

# Approximate Capacity of the MIMO Relay Channel

Xianglan Jin

Department of Electrical and Computer Engineering  
Pusan National University  
Busan 609-735, Republic of Korea

Young-Han Kim

Department of Electrical and Computer Engineering  
University of California, San Diego  
La Jolla, CA 92093-0407

**Abstract**—The capacity bounds are studied for the multiple-antenna real full-duplex Gaussian relay channel with  $t_1$  transmitting antennas at the sender,  $r_2$  receiving and  $t_2$  transmitting antennas at the relay, and  $r_3$  receiving antennas at the receiver. It is shown that compress-forward and partial decode-forward achieve within  $(1/2)(\min(t_1 + t_2, r_3) + r_2)$  bits and  $(1/2)\min(t_1, r_2)$  bits, respectively, from the cutset bound. Unlike the single-antenna case, partial decode-forward can be arbitrarily better than optimal selection between decode-forward and direct transmission. Similar gap results for half-duplex models are briefly discussed.

## I. INTRODUCTION

The relay channel, whereby point-to-point communication between a sender and a receiver is aided by a relay, is an important building block for cooperative wireless communication. Introduced by van der Meulen [1], this channel model has been studied extensively in the literature, including the now classical paper by Cover and El Gamal [2]. Nonetheless, even for the most basic Gaussian relay channel, the problem of characterizing the capacity in a computable form remains open under any nondegenerate channel gain and power constraint. Consequently, a large body of the literature has been devoted to the study of bounds on the capacity. Reminiscent of the max-flow min-cut theorem [3], the cutset bound was developed by Cover and El Gamal [2] that sets an upper bound on the capacity. There are myriads of coding schemes [4] and corresponding lower bounds on the capacity.

The main focus of this paper is on two canonical coding schemes, compress-forward [2, Th. 6] and partial decode-forward [2, Th. 7] for Gaussian relay channels; see also [5, Ch. 16] for the detailed descriptions of the coding schemes. For the single-antenna full-duplex real Gaussian relay channel, compress-forward and partial decode-forward, respectively, achieve within half a bit from the cutset bound [6], [7], providing a half-bit approximation of the capacity. Moreover, partial decode-forward, which is superposition of decode-forward and direct transmission, reduces to the better of the two [8].

Paralleling these results for the single-antenna model, we study the performance of compress-forward and partial decode-forward for multiple-antenna (also known as multiple-input multiple-output or MIMO) Gaussian relay channels. Capacity bounds for MIMO relay channels have been studied in numerous papers. For example, by convex programming techniques, Wang, Zhang, and Høst-Madsen [9] derived upper and lower bounds based on looser versions of cutset bound

and decode-forward. These results have been improved by more advanced coding schemes (compress-forward and partial decode-forward) often with suboptimal decoding rules [10], [11], [12]. The usual focus of this line of work, however, has been on the optimization of resources (power and bandwidth) for practical implementations and on numerical computation of resulting capacity bound (see also [13]). Consequently, no clean capacity approximation result has been obtained in the literature. One notable exception is a recent result by Kolte, Özgür, and El Gamal [14] on a general MIMO relay network, which shows that noisy network coding [15] achieves the cutset bound within an additive constant that depends on the network topology and the numbers of antennas.

In this paper, we further refine the result in [14] and other bounds on the capacity of the MIMO relay channel. Our findings are summarized as follows.

- For the full-duplex real Gaussian relay channel with  $t_1$  transmitting antenna at the sender,  $r_2$  receiving and  $t_2$  transmitting antennas at the relay, and  $r_3$  receiving antennas at the receiver, we show that compress-forward achieves  $(1/2)(\min(t_1 + t_2, r_3) + r_2)$  bits within the cutset bound.
- For the same model, we show that partial decode-forward achieves within  $(1/2)\min(t_1, r_2)$  bits of the cutset bound. Incidentally, unlike the single-antenna counterpart, partial decode-forward can achieve rates arbitrarily higher than the better of decode-forward and direct transmission.

We can also establish similar capacity approximation results for half-duplex models, sender frequency-division Gaussian relay channels and receiver frequency-division Gaussian relay channels, which will be presented in Section VI without proofs. Note that these results can be extended to complex channel models in a straightforward manner by doubling the gaps.

