Relaying via Hybrid Coding

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***Abstract*—Motivated by the recently developed hybrid coding scheme for joint source–channel coding, this paper proposes a new coding scheme for noisy relay networks. The proposed coding scheme operates in a similar manner to the noisy network coding scheme, except that each relay node uses the hybrid coding interface to transmit a symbol-by-symbol function of the received sequence and its quantized version. This coding scheme unifies both amplify–forward and noisy network coding and can strictly outperform both. The potential of the hybrid coding interface for relaying is demonstrated through the diamond relay network and two-way relay channel examples.**

I. Introduction

Over the past decade, analog/digital hybrid coding has been proposed as an alternative to Shannon's source–channel separation architecture [1], [2], for example,

- when the source should be encoded systematically [3],
- when the channel state information is available at the encoder [4], or
- when multiple sources are to be transmitted over a multiuser channel [5], [6].

In most cases, the focus has been on communication of Gaussian sources over an additive Gaussian noise channel under squared error (quadratic) distortion measures, for which separate source and channel coding performs rather poorly.

Motivated by recent developments on hybrid coding [7], [8] for joint source–channel coding over a single-hop network, we explore a new application of hybrid coding in relaying over noisy networks. There have been three dominant relaying paradigms: decode–forward, compress–forward, and amplify–forward.

- In decode–forward [9], the relay recovers the message either fully or partially and forwards it (digital-to-digital interface) while coherently cooperating with the source node. Decode–forward has been generalized to multiple relay networks, for example, in [10], [11] and further improved by combining with structured coding [12], [13].
- In amplify–forward [14], the relay sends a scaled version of its received sequence and forwards it (analog-to-analog interface).
- In compress–forward [9], the relay vector-quantizes its received sequence and forwards it (analog-to-digital interface). Compress–forward has been generalized to arbitrary noisy networks in [15] as noisy network coding, which also extends network coding [16].

In this paper we propose a new coding scheme for a noisy network that uses hybrid coding at relay nodes. The proposed scheme naturally extends both noisy network coding (single-letter performance bound that resembles the cutset bound) and amplify–forward (coherent transmission at the relays). More important than this conceptual unification is the performance improvement of hybrid coding. As will be demonstrated via Gaussian network examples, hybrid coding can strictly outperform the existing coding schemes, not only amplify–forward and noisy network coding, but also decode–forward and its extensions. Moreover, since the coding scheme is developed for a general (not necessarily Gaussian) network, the resulting performance is better than blindly applying the scalar Gaussian hybrid coding interfaces used in [5], [6].

Since the main objective of this paper is to explore the potential of hybrid coding in relay networks, we would rather focus on concrete examples. In particular, we consider the two-relay diamond network in Section III and the two-way relay channel in Section IV, and compare hybrid coding with existing coding schemes. Coding theorems for a general noisy network will be given in a separate occasion [17]. In the next section, we develop the necessary background on hybrid coding. Throughout the paper, we use the notation in [18].

II. Hybrid Coding

Consider the general point-to-point communication system depicted in Fig. 1, where the source $S \sim p(s)$ is to be communicated over the discrete memoryless channel $p(y|x)$. What is the sufficient and necessary condition such that the

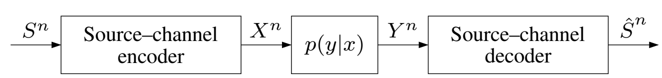

Fig. 1. Point-to-point communication system.

source can be reconstructed with expected distortion satisfying

$$\frac{1}{n}\sum_{i=1}^{n} \mathsf{E}(d(S_i, \hat{S}_i)) \leq D?$$

Shannon [1], [2] showed that a distortion D is achievable if

$$R(D) < C, \tag{1}$$

where $R(D) = \min_{p(\hat{s}|s):\, \mathsf{E}(d(S,\hat{S}))\leq D} I(S;\hat{S})$ is the rate–distortion function for the source S and distortion measure

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$d(s, \hat{s})$ and $C = \max_{p(x)} I(X; Y)$ is the capacity of the channel $p(y|x)$. The proof of this result uses separate source and channel coding, as illustrated in Fig. 2. Under this separate

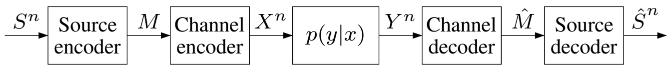

Fig. 2. Separate source and channel coding system architecture.

source and channel coding architecture, the source sequence is mapped into one of 2^{nR} indices M and then this index is mapped into a channel codeword X^n, which is transmitted over the channel. Upon receiving Y^n, the decoder finds an estimate $\hat{M}$ of M and reconstructs $\hat{S}^n$ from $\hat{M}$. The index M provides a digital interface between the source code and the channel code, which can be designed and operated separately. By the lossy source coding theorem and the channel coding theorem, the desired distortion D can be achieved, provided that the index rate R satisfies $R > R(D)$ and $R < C$.

