Sliding-Window Superposition Coding for Interference Networks

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***Abstract*—Superposition coding with successive cancellation decoding for interference channels is investigated as a low-complexity alternative to the rate-optimal simultaneous decoding. It is shown that regardless of the number of superposition layers and the code distribution of each layer, the standard rate-splitting scheme by Grant, Rimoldi, Urbanke, and Whiting for multiple access channels fails to achieve the simultaneous decoding inner bound on the capacity region for interference channels. A new coding scheme is proposed that uses coding over multiple blocks and sliding-window decoding. With at most two superposition layers, this scheme achieves the simultaneous decoding inner bound for any two-user-pair interference channels without using high-complexity simultaneous multiuser sequence detection. The proposed coding scheme can be also extended to achieve the performance of simultaneous decoding for general interference networks, including the Han–Kobayashi inner bound.**

I. Introduction

Consider an interference channel $p(y_1, y_2|x_1, x_2)$, in which sender $i \in \{1, 2\}$ wishes to communicate an independent message reliably to its respective receiver i. A $(2^{nR_1}, 2^{nR_2}, n)$ code for the interference channel consists of

- two message sets $[1 : 2^{nR_1}]$ and $[1 : 2^{nR_2}]$,
- two encoders, where encoder $i \in \{1, 2\}$ assigns a codeword $x_i^n(m_i)$ to each message $m_i \in [1 : 2^{nR_i}]$, and
- two decoders, where decoder $i \in \{1, 2\}$ assigns an estimate $\hat{m}_i$ or an error message e to each received sequence y_i^n.

We assume that the message pair (M_1, M_2) is uniformly distributed over $[1 : 2^{nR_1}] \times [1 : 2^{nR_2}]$. The average probability of error is defined as $P_e^{(n)} = \mathsf{P}\big\{(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)\big\}$. A rate pair (R_1, R_2) is said to be *achievable* if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$. The capacity region is the closure of the set of achievable rate pairs (R_1, R_2).

One important decoding scheme for interference channels is *simultaneous decoding*. It is a key component in the Han–Kobayashi coding scheme [1], whereby each receiver, instead of treating interference as noise, decodes for the intended message as well as part of the interfering message. Recently, Bandemer, El Gamal, and Kim showed that simultaneous nonunique decoding is rate-optimal for random code ensembles with superposition coding and time sharing [2].

Unfortunately, simultaneous decoding uses multiuser sequence detection at the core of its operation and it is not known how this can be implemented in low complexity. Consequently, several heuristic approaches have been developed that attempt to achieve "similar" performance; see, for example, [3], [4].

In this paper, we address this problem from a different angle and ask the following questions. Is simultaneous decoding really needed? Is there an alternative coding scheme that achieves the same performance at low complexity?

Treating interference as noise and successive cancellation decoding (with no rate-splitting) are the two main decoding schemes used in practice, both of which achieve strictly smaller rate regions than simultaneous decoding. Recently, Zhao, Tan, Avestimehr, Diggavi, and Pottie [5] studied successive cancellation decoding for more than two layers of Gaussian superposition codes, as an application of the rate-splitting scheme by Rimoldi and Urbanke [6] and Grant, Rimoldi, Urbanke, and Whiting [7] to interference channels. In Section II, we investigate this application in full generality by considering arbitrary code distributions for superposition coding, which is sometimes necessary as pointed out in [8]. We show that regardless of the number of layers and the code distribution of each layer, the standard single-block rate-splitting scheme fails to achieve the simultaneous decoding inner bound in interference channels.

Can we therefore conclude that simultaneous decoding is indeed necessary in optimal coding for the interference channel? Are point-to-point coding techniques, which can achieve capacity for multiple access and single-antenna Gaussian broadcast channels, fundamentally deficient for the interference channel? Inspired by a polar coding scheme in a parallel study [9] that achieves the simultaneous decoding inner bound, even with a low-complexity successive cancellation decoding algorithm, we develop in Section III a sliding-window superposition coding scheme. By coding over multiple blocks and sliding-window decoding, this scheme achieves the corner point in the simultaneous decoding inner bound, the exact same point that demonstrates the insufficiency of the standard single-block rate-splitting.