In the next section, we formally define the channel model, review the cutset upper bound, the compress-forward lower bound, and the partial decode-forward lower bound on the capacity, and present the main results. This will be followed by the the proofs of the propositions in two subsequent sections. The paper concludes with numerical simulations and a discussion on analogous gap results for half-duplex MIMO relay channels.

## II. PROBLEM SETUP AND MAIN RESULTS

We model the MIMO relay communication system as a (real) Gaussian vector relay channel (GV-RC) with sender node 1, relay node 2, and receiver node 3; see Fig. 1. The relay and the receiver have  $r_2$  and  $r_3$  receiving antennas with respective channel outputs

$$\begin{aligned} \mathbf{Y}_2 &= G_{21}\mathbf{X}_1 + \mathbf{Z}_2, \\ \mathbf{Y}_3 &= G_{31}\mathbf{X}_1 + G_{32}\mathbf{X}_2 + \mathbf{Z}_3, \end{aligned} \quad (1)$$

where  $G_{21}$ ,  $G_{31}$ , and  $G_{32}$  are channel gain matrices, and  $\mathbf{Z}_2 \sim \mathcal{N}(0, I_{r_2})$  and  $\mathbf{Z}_3 \sim \mathcal{N}(0, I_{r_3})$  are independent noise components. For simplicity, we will often use the shorthand notation

$$G_{3*} = [G_{31} \ G_{32}] \quad \text{and} \quad G_{*1} = \begin{bmatrix} G_{21} \\ G_{31} \end{bmatrix}.$$

We assume that the sender and the relay have  $t_1$  and  $t_2$  transmitting antennas, respectively, with average power constraint  $P$ . As in the standard relay channel model [2], the encoder is defined by  $\mathbf{X}_1^n(m)$ , the relay encoder is defined by  $\mathbf{X}_2^i(\mathbf{Y}_2^{i-1})$ , and the decoder is defined by  $\hat{m}(\mathbf{Y}_3^n)$ . We follow the standard definitions for the rate of a code, achievability of a given rate, and the capacity of the relay channel.

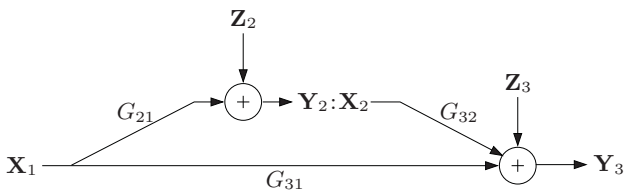


Fig. 1. Gaussian vector relay channel.

The following upper bound on the capacity is well known.

*Proposition 1 (Cutset bound [2, Th. 4]):* The capacity  $C$  of the GV-RC is upper bounded by

$$\begin{aligned} R_{\text{CS}} &= \sup_{F(\mathbf{x}_1, \mathbf{x}_2)} \min \{ I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3), I(\mathbf{X}_1; \mathbf{Y}_2, \mathbf{Y}_3 | \mathbf{X}_2) \} \\ &= \max_K \min \left\{ \frac{1}{2} \log |I_{r_3} + G_{3*} K G_{3*}^T|, \right. \\ &\quad \left. \frac{1}{2} \log |I_{r_2+r_3} + G_{*1} K_{1|2} G_{*1}^T| \right\} \\ &= \max_K \min \left\{ \frac{1}{2} \log |I_{r_3} + [G_{31} \ G_{32}] K [G_{31} \ G_{32}]^T|, \right. \\ &\quad \left. \frac{1}{2} \log |I_{t_1} + (G_{21}^T G_{21} + G_{31}^T G_{31}) K_{1|2}| \right\}, \end{aligned} \quad (2)$$

where the supremum is over all joint distributions  $F(\mathbf{x}_1, \mathbf{x}_2)$  such that  $\mathbf{E}(\mathbf{X}_j^T \mathbf{X}_j) \leq P$ ,  $j = 1, 2$ , the maximum is over all  $(t_1 + t_2) \times (t_1 + t_2)$  matrices

$$K = \begin{bmatrix} K_1 & K_{12} \\ K_{12}^T & K_2 \end{bmatrix} \succeq 0 \quad (3)$$

with  $t_j \times t_k$  submatrix components  $K_{jk}$  such that  $\text{tr}(K_j) \leq P$ ,  $j = 1, 2$  and  $K_{1|2} = K_1 - K_{12} K_2^{-1} K_{12}^T$ .