As another extreme point of communication, we can consider uncoded transmission as depicted in Fig. 3. In this simple

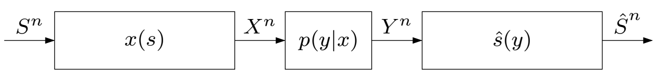

Fig. 3. Uncoded transmission.

analog interface between the source and the channel, the encoder transmits a symbol-by-symbol mapping of the source sequences. Despite its simplicity, this uncoded transmission can be sometimes optimal, for example, when communicating a Gaussian source over a Gaussian channel under a squared error (quadratic) distortion measure [19] or communicating a binary source over a binary symmetric channel under a Hamming distortion measure. In these two cases, the desired distortion D can be achieved if $C \geq R(D)$. The optimality conditions for uncoded transmission is studied by Gastpar, Rimoldi, and Vetterli [20].

Hybrid coding combines separate source and channel coding and uncoded transmission (see Fig. 4). Under this new

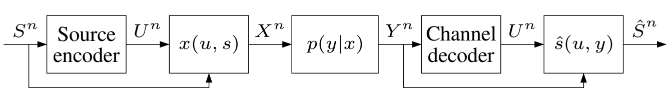

Fig. 4. Hybrid coding.

architecture, the source sequence S^n is mapped to one of 2^{nR} sequences $U^n(M)$ and then this sequence $U^n(M)$ (along with S^n) is mapped to X^n *symbol-by-symbol*, which is transmitted over the channel. Upon receiving Y^n, the decoder finds an estimate $U^n(\hat{M})$ of $U^n(M)$ and reconstructs $\hat{S}^n$ from U^n (and Y^n) again by a *symbol-by-symbol* mapping. Thus, the codeword $U^n(M)$ plays the roles of both the source codeword $\hat{S}^n(M)$ and the channel codeword $X^n(M)$ simultaneously. This dual role of $U^n(M)$ allows simple symbol-by-symbol interfaces $x(u, s)$ and $\hat{s}(u, y)$ that replace the channel encoder and the source decoder in the separation architecture. Moreover, the source encoder and the channel decoder can be operated separately. Roughly speaking, again by the lossy source coding theorem, the condition $R > I(U; S)$ guarantees a reliable source encoding operation and by the channel coding theorem, the condition $R < I(U; Y)$ guarantees a reliable channel decoding operation (over the channel $p(y|u) = \sum_s p(y|x(u, s))p(s)$). Thus, a distortion D is achievable if

$$I(S; U) < I(U; Y) \tag{2}$$

for some pmf $p(u|s)$ and functions $x(u, s)$ and $\hat{s}(u, y)$ such that $\mathsf{E}(d(S, \hat{S})) \leq D$. By taking $U = (X, \hat{S})$ where $\hat{S} \sim p(\hat{s}|s)$, $X \sim p(x)$ is independent of S and $\hat{S}$, and using the memoryless property of the channel, it can be easily shown that this condition simplifies to (1). On the other hand, by taking $U = \emptyset$, hybrid coding reduces to uncoded transmission.

Conceptually speaking, this hybrid coding scheme is as simple as separate source and channel coding. The precise analysis of its performance, however, involves a technical subtlety. In particular, because $U^n(M)$ is used as a source codeword, the index M depends on the entire codebook $\mathcal{C} = \{U^n(M) : M \in [1 : 2^{nR}]\}$. But the conventional random coding proof technique for a channel codeword $U^n(M)$ is developed for situations in which the index M and the (random) codebook $\mathcal{C}$ are independent of each other. For a detailed proof of achievability using hybrid coding with nonstandard analysis that overcomes this difficulty, refer to [7].

In the following, we show how hybrid coding can be used at the relays to achieve higher rates than the existing coding schemes.

III. Diamond Relay Network

As our first example, we consider the diamond relay network $p(y_2, y_3|x_1)p(y_4|x_2, x_3)$ depicted in Fig. 5. Node 1 wishes to send a message $M \in [1 : 2^{nR}]$ to node 4 with the help of the relay nodes 2 and 3.

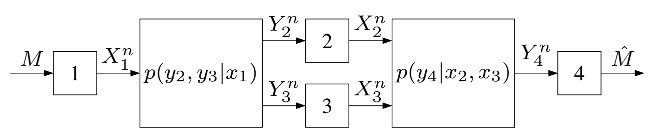

Fig. 5. Diamond network.