II. Insufficiency of Single-Block Rate-Splitting

In this section, we consider the symmetric Gaussian interference channels. We show a corner point of the simultaneous decoding inner bound is not achievable using rate-splitting with successive cancellation decoding. We assume average

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power constraint P. The channel outputs at the receivers for inputs X_1 and X_2 are

$$Y_1 = X_1 + gX_2 + Z_1,$$
$$Y_2 = gX_1 + X_2 + Z_2,$$

where g is a fixed constant and $Z_1, Z_2 \sim \mathrm{N}(0,1)$ are additive Gaussian noise components, independent of (X_1, X_2). We define the received *signal-to-noise ratio* as $S = P$ and the received *interference-to-noise ratio* as $I = g^2P$.

The (s, t, d_1, d_2, F) rate-splitting scheme and its achievable rate region are defined as follows.

Rate splitting. We represent the message M_1 by s independent parts $M_{11}, M_{12}, \dots, M_{1s}$ at rates $R_{11}, R_{12}, \dots, R_{1s}$, respectively, and the message M_2 by t independent parts $M_{21}, M_{22}, \dots, M_{2t}$ at rates $R_{21}, R_{22}, \dots, R_{2t}$, respectively.

Codebook generation. We use superposition coding. Fix a cdf $F = F(q)F(u^s|q)F(v^t|q)$ such that Q is finite, $\mathsf{E}[(U_s)^2] \le P$, and $\mathsf{E}[(V_t)^2] \le P$. Randomly and independently generate q^n according to $\prod_{k=1}^n F(q_k)$. Randomly and conditionally independently generate $2^{nR_{11}}$ sequences $u_1^n(m_{11})$, $m_{11} \in [1:2^{nR_{11}}]$, each according to a product cdf of $F(u_1|q)$. For $j \in [2:s]$, for each m_1^{j-1}, randomly and conditionally independently generate $2^{nR_{1j}}$ sequences $u_j^n(m_{1j}|m_1^{j-1})$, $m_{1j} \in [1:2^{nR_{1j}}]$, each according to a product cdf of $F(u_j|u^{j-1}, q)$. Randomly and conditionally independently generate $2^{nR_{21}}$ sequences $v_1^n(m_{21})$, $m_{21} \in [1:2^{nR_{21}}]$, each according to a product cdf of $F(v_1|q)$. For $j \in [2:t]$, for each m_2^{j-1}, randomly and conditionally independently generate $2^{nR_{2j}}$ sequences $v_j^n(m_{2j}|m_2^{j-1})$, $m_{2j} \in [1:2^{nR_{2j}}]$, each according to a product cdf of $F(v_j|v^{j-1}, q)$.

Encoding. To send message pair $(m_1, m_2) = (m_1^s, m_2^t)$, encoder 1 transmits $x_1^n(m_1^s) = u_s^n(m_{1s}|m_1^{s-1})$ and encoder 2 transmits $x_2^n(m_2^t) = v_t^n(m_{2t}|m_2^{t-1})$.

Decoding. We use successive cancellation decoding. Define the *decoding order* d_1 at decoder 1 as an ordering of elements in $\{U^s, V_{\mathcal{A}}\}$ and d_2 at decoder 2 as an ordering of elements in $\{U_{\mathcal{B}}, V^t\}$, where $\mathcal{A} \subseteq [1:t]$ and $\mathcal{B} \subseteq [1:s]$.

As an example, suppose that message M_1 is split into two parts while message M_2 is not split. The decoding orders are

$$d_1\colon U \to X_2 \to X_1,$$
$$d_2\colon U \to X_1 \to X_2,$$

where in case of a single split, we write $(U, X_1) = (U_1, U_2)$ and $X_2 = V_1$. This means that decoder 1 recovers M_{11}, M_2, and M_{12} successively and decoder 2 recovers M_{11}, M_{12} and M_2 successively. More precisely, upon receiving y_1^n at decoder 1, decoding proceeds in three steps:

1) Decoder 1 finds the unique message $\hat{m}_{11}$ such that $(u^n(\hat{m}_{11}), y_1^n, q^n) \in \mathcal{T}_\epsilon^{(n)}$.
2) If $\hat{m}_{11}$ is found, decoder 1 finds the unique $\hat{m}_2$ such that $(u^n(\hat{m}_{11}), x_2^n(\hat{m}_2), y_1^n, q^n) \in \mathcal{T}_\epsilon^{(n)}$.
3) If $(\hat{m}_{11}, \hat{m}_2)$ is found, find the unique $\hat{m}_{12}$ such that $(u^n(\hat{m}_{11}), x_2^n(\hat{m}_2), x_1^n(\hat{m}_{11}, \hat{m}_{12}), y_1^n, q^n) \in \mathcal{T}_\epsilon^{(n)}$.