The last equality in (2) is justified by the following simple fact, which will be used repeatedly throughout the paper: For  $\gamma \in (0, 1]$ ,  $r \times t$  matrix  $G$ , and  $t \times t$  matrix  $K \succeq 0$ ,

$$|I_r + \gamma G K G^T| = |I_t + \gamma G^T G K| \geq \gamma^{\min(t, r)} |I_r + G K G^T|. \quad (4)$$

We would like to compare the cutset bound with the following two lower bounds on the capacity.

*Proposition 2 (Compress-forward bound [2, Th. 6]):* The capacity  $C$  of the GV-RC is lower bounded by

$$R_{\text{CF}} = \sup \min \{ I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3) - I(\mathbf{Y}_2; \hat{\mathbf{Y}}_2 | \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_3), I(\mathbf{X}_1; \hat{\mathbf{Y}}_2, \mathbf{Y}_3 | \mathbf{X}_2) \}, \quad (5)$$

where the supremum is over all conditional distributions  $F(\mathbf{x}_1)F(\mathbf{x}_2)F(\hat{\mathbf{y}}_2 | \mathbf{y}_2, \mathbf{x}_2)$  such that  $\mathbf{E}(\mathbf{X}_j^T \mathbf{X}_j) \leq P$ ,  $j = 1, 2$ .

*Proposition 3 (Partial decode-forward bound [2, Th. 7]):* The capacity  $C$  of the GV-RC is lower bounded by

$$R_{\text{PDF}} = \sup \min \{ I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3), I(\mathbf{U}; \mathbf{Y}_2 | \mathbf{X}_2) + I(\mathbf{X}_1; \mathbf{Y}_3 | \mathbf{X}_2, \mathbf{U}) \}, \quad (6)$$

where the supremum is over all joint distributions  $F(\mathbf{u}, \mathbf{x}_1, \mathbf{x}_2)$  such that  $\mathbf{E}(\mathbf{X}_j^T \mathbf{X}_j) \leq P$ ,  $j = 1, 2$ .

The partial decode-forward lower bound can be relaxed in several directions. First, by limiting the input distribution to a more practical product form, we obtain the *noncoherent* partial decode-forward lower bound:

$$R_{\text{NPDF}} = \sup \min \{ I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3), I(\mathbf{U}; \mathbf{Y}_2 | \mathbf{X}_2) + I(\mathbf{X}_1; \mathbf{Y}_3 | \mathbf{X}_2, \mathbf{U}) \}, \quad (7)$$

where the supremum is over all *product* distributions  $F(\mathbf{u}, \mathbf{x}_1)F(\mathbf{x}_2)$  such that  $\mathbf{E}(\mathbf{X}_j^T \mathbf{X}_j) \leq P$ ,  $j = 1, 2$ . Second, by setting  $\mathbf{U} = \mathbf{X}_1$ , we obtain the decode-forward lower bound:

$$\begin{aligned} R_{\text{DF}} &= \sup \min \{ I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3), I(\mathbf{X}_1; \mathbf{Y}_2 | \mathbf{X}_2) \} \\ &= \max_K \min \left\{ \frac{1}{2} \log |I_{r_3} + G_{3*} K G_{3*}^T|, \right. \\ &\quad \left. \frac{1}{2} \log |I_{r_2} + G_{21} K_{1|2} G_{21}^T| \right\}, \end{aligned} \quad (8)$$

where the supremum is over all distributions  $F(\mathbf{x}_1, \mathbf{x}_2)$  such that  $\mathbf{E}(\mathbf{X}_j^T \mathbf{X}_j) \leq P$ ,  $j = 1, 2$ , and the maximum is over all  $(t_1 + t_2) \times (t_1 + t_2)$  matrices  $K \succeq 0$  of the form (3) such that  $\text{tr}(K_j) \leq P$ ,  $j = 1, 2$ . Third, by setting  $\mathbf{U} = \emptyset$  and  $\mathbf{X}_2 = 0$ , we obtain the direct-transmission lower bound:

$$\begin{aligned} R_{\text{DT}} &= \sup I(\mathbf{X}_1; \mathbf{Y}_3) \\ &= \max_{K_1} \frac{1}{2} \log |I_{r_3} + G_{31} K_1 G_{31}^T|, \end{aligned} \quad (9)$$

where the supremum is over all distributions  $F(\mathbf{x}_1)$  such that  $\mathbf{E}(\mathbf{X}_1^T \mathbf{X}_1) \leq P$  and the maximum is over all  $t_1 \times t_1$  matrices  $K_1$ .

