity region appear in [15], [16], [17], [18]. Specializing by removing the variability of the channel and assuming independent sources, the bound of Gohari and Anantharam [10] recovers the best known outer bounds. Inner and outer bounds to the capacity region have been shown to coincide for some important classes of broadcast channels. A survey of the known results appears in [15] and we refer the reader to this work for references. Recently, Kang and Kramer [6] studied the problem of broadcasting correlated sources over a general BC, assuming side information at the receivers and a degraded "source" structure, in which one of the two decoders is interested in reconstructing both source sequences. For this problem, the admissible source region is characterized in some special cases.

In this paper, we provide an alternative characterization of the (corrected version of) Han and Costa's coding theorem. There are several ingredients in our development. First, we provide a conceptually simple coding technique. This coding technique can be interesting on its own because it does not involve any random hashing (Slepian–Wolf binning), superposition coding, or rate splitting, as in the coding techniques of Marton and of Han and Costa. Second, the conceptually simple coding technique leads to a general inner bound on the admissible source region that includes Marton's inner bound on the broadcast channel capacity region and Gray and Wyner's distributed source coding region. This inner bound does not involve the common part of the two sources in the sense of Gacs, Körner, and Witsenhausen, providing a simpler description of sufficient conditions for admissible source pairs. Third, we show that our inner bound, apparently more general than Han and Costa's, is in fact equivalent to Han and Costa's inner bound. This equivalence implies that when two sources are sent over the broadcast channel, there is no special role played by the common part of the sources, confirming the standard engineering intuition. It is interesting to observe the striking contrast to the problem of sending arbitrarily correlated sources over multiple access channels. For that problem, the common part of the sources plays a pivotal role in Cover, El Gamal, and Salehi's random coding construction [2] by inducing coherent communication between separate encoders.

The rest of the paper is organized as follows. In the next section, we review Han and Costa's inner bound (with a recent correction by Kramer and Nair). Section III gives the alternative coding technique for broadcasting correlated sources and describes the associated sufficient conditions for admissible source pairs; the equivalence of two inner bounds is also established.

II. Han and Costa's coding Theorem

For convenience, we recall here Han and Costa's coding theorem for broadcasting correlated sources with a few minor changes in the adopted notation. An error in the derivation of [12, Theorem 1] was recently pointed out by Kramer and Nair [8], who showed that the correct form of Han and Costa's coding theorem is as follows

Theorem 2.1 ([12], [8]): Suppose that a BC $p(y_1, y_2|x)$ and a source (S_1, S_2) are given, and let $K = f(S_1) = g(S_2)$ be the common variable in the sense of Gacs and Körner (and also Witsenhausen). If there exists three auxiliary random variables U_0, U_1, U_2 (with values in finite sets) that satisfy the Markov chain property

$$S_1, S_2 \rightarrow U_0, U_1, U_2 \rightarrow X \rightarrow Y_1, Y_2$$

and the inequalities

$$\begin{aligned} H(S_1) &< I(U_0, U_1, S_1; Y_1) - I(U_0, U_1; S_2|S_1), \\ H(S_2) &< I(U_0, U_2, S_2; Y_2) - I(U_0, U_2; S_1|S_2), \\ H(S_1, S_2) &< I(K, U_0, U_1, S_1; Y_1) + I(U_2, S_2; Y_2|K, U_0) \\ &\quad - I(U_1, S_1; U_2, S_2|K, U_0), \qquad (2) \\ H(S_1, S_2) &< I(U_1, S_1; Y_1|K, U_0) + I(K, U_0, U_2, S_2; Y_2) \\ &\quad - I(U_1, S_1; U_2, S_2|K, U_0), \qquad (3) \\ H(S_1, S_2) &< I(U_0, U_1, S_1; Y_1) + I(U_0, U_2, S_2; Y_2) \\ &\quad - I(U_1, S_1; U_2, S_2|K, U_0) - I(S_1, S_2; K, U_0), \qquad (4) \end{aligned}$$

then the source (S_1, S_2) is admissible for the channel BC $p(y_1, y_2|x)$.

In the sequel, the subset of the admissible source region for the BC $p(y_1, y_2|x)$ characterized by the above theorem will be denoted as $\mathscr{S}_{HC}$. The coding scheme by Han and Costa requires three auxiliary random variables (U_0, U_1, U_2). Roughly speaking, U_0 carries the common part K and the hash indexes of S_1 and S_2, U_1 encodes the remaining uncertainty about the source sequence S_1, and U_2 encodes the remaining uncertainty about the source sequence S_2. For the accuracy of this interpretation, see [12].