We can use hybrid coding at the relay nodes as depicted in Fig. 6. This coding scheme yields the following lower bound on the capacity.

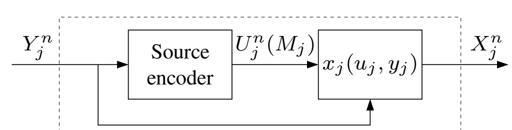

Fig. 6. Hybrid coding interface for relays.

Theorem 1: The capacity of the diamond network $p(y_2, y_3|x_1)p(y_4|x_2, x_3)$ is lower bounded as

$$
\begin{aligned}
C \geq \max\min\{&I(X_1; U_2, U_3, Y_4),\\
&I(X_1, U_2; U_3, Y_4) + I(X_1; U_2) - I(U_2; Y_2),\\
&I(X_1, U_3; U_2, Y_4) + I(X_1; U_3) - I(U_3; Y_3),\\
&I(X_1, U_2, U_3; Y_4) + I(X_1; U_2)\\
&+I(X_1, U_2; U_3) - I(U_2; Y_2) - I(U_3; Y_3)\},
\end{aligned}
$$

where the maximum is over all conditinal pmfs $p(x_1)p(u_2|y_2)p(u_3|y_3)$ and functions $x_2(u_2, y_2)$, $x_3(u_3, y_3)$.

Setting $U_j = \emptyset$ for $j = 2, 3$ recovers the generalized amplify–forward rate $R = \max I(X_1; Y_4)$, where the maximum is over all $p(x_1), x_2(y_2), x_3(y_3)$. On the other hand, if we take $U_j = (X_j, \hat{Y}_j)$ with $p(x_j)p(\hat{y}_j|y_j)$, Theorem 1 reduces to the noisy network coding lower bound

$$
\begin{aligned}
C \geq \max\min\{&I(X_1; \hat{Y}_2, \hat{Y}_3, Y_4|X_2, X_3),\\
&I(X_1, X_2; \hat{Y}_3, Y_4|X_3) - I(Y_2; \hat{Y}_2|X_1, X_2, X_3, \hat{Y}_3, Y_4),\\
&I(X_1, X_3; \hat{Y}_2, Y_4|X_2) - I(Y_3; \hat{Y}_3|X_1, X_2, X_3, \hat{Y}_2, Y_4),\\
&I(X_1, X_2, X_3; Y_4) - I(Y_2, Y_3; \hat{Y}_2, \hat{Y}_3|X_1, X_2, X_3, Y_4)\},
\end{aligned}
$$

where the maximum is over all conditional pmfs $p(x_1)p(x_2)p(x_3)\ p(\hat{y}_2|y_2)p(\hat{y}_3|y_3)$.

As a special case, suppose that the multiple access channel $p(y_2, y_3|x_1)$ and the broadcast channel $p(y_4|x_2, x_3)$ are deterministic. Then, the cutset bound [21] simplifies to

$$C \leq \max_{p(x_1)p(x_2,x_3)} R(Y_2, Y_3, Y_4|X_2, X_3)$$

where $R(Y_2, Y_3, Y_4|X_2, X_3) = \min\{H(Y_2, Y_3), H(Y_2) + H(Y_4|X_3), H(Y_3) + H(Y_4|X_2), H(Y_4)\}$. On the other hand, the general lower bound by Avestimehr, Diggavi, and Tse [22] simplifies to

$$C \geq \max_{p(x_1)p(x_2)p(x_3)} R(Y_2, Y_3, Y_4|X_2, X_3).$$

Note that the cutset bound and the general lower bound differ only in maximizing input pmfs. Now it can be easily shown that the hybrid coding lower bound in Theorem 1 simplifies to the expression of the same form

$$C \geq \max_{p(x_1)p(x_2|y_2)p(x_3|y_3)} R(Y_2, Y_3, Y_4|X_2, X_3),$$

which is between the cutset bound and the general lower bound in general. The following example demonstrates that the inclusion can be strict.

Example 1: Suppose that $p(y_2, y_3|x_1)$ is the Blackwell broadcast channel (i.e., $X_1 \in \{0, 1, 2\}$ and $p_{Y_2,Y_3|X_1}(0, 0|0) = p_{Y_2,Y_3|X_1}(0, 1|1) = p_{Y_2,Y_3|X_1}(1, 1|2) = 1$) and $p(y_4|x_2, x_3)$ is the binary erasure multiple access channel (i.e., $X_2, X_3 \in \{0, 1\}$ and $Y_4 = X_2 + X_3 \in \{0, 1, 2\}$). It can be easily seen that the general lower bound reduces to $C \geq 1.5$, while the capacity is $C = \log 3$, which coincides with the hybrid coding lower bound (with $X_2 = Y_2$ and $X_3 = Y_3$). Thus, hybrid coding strictly outperforms the coding scheme by Avestimehr, Diggavi, and Tse [22] (and noisy network coding [15]).