We are now ready to state the main results of the paper.

*Theorem 1:* For every  $G_{21}, G_{31}, G_{32}$ , and  $P$ ,

$$\begin{aligned} \Delta_{\text{CF}} &= R_{\text{CS}} - R_{\text{CF}} \\ &\leq \min_{\sigma^2} \frac{1}{2} \max[\min(t_1 + t_2, r_3) + r_2 \log(1 + 1/\sigma^2), \\ &\quad \min(t_1, r_2 + r_3) \log(1 + \sigma^2)] \end{aligned} \quad (10)$$

$$\leq \frac{1}{2} (\min(t_1 + t_2, r_3) + r_2). \quad (11)$$

This result improves upon a recent result by Kolte, Özgür, and El Gamal [14, Th. 1] on the performance of noisy network coding for general relay networks, when the latter is specialized to the 3-node relay channel.

*Theorem 2:* For every  $G_{21}, G_{31}, G_{32}$ , and  $P$ ,

$$\Delta_{\text{PDF}} = R_{\text{CS}} - R_{\text{PDF}} \leq \frac{1}{2} \min(t_1, r_2). \quad (12)$$

This result can be relaxed by using noncoherent partial decode-forward.

*Proposition 4:* For every  $G_{21}, G_{31}, G_{32}$ , and  $P$ ,

$$\begin{aligned} \Delta_{\text{NPDF}} &= R_{\text{CS}} - R_{\text{NPDF}} \\ &\leq \frac{1}{2} \max[\min(t_1, r_2), \min(t_1 + t_2, r_3)]. \end{aligned} \quad (13)$$

Unlike the single-antenna case [8] in which partial decode-forward is equivalent to the better of decode-forward and direct transmission, however, partial decode-forward for multiple antennas is much richer than decode-forward and direct transmission.

*Proposition 5:*

$$\sup_{G_{21}, G_{31}, G_{32}, P} [R_{\text{PDF}} - \max(R_{\text{DF}}, R_{\text{DT}})] = \infty.$$

### III. COMPRESS-FORWARD (PROOF OF THEOREM 1)

Let  $K \succeq 0$  be of the form (3) and attain the maximum in (2). Let  $\mathbf{X}_1 \sim \mathcal{N}(0, K_1)$  and  $\mathbf{X}_2 \sim \mathcal{N}(0, K_2)$  be independent of each other<sup>1</sup> and  $\hat{\mathbf{Y}}_2 = \mathbf{Y}_2 + \hat{\mathbf{Z}}_2$ , where  $\hat{\mathbf{Z}}_2 \sim \mathcal{N}(0, \sigma^2 I_{r_2})$  is independent of  $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Z}_2, \mathbf{Z}_3)$ . Then,

$$I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3) = \frac{1}{2} \log |I_{r_3} + G_{31} K_1 G_{31}^T + G_{32} K_2 G_{32}^T|.$$

Since

$$G_{3*} \begin{bmatrix} K_1 & -K_{12} \\ -K_{12}^T & K_2 \end{bmatrix} G_{3*}^T \succeq 0,$$

or equivalently,

$$G_{31} K_1 G_{31}^T + G_{32} K_2 G_{32}^T \succeq G_{32} K_{12}^T G_{31}^T + G_{31} K_{12} G_{32}^T, \quad (14)$$

we have

$$\begin{aligned} I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3) &\geq \frac{1}{2} \log |I_{r_3} + \frac{1}{2} (G_{31} K_1 G_{31}^T + G_{32} K_2 G_{32}^T \\ &\quad + G_{32} K_{12}^T G_{31}^T + G_{31} K_{12} G_{32}^T)| \\ &\geq \frac{1}{2} \log |I_{r_3} + G_{3*} K G_{3*}^T| - \frac{1}{2} \min(t_1 + t_2, r_3), \end{aligned} \quad (15)$$