Similarly, upon receiving y_2^n, decoding proceeds in three steps:

1) Decoder 2 finds the unique message $\hat{m}_{11}$ such that $(u^n(\hat{m}_{11}), y_2^n, q^n) \in \mathcal{T}_\epsilon^{(n)}$.
2) If $\hat{m}_{11}$ is found, decoder 2 finds the unique $\hat{m}_{12}$ such that $(u^n(\hat{m}_{11}), x_1^n(\hat{m}_{11}, \hat{m}_{12}), y_2^n, q^n) \in \mathcal{T}_\epsilon^{(n)}$.
3) If $(\hat{m}_{11}, \hat{m}_{12})$ is found, find the unique $\hat{m}_2$ such that $(u^n(\hat{m}_{11}), x_1^n(\hat{m}_{11}, \hat{m}_{12}), x_2^n(\hat{m}_2), y_2^n, q^n) \in \mathcal{T}_\epsilon^{(n)}$.

Following the standard analysis of the error probability [10, Sec. 4.5.1], $P_e^{(n)}$ tends to zero as $n \to \infty$ if

$$R_{11} < I(U; Y_1|Q) - \delta(\epsilon), \tag{1a}$$
$$R_2 < I(X_2; Y_1|U, Q) - \delta(\epsilon), \tag{1b}$$
$$R_{12} < I(X_1; Y_1|U, X_2, Q) - \delta(\epsilon), \tag{1c}$$
$$R_{11} < I(U; Y_2|Q) - \delta(\epsilon), \tag{1d}$$
$$R_{12} < I(X_1; Y_2|U, Q) - \delta(\epsilon), \tag{1e}$$
$$R_2 < I(X_2; Y_2|X_1, Q) - \delta(\epsilon). \tag{1f}$$

By Fourier–Motzkin elimination, (R_1, R_2) is achievable if

$$\begin{aligned} R_1 &< \min\{I(U; Y_1|Q),\, I(U; Y_2|Q)\} \\ &\quad + \min\{I(X_1; Y_1|U, X_2, Q),\, I(X_1; Y_2|U, Q)\}, \\ R_2 &< \min\{I(X_2; Y_1|U, Q),\, I(X_2; Y_2|X_1, Q)\}. \end{aligned} \tag{2}$$

We note some common misconception in the literature (see [11] and the references therein) that the bounds on R_{11} and R_{12} in (1) simplify to $R_1 < \min\{I(U; Y_1|Q) + I(X_1; Y_1|U, X_2, Q),\, I(X_1; Y_2|Q)\}$, which leads to an incorrect conclusion that the Han–Kobayashi inner bound can be achieved by rate-splitting and successive cancellation. As pointed out in [11], successive decoding requires individual rate constraints (1d) and (1e) instead of the sum-rate constraint $R_1 < I(X_1; Y_2|Q)$. Moreover, a proper application of the Fourier–Motzkin elimination procedure requires taking the minimum for four cases of sum-rates, which leads to (2).

For more layers of splitting and general decoding orders, decoding can be performed in a similar fashion. Thus, an (s, t, d_1, d_2, F) rate-splitting scheme is specified by

- the numbers s and t of independent parts in messages $M_1 = (M_{11}, \dots, M_{1s})$ and $M_2 = (M_{21}, \dots, M_{2t})$,
- the cdf $F = F(q)F(u^s|q)F(v^t|q)$, and
- the decoding orders d_1 and d_2.

Let $\mathscr{R}(s, t, d_1, d_2, F)$ denote the achievable rate region of the (s, t, d_1, d_2, F) rate-splitting scheme. Let $\mathscr{R}^*(s, t, d_1, d_2)$ be the closure of $\cup_F \mathscr{R}(s, t, d_1, d_2, F)$. Define $R_1^*(s, t, d_1, d_2) = \max\{R_1\colon (R_1, \mathsf{C}(S)) \in \mathscr{R}^*(s, t, d_1, d_2)\}$ as the maximal achievable rate R_1 such that R_2 is at individual capacity.