III. An alternative characterization of Han and Costa's result

The objective of this section is to prove the following Theorem:

Theorem 3.1: A source (S_1, S_2) is admissible for the discrete memoryless BC $p(y_1, y_2|x)$ if there exists three auxiliary random variables U_0, U_1, U_2 (with values in finite sets) that satisfy the Markov chain property

$$S_1, S_2 \rightarrow U_0, U_1, U_2 \rightarrow X \rightarrow Y_1, Y_2$$

and the inequalities

$$\begin{aligned} H(S_1) &< I(U_0, U_1, S_1; Y_1) - I(U_0, U_1; S_2|S_1), \\ H(S_2) &< I(U_0, U_2, S_2; Y_2) - I(U_0, U_2; S_1|S_2), \\ H(S_1, S_2) &< I(U_0, U_1, S_1; Y_1) + I(U_2, S_2; Y_2|U_0) \\ &\quad - I(U_1, S_1; U_2, S_2|U_0), \qquad (5) \\ H(S_1, S_2) &< I(U_1, S_1; Y_1|U_0) + I(U_0, U_2, S_2; Y_2) \\ &\quad - I(U_1, S_1; U_2, S_2|U_0), \qquad (6) \\ H(S_1, S_2) &< I(U_0, U_1, S_1; Y_1) + I(U_0, U_2, S_2; Y_2) \\ &\quad - I(U_1, S_1; U_2, S_2|U_0) - I(S_1, S_2; U_0). \qquad (7) \end{aligned}$$

The subset of the admissible source region for the BC $p(y_1, y_2|x)$ characterized by Theorem 3.1 is denoted by $\mathscr{S}$. Application of Theorem 3.1 yields the following results as special cases:

a) *The Gray–Wyner source coding problem [11]*: Consider the noiseless BC channel with $Y_1 = (X_0, X_1)$, $Y_2 = (X_0, X_2)$ with links of rate equal to $R_0 = H(X_0)$, $R_1 = H(X_1)$, $R_2 = H(X_2)$. By taking $U_1 = X_1$, $U_2 = X_2$, and $U_0 = (X_0, V)$, under the distribution $p(v|s_1, s_2)p(x_0)p(x_1)p(x_2)$, the inequalities in Theorem 3.1 simplify to the following rate region

$$\begin{aligned} R_0 + R_1 &> I(S_1, S_2; V) + H(S_1|V), \\ R_0 + R_2 &> I(S_1, S_2; V) + H(S_2|V), \\ R_0 + R_1 + R_2 &> I(S_1, S_2; V) + H(S_1|V) + H(S_2|V), \\ 2R_0 + R_1 + R_2 &> 2I(S_1, S_2; V) + H(S_1|V) \\ &\quad + H(S_2|V), \end{aligned}$$

which includes (and is in fact equivalent to) the Gray–Wyner rate region, which is characterized by the following set of inequalities:

$$R_0 > I(S_1, S_2; V),$$
$$R_1 > H(S_1|V),$$
$$R_2 > H(S_2|V).$$

b) *The Marton inner bound [16]*: Consider the special case of independent sources described in the Introduction, so take $S_1 = (W_0, W_1)$ and $S_2 = (W_0, W_2)$, with W_0, W_1, and W_2 having rates R_0, R_1, R_2, respectively. By choosing (U_0, U_1, U_2) to be independent of (W_0, W_1, W_2), Theorem 3.1 simplify to yield the achievability of Marton's inner bound to a general BC (see also Example 1 in [8]).

c) *More capable broadcast channels*: Consider the special case of a *more capable* broadcast channel such that $I(X; Y_1) \geq I(X; Y_2)$ for all $p(x)$ as defined in El Gamal [4]. The admissible source region for this class of channels is as follows.
Theorem 3.2 ([13]): A source (S_1, S_2) is admissible for a *more capable* discrete memoryless BC $p(y_1, y_2|x)$ if and only if there exists an auxiliary random variable U that satisfies the inequalities

$$H(S_2) \leq I(U; Y_2), \tag{8}$$
$$H(S_1, S_2) \leq I(U; Y_2) + I(X; Y_1|U), \tag{9}$$
$$H(S_1, S_2) \leq I(X; Y_1), \tag{10}$$

with

$$p(s_1, s_2, u, x, y_1, y_2) = p(s_1, s_2)p(u, x)p(y_1, y_2|x).$$

Observe that the sufficiency follows from Theorem 3.1 by choosing $U_0 = (U, S_2)$, $U_1 = X$ and $U_2 =$ const. with $p(u, x|s_1, s_2) = p(u, x)$. Interestingly, the direct part of the theorem can also be proved by concatenating the Slepian-Wolf binning scheme to the channel code in [4]. Thus, separation between source and channel coding is optimal for this special class of problems.