<sup>1</sup>This choice of input covariance matrices tightens the gap result in [14, Th. 1] that uses identity matrices.

where the last inequality follows by (4). Similarly, we have

$$I(\mathbf{Y}_2; \hat{\mathbf{Y}}_2 | \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_3) = \frac{r_2}{2} \log(1 + 1/\sigma^2) \quad (16)$$

and

$$\begin{aligned} I(\mathbf{X}_1; \hat{\mathbf{Y}}_2, \mathbf{Y}_3 | \mathbf{X}_2) &= \frac{1}{2} \log \frac{\left| \begin{bmatrix} (1 + \sigma^2) I_{r_2} & 0 \\ 0 & I_{r_3} \end{bmatrix} + G_{*1} K_1 G_{*1}^T \right|}{\left| \begin{bmatrix} (1 + \sigma^2) I_{r_2} & 0 \\ 0 & I_{r_3} \end{bmatrix} \right|}} \\ &= \frac{1}{2} \log \left| I_{t_1} + \left( \frac{1}{1 + \sigma^2} G_{21}^T G_{21} + G_{31}^T G_{31} \right) K_1 \right| \\ &\geq \frac{1}{2} \log |I_{t_1} + (G_{21}^T G_{21} + G_{31}^T G_{31}) K_{1|2}| \\ &\quad - \frac{1}{2} \log(1 + \sigma^2) \min(t_1, r_2 + r_3). \end{aligned} \quad (17)$$

Substituting (15)–(17) in (5) establishes (10). Finally, setting  $\sigma^2 = 1$  in (10) yields (11), which completes the proof of Theorem 1.

## IV. PARTIAL DECODE-FORWARD

### A. Coherent Partial Decode-Forward (Proof of Theorem 2)

Let  $K \succeq 0$  be of the form (3) and attain the maximum in (2). Let  $\mathbf{X}_1 \sim \mathcal{N}(0, K_1)$ ,  $\mathbf{X}_2 \sim \mathcal{N}(0, K_2)$ , and  $\mathbf{U} = G_{21} \mathbf{X}_1 + \mathbf{Z}'_2$ , where  $\mathbf{Z}'_2 \sim \mathcal{N}(0, \sigma^2 I_{r_2})$  is independent of  $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Z}_2, \mathbf{Z}_3)$ . Then,

$$I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3) = \frac{1}{2} \log |I_{r_3} + G_{3*} K G_{3*}^T| \quad (18)$$

and

$$\begin{aligned} I(\mathbf{U}; \mathbf{Y}_2 | \mathbf{X}_2) + I(\mathbf{X}_1; \mathbf{Y}_3 | \mathbf{X}_2, \mathbf{U}) &= \frac{1}{2} \log |I_{r_2} + G_{21} K_{1|2} G_{21}^T| + \frac{1}{2} \log \frac{|I_{r_3} + G_{31} \tilde{K} G_{31}^T|}{|I_{r_2} + G_{21} \tilde{K} G_{21}^T|} \\ &= \frac{1}{2} \log |I_{r_2} + G_{21} K_{1|2} G_{21}^T| \\ &\quad + \frac{1}{2} \log \frac{|I_{t_1} + (1/\sigma^2) G_{21}^T G_{21} + G_{31}^T G_{31}| K_{1|2}|}{|I_{t_1} + (1 + 1/\sigma^2) G_{21}^T G_{21} K_{1|2}|}, \end{aligned} \quad (19)$$

where

$$\tilde{K} = \text{Cov}(\mathbf{X}_1 | \mathbf{U}, \mathbf{X}_2) = K_{1|2} \left( I_{t_1} + \frac{1}{\sigma^2} G_{21}^T G_{21} K_{1|2} \right)^{-1}.$$