Now we are ready to state the main result of this section. Assume that the symmetric Gaussian interference channel has *strong but not very strong* interference, i.e., $S < I < S(S+1)$. The capacity region is the set of rate pairs (R_1, R_2) such that

$$R_1 \le \mathsf{C}(S),$$
$$R_2 \le \mathsf{C}(S),$$
$$R_1 + R_2 \le \mathsf{C}(I + S),$$

which is achieved by simultaneous decoding with $X_1, X_2 \sim N(0, P)$ and $Q = \emptyset$ [1], [12]. Theorem 1 states that the corner point of this region is not achievable using any (s, t, d_1, d_2, F) rate-splitting scheme; see Appendix for the proof.

Theorem 1: For the symmetric Gaussian interference channel with $S < I < S(S+1)$,

$$R_1^*(s, t, d_1, d_2) < \mathsf{C}\left(\frac{I}{1+S}\right)$$

for any finite s, t and decoding orders d_1, d_2.

The idea of the standard rate-splitting scheme for the multiple access channel is to represent each message by multiple parts and encode them into superimposed layers. Combined with successive cancellation decoding, this superposition coding scheme transforms the multiple access channel into a sequence of point-to-point channels. For the interference channel, which consists of two underlying multiple access channels $p(y_i|x_1, x_2), i = 1, 2$, however, this idea no longer works. Here rate-splitting induces two sequences of point-to-point channels that have different qualities in general. To ensure reliable communication, the messages have to be loaded at the rate of the worse channel on each layer, which in general incurs a total rate loss. Theorem 1 essentially states that there is no split of the messages that "equalizes" the qualities of the two point-to-point channels on each layer, even when the decoding orders of the layers are optimized. Rate-splitting is alternatively viewed as mapping a boundary point of one multiple access rate region to a corner point of another multiple access rate region in a higher dimensional space [6], [7]. Theorem 1 shows that there is no such mapping in general under which the corresponding corner points for the two multiple access channels coincide.

III. Sliding-Window Superposition Coding

In this section, we propose a sliding-window superposition coding scheme that resolves the difficulty in single-block rate splitting. We present the coding scheme for the general discrete memoryless interference channels. Due to space limitations, we only describe how to achieve the corner point

$$(R_1, R_2) = (I(X_1; Y_2), I(X_2; Y_2|X_1)) \quad (3)$$

of the simultaneous decoding inner bound when $Q = \emptyset$ and the sum-rates are equal, i.e.,

$$I(X_1, X_2; Y_1) = I(X_1, X_2; Y_2) \quad (4)$$

as shown in Figure 1. For the symmetric Gaussian interference channel, this is the exact same point that demonstrated the insufficiency of single-block rate-splitting in Section II.

Theorem 2: A rate pair (R_1, R_2) is achievable with the sliding-window superposition coding scheme if

$$R_1 < \min\{I(U; Y_1) + I(X_1; Y_1|U, X_2),\ I(X_1; Y_2)\} := I_1,$$
$$R_2 < \min\{I(X_2; Y_1|U),\ I(X_2; Y_2|X_1)\} := I_2$$

for some pmf $p(u, x_1)p(x_2)$. In addition, there exists a pmf $p(u, x_1)p(x_2)$ such that $(I_1, I_2) = (I(X_1; Y_2), I(X_2; Y_2|X_1))$; in other words, the corner point (3) is achievable.

Roughly speaking, instead of splitting the message M_1 into two parts and recovering the two parts separately, we send M_1 without split over two consecutive blocks and recover it using sliding-window decoding. Details are as follows.

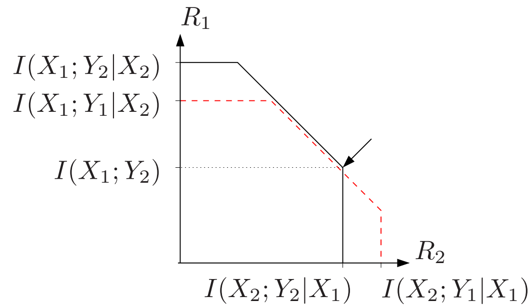

Fig. 1. The simultaneous decoding inner bound and the corner point (3) (the one with the arrow) that will be illustrated to be achievable by sliding-window superposition coding.