In the above example, the capacity is achieved by the extreme case of hybrid coding, namely, amplify–forward. In general, hybrid coding can strictly outperform amplify–forward as well, as demonstrated in the following.

Example 2: Consider the Gaussian diamond channel

$$
\begin{aligned}
Y_2 &= g_{21}X_1 + Z_2,\\
Y_3 &= g_{31}X_1 + Z_3,\\
Y_4 &= g_{42}X_2 + g_{43}X_3 + Z_4,
\end{aligned}
$$

where the noise components Z_k, $k = 2, 3, 4$, are i.i.d. $\mathrm{N}(0, 1)$. We assume power constrain P on each sender and denote the SNR for the signal from node k to node j as $S_{jk} = g_{jk}^2 P$.

Let $X_1 \sim \mathrm{N}(0, P)$. For $j = 2, 3$, let $\hat{Y}_j = Y_j + \hat{Z}_j$ and $U_j = (V_j, \hat{Y}_j)$, where $V_j \sim \mathrm{N}(0, 1)$ and $\hat{Z}_j \sim \mathrm{N}(0, \sigma_j^2)$ are independent of each other and of (X_1, Y_2, Y_3). For $\alpha_j, \beta_j \in [0, 1]$ such that $\alpha_j + \beta_j \leq 1$, let $X_j = \sqrt{\frac{\alpha_j P}{S_{1j}+1}}Y_j + \sqrt{\frac{\beta_j P}{\sigma_j^2}}\hat{Z}_j + \sqrt{(1-\alpha_j-\beta_j)P}V_j$ so that $X_j \sim \mathrm{N}(0, P)$. Then under this choice of the conditional distribution, the hybrid coding lower bound in Theorem 1 simplifies as

$$
\begin{aligned}
C \geq \min\Big\{&\frac{1}{2}\log\frac{N_1}{D} + \mathsf{C}\Big(\frac{S_{21}(1+\sigma_3^2) + S_{31}(1+\sigma_2^2)}{(1+\sigma_2^2)(1+\sigma_3^2)}\Big),\\
&\frac{1}{2}\log\frac{N_2}{D} + \mathsf{C}\Big(\frac{S_{31}}{1+\sigma_3^2}\Big) - \mathsf{C}\Big(\frac{1}{\sigma_2^2}\Big),\\
&\frac{1}{2}\log\frac{N_3}{D} + \mathsf{C}\Big(\frac{S_{21}}{1+\sigma_2^2}\Big) - \mathsf{C}\Big(\frac{1}{\sigma_3^2}\Big),\\
&\frac{1}{2}\log\frac{N_4}{D} - \mathsf{C}\Big(\frac{1}{\sigma_2^2}\Big) - \mathsf{C}\Big(\frac{1}{\sigma_3^2}\Big)\Big\},
\end{aligned}
\tag{3}
$$

where $N_1 = K_{\tilde{Y}_4} - K_{\tilde{Y}_4|\hat{Y}_2,\hat{Y}_3}K^{-1}_{\hat{Y}_2,\hat{Y}_3}K'_{\tilde{Y}_4|\hat{Y}_2,\hat{Y}_3}$ with

$$
\begin{aligned}
K_{\tilde{Y}_4|\hat{Y}_2,\hat{Y}_3} = \Big[&\sqrt{\alpha_2 S_{42}(S_{21}+1)} + \sqrt{\beta_2 S_{42}\sigma_2^2}\\
&+\sqrt{\frac{\alpha_3 S_{43}S_{21}S_{31}}{S_{31}+1}}, \sqrt{\alpha_3 S_{43}(S_{31}+1)}\\
&+\sqrt{\beta_3 S_{43}\sigma_3^2} + \sqrt{\frac{\alpha_2 S_{42}S_{21}S_{31}}{S_{21}+1}}\Big],\\
K_{\hat{Y}_2,\hat{Y}_3} = &\begin{bmatrix} S_{21}+\sigma_2^2+1 & \sqrt{S_{21}S_{31}}\\ \sqrt{S_{21}S_{31}} & S_{31}+\sigma_3^2+1\end{bmatrix},\\
K_{\tilde{Y}_4} = &\,1 + (\alpha_2+\beta_2)S_{42} + (\alpha_3+\beta_3)S_{43}\\
&+2\sqrt{\frac{\alpha_2\alpha_3 S_{43}S_{42}S_{21}S_{31}}{(S_{31}+1)(S_{21}+1)}},\\
N_2 = &\,1 + S_{42}\left(1 - \frac{\alpha_2 S_{21}S_{31}}{(S_{21}+1)(S_{31}+1+\sigma_3^2)}\right)\\
&+\frac{S_{43}}{S_{31}+\sigma_3^2+1}\Big(\sqrt{\alpha_3\sigma_3^2} - \sqrt{\beta_3(S_{31}+1)}\Big)^2\\
&+\frac{2}{S_{31}+\sigma_3^2+1}\sqrt{\frac{\alpha_2 S_{42}S_{43}S_{21}S_{31}\sigma_3^2}{(S_{21}+1)(S_{31}+1)}}\\
&\times\Big(\sqrt{\alpha_3\sigma_3^2} - \sqrt{\beta_3(S_{31}+1)}\Big),
\end{aligned}
$$