d) *Degraded "source" sets* [6]: Consider the case in which decoder 1 is interested in reconstructing both source sequences (S_1^n, S_2^n) with vanishing error probability. This setup can be captured by considering transmission of a pair of sources $\tilde{S}_1 = (S_1, S_2)$, $\tilde{S}_2 = S_2$. Kang and Kramer characterized the admissible source region for this special class of problems:
Theorem 3.3 ([6]): A source (S_1, S_2) is admissible for a discrete memoryless BC $p(y_1, y_2|x)$ with *degraded "source" sets* if and only if there exists an auxiliary random variable U that satisfies the inequalities (8), (9), (10) with

$$p(s_1, s_2, u, x, y_1, y_2) = p(s_1, s_2)p(u, x)p(y_1, y_2|x).$$

Again, by choosing $U_0 = (U, S_2)$, $U_1 = X$ and $U_2 =$ const. with $p(u, x|s_1, s_2) = p(u, x)$, Theorem 3.1 simplify to yield the achievability of the region in Theorem 3.3. Observe that separation between source and channel coding is also optimal for this special class of problems.

Comparing the statement of Theorem 2.1 to that of Theorem 3.1, observe that inequalities (5)-(7) do not involve the common part K of the source (S_1, S_2) as in their counterparts in Theorem 2.1. It is interesting to observe that if we define a new random variable as $\tilde{U}_0 = (U_0, K)$, where K denotes the common part of the source (S_1, S_2), then the inequalities in Theorem 3.1 for the triple $(\tilde{U}_0, U_1, U_2)$ simplify and reduce to those in Theorem 2.1. Thus, we conclude that Theorem 3.1 includes Han and Costa's region. A priori it is unclear whether or not the inclusion is strict. The following proposition, however, shows that this is not the case, so the difference in the two sets of inequalities is purely formal.

Proposition 3.4: $\mathscr{S}_{HC} = \mathscr{S}$.

The proof of the above proposition will be given in the Appendix.

The achievability of Theorem 3.1 is established by means of a random coding argument. In the remaining of this section, we first describe the random codebook generation and encoding-decoding scheme, then we outline the analysis of the probability of error.

Random codebook generation: Fix a joint distribution

$$p(u_0, u_1, u_2, x, y_1, y_2|s_1, s_2)$$
$$= p(u_0, u_1, u_2|s_1, s_2)p(x|u_0, u_1, u_2)p(y_1, y_2|x),$$

and compute $p(u_0)$, $p(u_1|s_1)$ and $p(u_2|, s_2)$ for the given source distribution $p(s_1, s_2)$. Randomly and independently generate 2^{nR_0} sequences $u_0^n(l)$, $l \in [1 : 2^{nR_0}]$, each according to $\prod_{k=1}^n p(u_{0,k})$. For each source sequence s_1^n randomly and independently generate 2^{nR_1} sequences $u_1^n(s_1^n, i)$, $i \in [1 : 2^{nR_1}]$, according to $\prod_{k=1}^n p(u_{1,k}|s_{1,k})$. The same procedure, using $\prod_{k=1}^n p(u_{2,k}|s_{2,k})$, is repeated for generating 2^{nR_2} sequences $u_2^n(s_1^n, j)$, $j \in [1 : 2^{nR_2}]$. The rates (R_0, R_1, R_2) are chosen so that the ensemble of generated sequences (u_0^n, u_1^n, u_2^n) cover the set of jointly typical sequences for (U_0, U_1, U_2). The conditions for covering are given in the following Lemma.