Codebook generation. Fix the pmf $p(u, x_1)p(x_2)$ that attains the target rate pair. Randomly and independently generate a codebook for each block. Assume $m_{10} = m_{1b} = 1$ by convention. For $j \in [1:b]$, randomly and independently generate 2^{nR_1} sequences $u^n(m_{1,j-1}), m_{1,j-1} \in [1:2^{nR_1}]$, each according to a product of $p(u)$. For each $m_{1,j-1}$, randomly and conditionally independently generate 2^{nR_1} sequences $x_1^n(m_{1j}|m_{1,j-1}), m_{1j} \in [1:2^{nR_1}]$, each according to a product of $p(x_1|u)$. Randomly and independently generate 2^{nR_2} sequences $x_2^n(m_{2j}), m_{2j} \in [1:2^{nR_2}]$, each according to a product of $p(x_2)$. This defines the codebook

$$\mathcal{C}_j = \{u^n(m_{1,j-1}), x_1^n(m_{1j}|m_{1,j-1}), x_2^n(m_{2j}), m_{1,j-1}, m_{1j} \in [1:2^{nR_1}], m_{2j} \in [1:2^{nR_2}]\}, \quad j \in [1:b].$$

Encoding. Sender 1 transmits $x_1^n(m_{1j}|m_{1,j-1})$ and sender 2 transmits $x_2^n(m_{2j})$ in block $j \in [1:b]$.

Decoding and analysis of error. Decoder 1 (and 2, respectively) successively recovers $\hat{m}_{2j}$ and $\hat{m}_{1j}$ ($\hat{m}_{1j}$ and $\hat{m}_{2j}$), $j \in [1:b]$, where the decoding of $\hat{m}_{1j}$ is done by a sliding-window decoding over blocks j and $j+1$. Table I reveals the scheduling of the messages.

Let the received sequences in block j be $y_1^n(j)$ and $y_2^n(j)$, $j \in [1:b]$. For receiver 1, in block 1, it finds the unique message $\hat{m}_{21}$ such that

$$(x_2^n(\hat{m}_{21}), y_1^n(1), u^n(1)) \in \mathcal{T}_\epsilon^{(n)}$$

(and declares an error if there is none or more than one). By standard analysis, the probability of error tends to zero if

block j	1	2	3	$\cdots$	$b-1$	b
U	1	m_{11}	m_{12}	$\cdots$	$\cdots$	$m_{1,b-1}$
X_1	m_{11}	m_{12}	$\cdots$	$\cdots$	$m_{1,b-1}$	1
X_2	m_{21}	m_{22}	$\cdots$	$\cdots$	$m_{2,b-1}$	m_{2b}
Y_1	$\emptyset$	$\hat{m}_{11}$	$\hat{m}_{12}$	$\cdots$	$\cdots$	$\hat{m}_{1,b-1}$
	$\hat{m}_{21}$	$\hat{m}_{22}$	$\cdots$	$\cdots$	$\hat{m}_{2,b-1}$	$\hat{m}_{2b}$
Y_2	$\emptyset$	$\hat{m}_{11}$	$\hat{m}_{12}$	$\cdots$	$\cdots$	$\hat{m}_{1,b-1}$
	$\emptyset$	$\hat{m}_{21}$	$\hat{m}_{22}$	$\cdots$	$\cdots$	$\hat{m}_{2,b-1}, \hat{m}_{2b}$

TABLE I
Sliding-window superposition coding scheme.

$R_2 < I(X_2; Y_1|U) - \delta(\epsilon)$. In block $j+1$, $j \in [1 : b-1]$, it finds the unique message $\hat{m}_{1j}$ such that

$$(u^n(\hat{m}_{1,j-1}), x_1^n(\hat{m}_{1j}|\hat{m}_{1,j-1}), x_2^n(\hat{m}_{2j}), y_1^n(j)) \in \mathcal{T}_\epsilon^{(n)}, \quad (u^n(\hat{m}_{1j}), y^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$$

simultaneously. The probability of error tends to zero if $R_1 < I(X_1; Y_1|U, X_2) + I(U; Y_1) - 2\delta(\epsilon)$. Then it finds the unique $\hat{m}_{2,j+1}$ such that

$$(x_2^n(\hat{m}_{2,j+1}), y_1^n(j+1), u^n(\hat{m}_{1j})) \in \mathcal{T}_\epsilon^{(n)}.$$