N_3 is defined as N_2 with subscript '2' replaced by '3',

$$N_4 = 1 + S_{42} + S_{43} + 2\sqrt{\frac{\alpha_2\alpha_3 S_{21}S_{31}S_{42}S_{43}}{(S_{21}+1)(S_{31}+1)}},$$
$$D = \frac{S_{42}}{\sigma_2^2+1}\Big(\sqrt{\frac{\sigma_2^2\alpha_2}{S_{21}+1}} - \sqrt{\beta_2}\Big)^2 + \frac{S_{43}}{\sigma_3^2+1}\Big(\sqrt{\frac{\sigma_3^2\alpha_3}{S_{31}+1}} - \sqrt{\beta_3}\Big)^2 + 1.$$

for some $\alpha_j, \beta_j \in [0,1]$ such that $\alpha_j + \beta_j \le 1$ and $\sigma_j^2 > 0$, $j = 1, 2$. If we let $\alpha_j = \beta_j = 0$, $j = 1, 2$, then this bound reduces to the noisy network coding lower bound

$$\begin{aligned} C \le \min\Big\{ &\mathsf{C}\Big(\frac{S_{21}(1+\sigma_3^2) + S_{31}(1+\sigma_2^2)}{(1+\sigma_2^2)(1+\sigma_3^2)}\Big), \\ &\mathsf{C}(S_{21}) + \mathsf{C}\Big(\frac{S_{31}}{1+\sigma_3^2}\Big) - \mathsf{C}\Big(\frac{1}{\sigma_2^2}\Big), \\ &\mathsf{C}(S_{31}) + \mathsf{C}\Big(\frac{S_{21}}{1+\sigma_2^2}\Big) - \mathsf{C}\Big(\frac{1}{\sigma_3^2}\Big), \\ &\mathsf{C}(S_{42}+S_{43}) - \mathsf{C}\Big(\frac{1}{\sigma_2^2}\Big) - \mathsf{C}\Big(\frac{1}{\sigma_3^2}\Big)\Big\}, \end{aligned}$$

for some $\sigma_j^2 > 0$, $j = 1, 2$. On the other hand, if we let $\alpha_j = 1$, $\beta_j = 0$, and $\sigma_j^2 \to \infty$ for $j = 1, 2$, then the hybrid coding lower bound reduces to the amplify–forward lower bound that consists of all rates R such that

$$C \ge \mathsf{C}\left(\frac{\big(\sqrt{S_{21}S_{42}(S_{31}+1)} + \sqrt{S_{31}S_{43}(S_{21}+1)}\big)^2}{S_{42}(S_{31}+1) + S_{43}(S_{21}+1) + (S_{21}+1)(S_{31}+1)}\right).$$

Note that our choice of U_j is richer than traditional Gaussian hybrid coding [5], [6]. This follows since our coding scheme is developed for a general discrete memoryless network.

Fig. 7 compares the amplify–forward, noisy network coding, and hybrid coding lower bounds on the capacity (with optimized parameters) when $g_{21} = g_{43} = g$, $g_{31} = g_{42} = 1$, and $P = 1$. Hybrid coding strictly outperforms both amplify–forward and noisy network coding for this choice of parameters. Note that hybrid coding also outperforms decode–forward, which achieves $R_{\rm DF} = \mathsf{C}(S_{31}) = 1/2$ (not shown).