Let  $\sigma^2 = 1$ . Then, (19) simplifies as

$$\begin{aligned} I(\mathbf{U}; \mathbf{Y}_2 | \mathbf{X}_2) + I(\mathbf{X}_1; \mathbf{Y}_3 | \mathbf{X}_2, \mathbf{U}) &= \frac{1}{2} \log |I_{t_1} + (G_{21}^T G_{21} + G_{31}^T G_{31}) K_{1|2}| \\ &\quad + \frac{1}{2} \log \frac{|I_{r_2} + G_{21} K_{1|2} G_{21}^T|}{|I_{t_1} + 2G_{21}^T G_{21} K_{1|2}|} \\ &\geq \frac{1}{2} \log |I_{t_1} + (G_{21}^T G_{21} + G_{31}^T G_{31}) K_{1|2}| - \frac{1}{2} \min(t_1, r_2). \end{aligned} \quad (20)$$

Comparing (18) and (20) with (2) completes the proof of Theorem 2.

### B. Noncoherent Partial Decode-Forward (Proposition 4)

Let  $\mathbf{X}_1$  and  $\mathbf{X}_2$  be defined as in Section III and  $\mathbf{U}$  be defined as in Section IV-A. Then, by (15),

$$I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_3) \geq \frac{1}{2} \log |I_{r_3} + G_{3*} K G_{3*}^T| - \frac{1}{2} \min(t_1 + t_2, r_3).$$

Also, by substituting  $K_1 = K_{1|2}$  in (20) (recall that  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are independent), we have

$$\begin{aligned} I(\mathbf{U}; \mathbf{Y}_2 | \mathbf{X}_2) + I(\mathbf{X}_1; \mathbf{Y}_3 | \mathbf{X}_2, \mathbf{U}) \\ \geq \frac{1}{2} \log |I_{t_1} + (G_{21}^T G_{21} + G_{31}^T G_{31}) K_1| - \frac{1}{2} \min(t_1, r_2). \end{aligned}$$

This completes the proof of Proposition 4.

### C. Decode-Forward and Direct Transmission (Proposition 5)

Consider the GV-RC with  $G_{31} = \text{diag}(g, 1)$ ,  $G_{21} = \text{diag}(1, g)$ ,  $G_{32} = \text{diag}(g, g)$ ,  $g > 1$ , which is equivalent to a product of two mismatched single-antenna relay channels, one with the direct channel stronger than the sender-to-relay channel and the other in the opposite direction. Let  $K_1 = K_2 = (P/2)I_2$  in (18) and (20). Then,

$$\begin{aligned} R_{\text{PDF}} &\geq \min \left\{ \frac{1}{2} \log \left( 1 + g^2 P \right) \left( 1 + (1 + g^2) \frac{P}{2} \right), \right. \\ &\quad \left. \frac{1}{2} \log \left( 1 + (1 + g^2) \frac{P}{2} \right)^2 - 1 \right\} \\ &= \frac{1}{2} \log \left( 1 + (1 + g^2) \frac{P}{2} \right)^2 - 1. \end{aligned} \quad (21)$$

In comparison, it can be readily checked that

$$\begin{aligned} R_{\text{DF}} &\leq R_{\text{DT}} \\ &= \max_{P_1 + P_2 \leq P} \frac{1}{2} \log(1 + P_1)(1 + g^2 P_2) \\ &\leq \frac{1}{2} \log(1 + P)(1 + g^2 P). \end{aligned}$$

Therefore, we have

$$R_{\text{PDF}} - \max(R_{\text{DF}}, R_{\text{DT}}) \geq \frac{1}{2} \log \frac{(1 + (1 + g^2) \frac{P}{2})^2}{(1 + P)(1 + g^2 P)} - 1,$$

which tends to infinity as  $g \rightarrow \infty$ . This completes the proof of Proposition 5.

## V. NUMERICAL RESULTS

In this section, we numerically confirm the maximum and average of the gaps  $\Delta_{\text{CF}}$  and  $\Delta_{\text{PDF}}$  on the large numbers of the Gaussian vector relay channels. We use the auxiliary random variables  $\hat{\mathbf{Y}}_2 = \mathbf{Y}_2 + \hat{\mathbf{Z}}_2$ ,  $\hat{\mathbf{Z}}_2 \sim \mathcal{N}(0, I_{r_2})$  and  $\mathbf{U} = G_{21} \mathbf{X}_1 + \mathbf{Z}'_2$ ,  $\mathbf{Z}'_2 \sim \mathcal{N}(0, I_{r_2})$  for the compress-forward schemes and partial decode-forward. Fig. 2 shows the maximum and the average of both  $\Delta_{\text{CF}}$  and  $\Delta_{\text{PDF}}$  on the  $2 \times 2$  Gaussian vector relay channels, where  $t_1 = t_2 = r_2 = r_3 = 2$  and the channel gain matrices  $G_{21}$ ,  $G_{31}$ , and  $G_{32}$  have independent  $\mathcal{N}(0, 1)$  entries. Our simulation result is consistent with the theoretical bounds  $\Delta_{\text{CF}} \leq 2$  and  $\Delta_{\text{PDF}} \leq 1$ .