The probability of error tends to zero if $R_2 < I(X_2; Y_1|U) - \delta(\epsilon)$. For receiver 2, in block $j+1$, $j \in [1 : b-1]$, it finds the unique $\hat{m}_{1j}$ such that

$$(u^n(\hat{m}_{1,j-1}), x_1^n(\hat{m}_{1j}|\hat{m}_{1,j-1}), y_2^n(j)) \in \mathcal{T}_\epsilon^{(n)}, \quad (u^n(\hat{m}_{1j}), y_2^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$$

simultaneously. The probability of error tends to zero if $R_1 < I(X_1; Y_2|U) + I(U; Y_2) - 2\delta(\epsilon) = I(X_1; Y_2) - 2\delta(\epsilon)$. Then it finds the unique $\hat{m}_{2j}$ such that

$$(x_2^n(\hat{m}_{2j}), y_2^n(j), u^n(\hat{m}_{1,j-1}), x_1^n(\hat{m}_{1j}|\hat{m}_{1,j-1})) \in \mathcal{T}_\epsilon^{(n)}.$$

The probability of error tends to zero if $R_2 < I(X_2; Y_2|X_1) - \delta(\epsilon)$. In the end, receiver 2 finds the unique $\hat{m}_{2b}$ such that

$$(x_2^n(\hat{m}_{2b}), y_2^n(b), u^n(\hat{m}_{1,b-1}), x_1^n(1|\hat{m}_{1,b-1})) \in \mathcal{T}_\epsilon^{(n)},$$

The probability of error tends to zero if $R_2 < I(X_2; Y_2|X_1) - \delta(\epsilon)$.

Finally, we note that $I(X_2; Y_1) \leq I(X_2; Y_2|X_1) \leq I(X_2; Y_1|X_1)$, which guarantees the existence of $p(u|x_1)$ such that $I(X_2; Y_1|U) = I(X_2; Y_2|X_1)$. Combined with (4), this implies that $I(X_1; Y_2) = I(U; Y_1) + I(V; Y_1|X_2, U)$ and the corner point is achievable. For the symmetric Gaussian interference channels, the corner point is achieved when $U \sim \mathrm{N}(0, \alpha P), V \sim \mathrm{N}(0, (1-\alpha)P)$ and $X_1 = U + V$, where U and V are independent and $\alpha = (S^2 + S - I)/S^2$.

IV. Discussion

We have shown that any single-block rate-splitting scheme is strictly suboptimal for the two-user-pair symmetric Gaussian interference channels. However, by sending the messages over multiple blocks and using sliding-window decoding, we are able to achieve the simultaneous decoding inner bound. Compared to implementing simultaneous decoding that requires multiuser sequence detection, the proposed sliding-window superposition coding scheme has a simpler implementation, since messages can be decoded one at a time without any need for multiuser sequence detection.

For the general K-sender L-receiver interference networks, where each sender transmits an independent message and each receiver recovers a subset of the K messages, one can similarly send the messages over more than two blocks. By carefully scheduling the decoding orders for each receiver, the sliding-window superposition coding scheme can be shown to achieve the simultaneous decoding inner bound for the interference networks, which includes the Han–Kobayashi inner bound for two-user-pair interference channels as a special case.

Although heterogeneous (UX) superposition coding was considered throughout this paper, a similar conclusion holds for homogeneous (UV) superposition coding [13], which can be simpler to implement in practice. For the Gaussian interference channels, one can split X_1 into independent U and V, and take the function $X_1 = U + V$.

Appendix
Proof of Theorem 1

For the simplicity of notation, we prove the claim for $Q = \emptyset$. The case for general Q follows the same logic. We need the following three lemmas, the proofs of which are skipped due to space limitations.

Lemma 1: For any (s, t, d_1, d_2, F) rate-splitting scheme that achieves $R_1^*(s, t, d_1, d_2)$, we can assume without loss of generality that $s = t$ and the decoding orders are

$$d_1^*: U_1 \to V_1 \to U_2 \to V_2 \to \cdots \to U_{s-1} \to V_{s-1} \to U_s,$$
$$d_2^*: U_1 \to U_2 \to \cdots \to U_s \to V_1 \to V_2 \to \cdots \to V_s.$$

Lemma 2: A necessary condition for $(2, 2, d_1^*, d_2^*, F)$ rate-splitting scheme to attain the corner point is that the distribution F is such that $X_1, X_2 \sim \mathrm{N}(0, P)$.