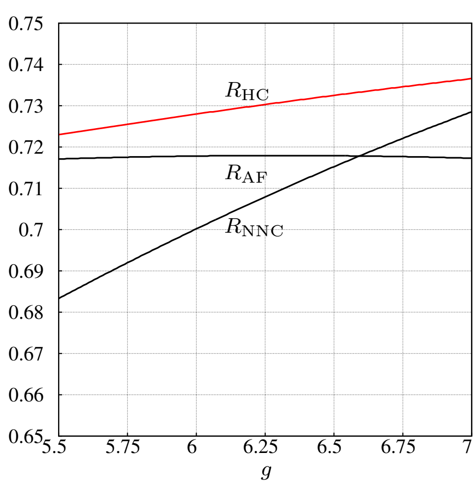

Fig. 7. Comparison of the amplify–forward lower bound $R_{\rm AF}$, noisy network coding lower bound $R_{\rm NNC}$, and hybrid coding lower bound $R_{\rm HC}$ on the capacity for the Gaussian diamond network as a function of the channel gain $g = g_{21} = g_{43}$, when $g_{31} = g_{42} = 1$ and $P = 1$.

IV. Two-Way Relay Channel

As another example to illustrate the hybrid coding scheme, we consider the discrete memoryless two-way relay channel (DM-TWRC) without direct links $p(y_3|x_1,x_2)p(y_1,y_2|x_3)$. Node 1 wishes to send message $M_1 \in [1:2^{nR_1}]$ to node 2 and node 2 wishes to send message $M_2 \in [1:2^{nR_2}]$ to node 1 with the help of the relay node 3.

Hybrid coding at the relay node 3 yields the inner bound on the capacity region that consists of all rate pairs (R_1, R_2) such that

$$\begin{aligned} R_1 &< I(X_1; Y_2, U_3 | X_2), \\ R_1 &< I(X_1, U_3; X_2, Y_2) + I(X_1; U_3) - I(Y_3; U_3), \\ R_2 &< I(X_2; Y_1, U_3 | X_1), \\ R_2 &< I(X_2, U_3; X_1, Y_1) + I(X_2; U_3) - I(Y_3; U_3) \end{aligned} \tag{4}$$

for some pmf $p(x_1)p(x_2)p(u_3|y_3)$ and function $x_3(u_3, y_3)$. By setting $U_3 = (\hat{Y}_3, X_3)$ for $p(\hat{y}_3, x_3|y_3) = p(\hat{y}_3|y_3)p(x_3)$, this inner bound reduces to the noisy network coding inner bound consisting of all rate pairs (R_1, R_2) such that

$$\begin{aligned} R_1 &< I(X_1; Y_2, \hat{Y}_3 | X_2, X_3), \\ R_1 &< I(X_3; Y_2) - I(Y_3; \hat{Y}_3 | X_1, X_2), \\ R_2 &< I(X_2; Y_1, \hat{Y}_3 | X_1, X_3), \\ R_2 &< I(X_3; Y_1) - I(Y_3; \hat{Y}_3 | X_1, X_2) \end{aligned}$$

for some $p(x_1)p(x_2)p(u_3|y_3)$ and $x_3(u_3, y_3)$.

Example 3: Now consider the Gaussian TWRC

$$\begin{aligned} Y_1 &= g_{13}X_3 + Z_1, \\ Y_2 &= g_{23}X_3 + Z_2, \\ Y_3 &= g_{31}X_1 + g_{32}X_2 + Z_3, \end{aligned}$$

where the noise components Z_k, $k = 1, 2, 3$, are i.i.d. $\mathrm{N}(0, 1)$ noise. We assume expected power constraint P on each sender. Denote the received SNR $S_{jk} = g_{jk}^2 P$.

Let $X_1 \sim \mathrm{N}(0, P)$ and $X_2 \sim \mathrm{N}(0, P)$ be independent of each other. Let $\hat{Y}_3 = Y_3 + \hat{Z}_3$ and $U_3 = (V_3, \hat{Y}_3)$, where $V_3 \sim \mathrm{N}(0, 1)$ and $\hat{Z}_3 \sim \mathrm{N}(0, \sigma^2)$ are independent of each other and of (X_1, X_2, Y_3). For $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \le 1$, let

$$X_3 = \sqrt{\frac{\alpha P}{S_{31} + S_{32} + 1}}\, Y_3 + \sqrt{\frac{\beta P}{\sigma^2}}\, \hat{Z}_3 + \sqrt{(1-\alpha-\beta)P}\, V_3,$$

so that $X_3 \sim \mathrm{N}(0, P)$. Then under this choice of the conditional distribution, the hybrid coding inner bound in (4) simplifies to the set of all rate pairs (R_1, R_2) such that the inequalities in (5) are satisfied for some $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \le 1$ and $\sigma^2 > 0$. It can be easily shown that, if we let $\alpha = \beta = 0$, then the hybrid coding inner bound reduces to the noisy network coding inner bound [15]. On the other hand, if we let $\alpha = 1$, $\beta = 0$, and $\sigma^2 \to \infty$, then the hybrid coding inner bound reduces to the amplify–forward inner bound [23].