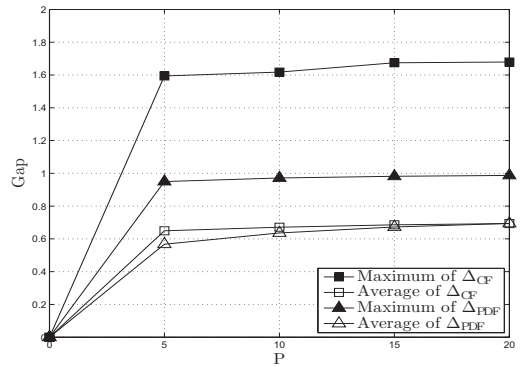


Fig. 2. The gaps between the cutset bound and the compress-forward and partial decode-forward lower bounds for randomly generated  $2 \times 2$  Gaussian vector relay channels.

## VI. DISCUSSION

In most wireless communication systems, the relay cannot send and receive in the same time slot or in the same frequency band. Half-duplex relay channel models are often investigated to study these systems. There are two different types of half-duplex models. One is the sender frequency-division (SFD) Gaussian vector relay channel (Fig. 3), in which the channel from the sender to the relay,  $\mathbf{X}_1 \rightarrow \mathbf{Y}_2$ , is orthogonal to the multiple access channel from the sender and the relay to the receiver,  $(\mathbf{X}_1, \mathbf{X}_2) \rightarrow \mathbf{Y}_3$ . The other is the receiver frequency-division (RFD) Gaussian vector relay channel (Fig. 4), in which the channel  $\mathbf{X}_2 \rightarrow \mathbf{Y}_3$  is orthogonal to the broadcast channel  $\mathbf{X}_1 \rightarrow (\mathbf{Y}_2, \mathbf{Y}_3)$ .

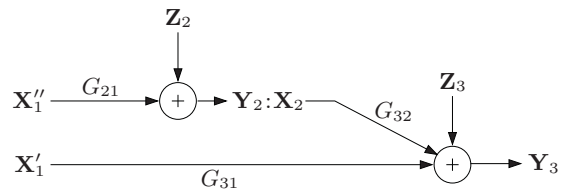


Fig. 3. Sender frequency-division Gaussian vector relay channel.

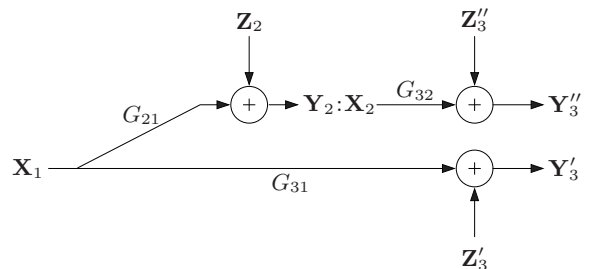


Fig. 4. Receiver frequency-division Gaussian vector relay channel.

For the sender frequency-division channel, it is known [16] that the capacity is achieved by partial decode-forward. We include this result by El Gamal and Zahedi as the performance benchmark.