Lemma 3: Let $F(u, x)$ be any distribution such that $X \sim \mathrm{N}(0, P)$ and $I(U; Y) = 0$, where $Y = X + Z$ with $Z \sim \mathrm{N}(0, 1)$ independent of X. Then, $I(U; X) = 0$.

Now we establish the insufficiency. It is straightforward to check for the case $s = 1$. For $s = 2$, we prove by contradiction. We write $(U, X_1) = (U_1, U_2)$ and $(V, X_2) = (V_1, V_2)$. The achievable rate region of the $(2, 2, d_1^*, d_2^*, F)$ rate-splitting scheme is the set of rate pairs (R_1, R_2) such that

$$\begin{aligned} R_1 &< \min\{I(U; Y_1),\, I(U; Y_2)\} \\ &\quad + \min\{I(X_1; Y_1|U, V),\, I(X_1; Y_2|U)\} := I_1, \\ R_2 &< \min\{I(V; Y_1|U),\, I(V; Y_2|X_1)\} \\ &\quad + I(X_2; Y_2|X_1, V) := I_2. \end{aligned}$$

Assume that the corner point of the capacity region is achieved by the $(2, 2, d_1^*, d_2^*, F)$ rate-splitting scheme, that is,

$$I_1 = \mathsf{C}(I/(1+S)), \tag{5}$$
$$I_2 = \mathsf{C}(S). \tag{6}$$

Then, by Lemma 2, we must have $X_1 \sim \mathrm{N}(0, P)$ and $X_2 \sim \mathrm{N}(0, P)$. Consider

$$\begin{aligned} I_1 &= \min\{I(U; Y_1),\, I(U; Y_2)\} \\ &\qquad + \min\{I(X_1; Y_1|U, V),\, I(X_1; Y_2|U)\}, \\ &\leq I(U; Y_1) + I(X_1; Y_2|U) \qquad (7) \\ &= h(Y_1) - h(Y_1|U) + h(Y_2'|U) - h(Y_2'|X_1), \qquad (8) \end{aligned}$$

where $Y_2' = Y_2/g = X_1 + (X_2 + Z_2)/g$. Since $\frac{1}{2}\log(2\pi e(S+1)/g^2) = h(Y_2'|X_1) \leq h(Y_2'|U) \leq h(Y_2') = \frac{1}{2}\log(2\pi e(I + S + 1)/g^2)$, there exists an $\alpha \in [0, 1]$ such that $h(Y_2'|U) = (1/2)\log(2\pi e(\alpha I + S + 1)/g^2)$. Moreover, since $X_2 \sim \mathrm{N}(0, P)$

and $I < S(1+S)$, the channel $X_1 \to Y_1$ is a degraded version of the channel $X_1 \to Y_2'$, i.e., $Y_1 = Y_2' + Z'$, where $Z' \sim \mathrm{N}(0, I+1-(S+1)/g^2)$ is independent of X_1 and X_2. By the entropy power inequality, $2^{2h(Y_1|U)} \geq 2^{2h(Y_2'|U)} + 2^{2h(Z'|U)} = 2\pi e(\alpha S + I + 1)$. Therefore, it follows from (8) that

$$\begin{aligned} I_1 &\leq h(Y_1) - h(Y_1|U) + h(Y_2'|U) - h(Y_2'|X_1) \\ &\leq \frac{1}{2}\log\left(\frac{(I+S+1)(\alpha I + S + 1)}{(\alpha S + I + 1)(1+S)}\right) \\ &\stackrel{(a)}{\leq} \mathsf{C}\left(I/(1+S)\right), \end{aligned}$$

where (a) follows since $S < I$. To match the standing assumption in (5), we must have equality in (a), which forces $\alpha = 1$ and $h(Y_2'|U) = (1/2)\log(2\pi e(I+S+1)/g^2) = h(Y_2')$, i.e., $I(U; Y_2') = 0$. Note that $X_1, X_2 \sim \mathrm{N}(0, P)$ and the channel from X_1 to Y_2' is a Gaussian channel. Applying Lemma 3 yields