Fig. 8 compares the cutset bound to the decode–forward, amplify–forward, noisy network coding, and hybrid coding bounds on the sum-capacity (with optimized parameters). Here nodes 1 and 2 are unit distance apart and node 3 is at distance

$$
\begin{aligned}
R_1 &< \frac{1}{2}\log \frac{\left(\frac{\alpha S_{23}(S_{31}+1)}{S_{31}+S_{32}+1} + \beta S_{23} + 1\right)(S_{31}+1+\sigma^2) - S_{23}\left(\sqrt{\frac{\alpha(S_{31}+1)}{S_{31}+S_{32}+1}} + \sqrt{\beta\sigma^2}\right)^2}{\left(\frac{\alpha S_{23}}{S_{31}+S_{32}+1} + \beta S_{23} + 1\right)(1+\sigma^2) - S_{23}\left(\sqrt{\frac{\alpha}{S_{31}+S_{32}+1}} + \sqrt{\beta\sigma^2}\right)^2},\\
R_1 &< \frac{1}{2}\log \frac{\left(\frac{\alpha S_{23}(S_{31}+1)}{S_{31}+S_{32}+1} + (1-\alpha) S_{23} + 1\right)(1+\sigma^2)}{\left(\frac{\alpha S_{23}}{S_{31}+S_{32}+1} + \beta S_{23} + 1\right)(1+\sigma^2) - S_{23}\left(\sqrt{\frac{\alpha}{S_{31}+S_{32}+1}} + \sqrt{\beta\sigma^2}\right)^2} - \mathsf{C}(1/\sigma^2),\\
R_2 &< \frac{1}{2}\log \frac{\left(\frac{\alpha S_{13}(S_{32}+1)}{S_{31}+S_{32}+1} + \beta S_{13} + 1\right)(S_{32}+1+\sigma^2) - S_{13}\left(\sqrt{\frac{\alpha(S_{32}+1)}{S_{31}+S_{32}+1}} + \sqrt{\beta\sigma^2}\right)^2}{\left(\frac{\alpha S_{13}}{S_{31}+S_{32}+1} + \beta S_{13} + 1\right)(1+\sigma^2) - S_{13}\left(\sqrt{\frac{\alpha}{S_{31}+S_{32}+1}} + \sqrt{\beta\sigma^2}\right)^2},\\
R_2 &< \frac{1}{2}\log \frac{\left(\frac{\alpha S_{13}(S_{32}+1)}{S_{31}+S_{32}+1} + (1-\alpha) S_{13} + 1\right)(1+\sigma^2)}{\left(\frac{\alpha S_{13}}{S_{31}+S_{32}+1} + \beta S_{13} + 1\right)(1+\sigma^2) - S_{13}\left(\sqrt{\frac{\alpha}{S_{31}+S_{32}+1}} + \sqrt{\beta\sigma^2}\right)^2} - \mathsf{C}(1/\sigma^2)
\end{aligned}
\tag{5}
$$

$r \in [0,1]$ from node 1 along the line between nodes 1 and 2; the channel gain are of the form $g_{jk} = r_{jk}^{-3/2}$, where r_{jk} is the distance between nodes j and k, hence $g_{13} = g_{31} = r^{-3/2}$, $g_{23} = g_{32} = (1-r)^{-3/2}$, and the power $P = 10$. Note that hybrid coding strictly outperforms amplify–forward and noisy network coding for every r. This result is surprising since unlike the diamond network example, there is no coherence gain for hybrid coding. Here the gain is due to the fact that hybrid coding provides differentiated information to separate receivers. Hybrid coding also outperforms decode–forward when the relay is sufficiently far from both destination nodes.

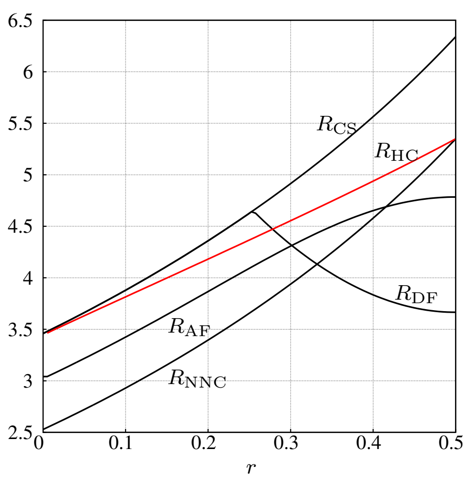

Fig. 8. Comparison of the cutset bound R_{CS}, decode–forward lower bound R_{DF}, amplify–forward lower bound R_{AF}, noisy network coding lower bound R_{NNC}, and hybrid coding lower bound R_{HC} on the sum-capacity for the Gaussian TWRC.