*Proposition 6 ([16]):* The capacity of the SFD GV-RC is

$$\begin{aligned} C &= R_{\text{CS}} \\ &= R_{\text{PDF}} \\ &= \sup_{F(\mathbf{x}'_1, \mathbf{x}_2)F(\mathbf{x}'_1)} \min\{I(\mathbf{X}'_1, \mathbf{X}_2; \mathbf{Y}_3), \\ &\quad I(\mathbf{X}'_1; \mathbf{Y}_2) + I(\mathbf{X}'_1; \mathbf{Y}_3 | \mathbf{X}_2)\} \\ &= \max_K \min \left\{ \frac{1}{2} \log \left| I + G_{3*} \begin{bmatrix} K'_1 & K_{12} \\ K_{12}^T & K_2 \end{bmatrix} G_{3*}^T \right|, \right. \\ &\quad \left. \frac{1}{2} \log |I + G_{21} K_1'' G_{21}^T| \right. \\ &\quad \left. + \frac{1}{2} \log |I + G_{31} K_{1|2}' G_{31}^T| \right\} \end{aligned}$$

where the supremum is over all  $F(\mathbf{x}'_1, \mathbf{x}_2)F(\mathbf{x}'_1)$  such that  $\mathbf{E}(\mathbf{X}'_1 \mathbf{X}'_1^T) + \mathbf{E}(\mathbf{X}_1'' \mathbf{X}_1''^T) \leq P$  and  $\mathbf{E}(\mathbf{X}_2^T \mathbf{X}_2) \leq P$ , and the maximum is over all  $(t_1 + t_2) \times (t_1 + t_2)$  matrices

$$K = \begin{bmatrix} K'_1 + K_1'' & K_{12} \\ K_{12}^T & K_2 \end{bmatrix}$$

such that  $\text{tr}(K'_1 + K_1'') \leq P$ ,  $\text{tr}(K_2) \leq P$ ,  $\begin{bmatrix} K'_1 & K_{12} \\ K_{12}^T & K_2 \end{bmatrix} \succeq 0$ , and  $K_{1|2}' = K'_1 - K_{12} K_2^{-1} K_{12}^T$ .

We have the following results for compress-forward and noncoherent partial decode-forward.

*Proposition 7 (Gap results for the SFD GV-RC):* For every  $G_{21}$ ,  $G_{31}$ ,  $G_{32}$ , and  $P$ ,

$$\begin{aligned} \Delta_{\text{CF}} &\leq \frac{1}{2} [\min(t_1 + t_2, r_3) + r_2], \\ \Delta_{\text{NPDF}} &\leq \frac{1}{2} \min(t_1 + t_2, r_3). \end{aligned}$$

For the receiver frequency-division channel, the capacity is not known in general. We first specialize the cutset bound for this case.

*Proposition 8:* The capacity  $C$  of the RFD GV-RC is upper bounded by

$$\begin{aligned} R_{\text{CS}} &= \max_{F(\mathbf{x}_1)F(\mathbf{x}_2)} \min\{I(\mathbf{X}_1; \mathbf{Y}_3) + I(\mathbf{X}_2; \mathbf{Y}_3), \\ &\quad I(\mathbf{X}_1; \mathbf{Y}_2, \mathbf{Y}_3)\} \\ &= \max_{K_1} \min \left\{ \frac{1}{2} \log |I + G_{31} K_1 G_{31}^T| \right. \\ &\quad \left. + \max_{K_2} \frac{1}{2} \log |I + G_{32} K_2 G_{32}^T|, \right. \\ &\quad \left. \frac{1}{2} \log |I + (G_{21}^T G_{21} + G_{31}^T G_{31}) K_1| \right\}, \end{aligned}$$

where the supremum is over all  $F(\mathbf{x}_1)F(\mathbf{x}_2)$  such that  $\mathbf{E}(\mathbf{X}_j^T \mathbf{X}_j) \leq P$ ,  $j = 1, 2$ , and the maxima are over all  $K_1, K_2 \succeq 0$  such that  $\text{tr}(K_j) \leq P$ ,  $j = 1, 2$ .

We have the following results for compress-forward and noncoherent partial decode-forward.

*Proposition 9 (Gap results for the RFD GV-RC):* For every  $G_{21}$ ,  $G_{31}$ ,  $G_{32}$ , and  $P$ ,

$$\begin{aligned} \Delta_{\text{CF}} &\leq \frac{1}{2} \max [\min(t_1, r_2 + r_3), r_2], \\ \Delta_{\text{PDF}} &= \Delta_{\text{NPDF}} \leq \frac{1}{2} \min(t_1, r_2). \end{aligned}$$

As in the full-duplex case, it can be also shown that partial decode-forward is strictly better than the better of decode-forward and direct transmission.

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