Lemma 3.5 (Covering Lemma): For each $\epsilon > 0$, let $\mathscr{T}_\epsilon^{(n)}(S_1, S_2, U_0, U_1, U_2)$ denote the set of ϵ-jointly typical sequences of length n independently generated according to $p(s_1, s_2, u_0, u_1, u_2, x, y_1, y_2)$. For each fixed ϵ-jointly typical sequences (s_1^n, s_2^n), let $u_0^n(l)$, $u_1^n(s_1^n, i)$ and $u_0^n(s_2^n, j)$, $(l, i, j) \in [1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]$, be generated as described above. Then, we have

$$\Pr\{s_1^n, s_2^n, u_0^n(l), u_1^n(s_1^n, i), u_2^n(s_2^n, j) \notin \mathscr{T}_\epsilon^{(n)}(S_1, S_2, U_0, U_1, U_2)$$
$$\text{for all } (l, i, j) \in [1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]\} \to 0$$

if the following inequalities are satisfied

$$\begin{aligned} R_0 &> I(U_0; S_1, S_2), \\ R_1 &> I(U_1; S_2|S_1), \\ R_2 &> I(U_2; S_1|S_2), \\ R_0 + R_1 &> I(U_0; S_1, S_2) + I(U_1; U_0, S_2|S_1), \\ R_0 + R_2 &> I(U_0; S_1, S_2) + I(U_2; U_0, S_1|S_2), \\ R_1 + R_2 &> I(U_1, S_1; U_2, S_2) - I(S_1; S_2), \\ R_0 + R_1 + R_2 &> I(U_0; S_1, S_2) + I(U_1; U_0, S_2|S_1) \\ &\quad + I(U_2; U_0, S_1|S_2) + I(U_1; U_2|U_0, S_1, S_2). \end{aligned} \tag{11}$$

The above Lemma can be showed using techniques similar to that in El Gamal and van der Meulen [5].

Encoding: For each typical sequence (s_1^n, s_2^n), choose a triple $(l, i, j) \in [1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]$ such that $(s_1^n, s_2^n, u_0^n(l), u_1^n(s_1^n, i), u_2^n(s_2^n, j))$ is in the jointly typical set of $(S_1, S_2, U_0, U_1, U_2)$. Declare an error if no such triple can be found. Then, generate a sequence x^n according to $\prod_{k=1}^n p(x_k|u_{0,k}(l), u_{1,k}(s_1^n, i), u_{2,k}(s_2^n, j))$. The sequence x^n so generated is the codeword corresponding to the source sequence (s_1^n, s_2^n).

Decoding: Upon observing the received sequence y_1^n, the decoder declares that s_1^n is the transmitted source sequence, if this is the unique sequence such that $(s_1^n, u_0(m), u_1^n(s_1, i), y_1^n)$ is typical for some $m \in [1 : 2^{nR_0}]$ and $i \in [1 : 2^{nR_1}]$. Otherwise, declare an error. Similarly, the decoder observing y_2^n declares that that s_2^n was sent, if this is the unique sequence such that $(s_2^n, u_0(m), u_2^n(s_2, j), y_2^n)$ is typical for some $m \in [1 : 2^{nR_0}]$ and $j \in [1 : 2^{nR_2}]$.

Error events: If the covering conditions (11) are satisfied, then the probability of an error upon encoding vanishes as n tends to infinity. Thus, we focus on the analysis of the decoding errors. Given the symmetry in the code generation, assume without loss of generality that $(m, i, j) = (1, 1, 1)$ for the transmitted source sequence (s_1^n, s_2^n). The error event in decoding upon receiving the sequence y_1^n can be divided into two parts:

1) There exists a sequence $\hat{s}_1^n \neq s_1^n$ and an $m \neq 1$ such that $(\hat{s}_1^n, u_0(m), u_1^n(\hat{s}_1^n, i), y_1^n)$ is typical for some i. Using standard typicality arguments, the probability of this event can be made arbitrarily small as n tends to infinity if

$$H(S_1) + R_0 + R_1 < I(U_0, U_1, S_1; Y_1) + I(U_0; U_1, S_1). \tag{12}$$

2) There exists a sequence $\hat{s}_1^n \neq s_1^n$ such that $(\hat{s}_1^n, u_0(1), u_1^n(\hat{s}_1^n, i), y_1^n)$ is typical for some i. The probability of this event vanishes as n increases if the following inequality is satisfied

$$H(S_1) + R_1 < I(U_1, S_1; U_0, Y_1). \tag{13}$$

Similarly, the probability of decoding error upon receiving y_2^n can be made arbitrarily small for n tending to infinity if