REFERENCES

[1] C. E. Shannon, "A mathematical theory of communication," *Bell System Tech. J.*, vol. 27, pp. 379–423, 623–656, 1948.

[2] ——, "Coding theorems for a discrete source with a fidelity criterion," in *IRE Int. Conv. Rec., part 4*, 1959, vol. 7, pp. 142–163, reprinted with changes in *Information and Decision Processes,* R. E. Machol, Ed. New York: McGraw-Hill, 1960, pp. 93-126.

[3] S. Shamai, S. Verdu, and R. Zamir, "Systematic lossy source/channel coding," *IEEE Trans. Inf. Theory*, vol. 44, no. 2, pp. 564–579, 1998.

[4] M. P. Wilson, K. Narayanan, and G. Caire, "Joint source channel coding with side information using hybrid digital analog codes," *IEEE Trans. Inf. Theory*, vol. 56, no. 10, pp. 4922–4940, Oct. 2010.

[5] A. Lapidoth and S. Tinguely, "Sending a bivariate Gaussian over a Gaussian MAC," *IEEE Trans. Inf. Theory*, vol. 56, no. 6, pp. 2714–2752, 2010.

[6] C. Tian, S. Diggavi, and S. Shamai, "The achievable distortion region of bivariate Gaussian source on Gaussian broadcast channel," 2010, submitted to *IEEE Trans. Inf. Theory,* 2010.

[7] S. H. Lim, P. Minero, and Y.-H. Kim, "Lossy communication of correlated sources over multiple access channels," in *Proc. 48th Annual Allerton Conference on Communications, Control, and Computing*, Monticello, IL, Sep. 2010.

[8] P. Minero, S. H. Lim, and Y.-H. Kim, "Hybrid coding: A simple source–channel interface for network communication," in *Proc. ITA Workshop*, La Jolla, CA, Feb. 2011.

[9] T. M. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inf. Theory*, vol. 25, no. 5, pp. 572–584, Sep. 1979.

[10] M. R. Aref, "Information flow in relay networks," Ph.D. Thesis, Stanford University, Stanford, CA, Oct. 1980.

[11] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3037–3063, Sep. 2005.

[12] B. Nazer and M. Gastpar, "Computation over multiple-access channels," *IEEE Trans. Inf. Theory*, vol. 53, no. 10, pp. 3498–3516, Oct. 2007.

[13] W. Nam, S.-Y. Chung, and Y. H. Lee, "Capacity of the Gaussian two-way relay channel to within 1/2 bit," *IEEE Trans. Inf. Theory*, vol. 56, no. 11, pp. 5488–5494, Nov. 2010.

[14] B. Schein and R. Gallager, "The Gaussian parallel relay channel," in *Proc. IEEE International Symposium on Information Theory*, Sorrento, Italy, June 2000, p. 22.

[15] S. H. Lim, Y.-H. Kim, A. El Gamal, and S.-Y. Chung, "Noisy network coding," 2011, to appear in *IEEE Trans. Inf. Theory,* Apr. 2011.

[16] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inf. Theory*, vol. 46, pp. 1204–1216, 2000.

[17] Y.-H. Kim, S. H. Lim, and P. Minero, "Hybrid coding," *in preparation*.

[18] A. El Gamal and Y.-H. Kim, "Lecture notes on network information theory," 2010. [Online]. Available: http://arxiv.org/abs/1001.3404

[19] T. J. Goblick, "Theoretical limitations on the transmission of data from analog sources," *IEEE Trans. Inf. Theory*, vol. 11, pp. 558–567, 1965.

[20] M. Gastpar, B. Rimoldi, and M. Vetterli, "To code, or not to code: lossy source-channel communication revisited," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1147–1158, 2003.

[21] A. El Gamal, "On information flow in relay networks," in *Proc. IEEE National Telecom Conference*, Nov. 1981, vol. 2, pp. D4.1.1–D4.1.4.

[22] S. Avestimehr, S. Diggavi, and D. Tse, "Wireless network information flow," 2007, submitted to *IEEE Trans. Inf. Theory,* 2007. [Online]. Available: http://arxiv.org/abs/0710.3781/

[23] B. Rankov and A. Wittneben, "Achievable rate region for the two-way relay channel," in *Proc. IEEE Int. Symp. Inf. Theory*, Seattle, WA, 2006.