$$H(S_2) + R_2 < I(U_2, S_2; U_0, Y_2), \tag{14}$$

and

$$H(S_2) + R_0 + R_2 < I(U_0, U_2, S_2; Y_2) + I(U_0; U_2, S_2). \tag{15}$$

Fourier–Motzkin elimination: In summary, we conclude that the error probabilities $P_{e_1}^{(n)}$ and $P_{e_2}^{(n)}$ can be made arbitrarily small by letting n grow to infinity if conditions (11), (12), (13), (14), and (15) are satisfied. The final step in our derivation consists of eliminating the auxiliary variables R_0, R_1, R_2 from these inequalities. Application of the standard Fourier–Motzkin elimination algorithm results in a large number of inequalities involving $H(S_1)$, $H(S_2)$, and $H(S_1, S_2)$. Some of them are inactive because they are implied by other inequalities. Also there are inequalities that involve the mutual information terms $I(U_1; U_2)$ between auxiliary random variables U_0, U_1, U_2. For example, the Fourier–Motzkin elimination gives an inequality

$$H(S_1) < I(U_1, S_1; U_0, Y_1) - I(U_1; S_2|S_1), \tag{16}$$

which is not implied by other inequalities. To further remove inequalities of this type, the following trick can be employed. Take $\tilde{U}_0 = (U_0, W)$ where W is independent of $(S_1, S_2, U_0, U_1, U_2, Y_1, Y_2)$. Then (16) can be written as

$$H(S_1) < I(U_1, S_1; U_0, Y_1) - I(U_1; S_2|S_1) + H(W),$$

while the inequalities that do not involve such mutual information terms are unaffected. Now by taking $H(W) \to \infty$, we can make inequalities like (16) trivial (and thus inactive). It can be shown that all inequalities from the Fourier–Motzkin elimination either are inactive or can be made trivial this way, except for those appearing in the statement of Theorem 3.1. This proves the desired admissibility.

IV. Acknowledgment

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Appendix: Proof of Proposition 3.4

To prove that $\mathscr{S} \subseteq \mathscr{S}_{HC}$, suppose that $(S_1, S_2) \in \mathscr{S}$, so that the inequalities in Theorem 3.1 are satisfied for some triple of random variables (U_0, U_1, U_2). The goal is to show that (S_1, S_2) also satisfy the inequalities in Theorem 2.1 for the same triple of auxiliary random variables. Expanding the left hand side of (5), we obtain that

$$\begin{aligned} &H(S_1, S_2) \\ &< I(U_0, U_1, S_1; Y_1) + I(U_2, S_2; Y_2|U_0) \\ &\quad - I(U_1, S_1; U_2, S_2|U_0), \\ &= H(Y_1) - H(Y_1|U_0, U_1, S_1) - H(S_2, U_2|Y_2, U_0) \\ &\quad + H(S_2, U_2|U_0, U_1, S_1) \\ &\leq H(Y_1) - H(Y_1|K, U_0, U_1, S_1) - H(S_2, U_2|Y_2, K, U_0) \\ &\quad + H(S_2, U_2|K, U_0, U_1, S_1), \\ &= I(K, U_0, U_1, S_1; Y_1) + I(U_2, S_2; Y_2|K, U_0) \end{aligned}$$

$$- I(U_1, S_1; U_2, S_2|K, U_0),$$

where the second inequality follows from the fact that conditioning reduces the entropy and that $K = f(S_1) = g(S_2)$ is a deterministic function of the source (S_1, S_2). Thus, (S_1, S_2) satisfies (2). Proceeding in a similar way, it is immediate to show that (S_1, S_2) satisfies (3). Finally, we expand the left hand side of (7) and obtain that

$$\begin{aligned}
&H(S_1, S_2)\\
&< I(U_0, U_1, S_1; Y_1) + I(U_0, U_2, S_2; Y_2)\\
&\quad - I(U_1, S_1; U_2, S_2|U_0) - I(S_1, S_2; U_0).\\
&= H(Y_1) - H(Y_1|U_0, U_1, S_1) - H(Y_2) - H(Y_2|U_0, U_2, S_2)\\
&\quad - H(U_1|S_1, U_0) + H(U_1, S_1|U_0, U_2, S_2) - H(S_1)\\
&\leq H(Y_1) - H(Y_1|K, U_0, U_1, S_1) - H(Y_2)\\
&\quad - H(Y_2|K, U_0, U_2, S_2) - H(U_1|S_1, K, U_0)\\
&\quad + H(U_1, S_1|K, U_0, U_2, S_2) - H(S_1)\\
&= I(K, U_0, U_1, S_1; Y_1) + I(K, U_0, U_2, S_2; Y_2)\\
&\quad - I(U_1, S_1; U_2, S_2|K, U_0) - I(S_1, S_2; K, U_0),
\end{aligned}$$

so (S_1, S_2) satisfies (4). Thus, we conclude that $(S_1, S_2) \in \mathscr{S}_{HC}$.

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