$$I(U; X_1) = 0. \tag{9}$$

Now, I_2 can be simplified to

$$\begin{aligned} I_2 &= \min\{I(V; Y_1|U), I(V; Y_2|X_1)\} + I(X_2; Y_2|X_1, V) \\ &\stackrel{(b)}{=} \min\{I(V; Y_1), I(V; Y_2|X_1)\} + I(X_2; Y_2|X_1, V) \quad (10) \\ &\leq I(V; Y_1) + I(X_2; Y_2|X_1, V) \\ &= h(\tilde{Y}_1) - h(\tilde{Y}_1|V) + h(\tilde{Y}_2|V) - h(Y_2|X_1, X_2), \quad (11) \end{aligned}$$

where (b) follows since $I(U; Y_1|V) \leq I(U; Y_1|X_2) = I(U; X_1 + Z_1) \leq I(U; X_1) = 0$, which implies $I(V; Y_1|U) = I(V; Y_1)$. In (11), we denote $\tilde{Y}_1 = Y_1/g = X_2 + (X_1 + Z_1)/g$ and $\tilde{Y}_2 = X_2 + Z_2$. Since $\frac{1}{2}\log(2\pi e) = h(\tilde{Y}_2|X_2) \leq h(\tilde{Y}_2|V) \leq h(\tilde{Y}_2) = \frac{1}{2}\log(2\pi e(1+S))$, there exists a $\beta \in [0,1]$ such that $h(\tilde{Y}_2|V) = (1/2)\log(2\pi e\ (1+\beta S))$. Moreover, since $X_1 \sim \mathrm{N}(0, P)$ and $I < S(1+S)$, $\tilde{Y}_1$ is a degraded version of $\tilde{Y}_2$, i.e., $\tilde{Y}_1 = \tilde{Y}_2 + \tilde{Z}$, where $\tilde{Z} \sim \mathrm{N}(0, (1+S)/g^2 - 1)$ is independent of X_1 and X_2. Applying the entropy power inequality, we have $2^{2h(\tilde{Y}_1|V)} \geq 2^{2h(\tilde{Y}_2|V)} + 2^{2h(\tilde{Z}|V)} = 2\pi e(\beta S + (1+S)/g^2)$. Therefore, it follows from (11) that

$$\begin{aligned} I_2 &\leq h(\tilde{Y}_1) - h(\tilde{Y}_1|V) + h(\tilde{Y}_2|V) - h(Y_2|X_1, X_2) \\ &\leq \frac{1}{2}\log\left(\frac{(I+S+1)(1+\beta S)}{g^2(\beta S + (1+S)/g^2)}\right) \stackrel{(c)}{\leq} \mathsf{C}(S), \end{aligned}$$

where (c) follows from the channel condition $I < (1+S)S$. To match the standing assumption in (6), we must have equality in (c), which forces $\beta = 1$ and $h(\tilde{Y}_2|V) = (1/2)\log(2\pi e(1+S)) = h(\tilde{Y}_2)$, i.e., $I(V; \tilde{Y}_2) = 0$. Note that $X_2 \sim \mathrm{N}(0, P)$ and the channel from X_2 to $\tilde{Y}_2$ is a Gaussian channel. Applying Lemma 3 yields

$$I(V; X_2) = 0. \tag{12}$$

However, conditions (9) and (12) implies

$$\begin{aligned} I(X_1; Y_1|U, V) &= I(U, V, X_1; Y_1) - I(U, V; Y_1) \\ &= I(X_1; Y_1) + I(V; Y_1|X_1) \\ &\quad - I(U; Y_1) - I(V; Y_1|U) \\ &\stackrel{(d)}{=} I(X_1; Y_1), \end{aligned}$$

where (d) follows since $I(U; Y_1) \leq I(U; X_1) = 0$ and $I(V; Y_1|U) \leq I(V; Y_1|X_1) \leq I(V; X_2) = 0$. Therefore,

$$I_1 = I(X_1; Y_1) = \mathsf{C}(S/(1+I)) < \mathsf{C}(I/(1+S)),$$

which contradicts (5) and completes the proof for $s = 2$.

Finally, consider the case when $s > 2$. Switching the order of the minimum and the sum, we can bound

$$\begin{aligned} R_1 &< \min\{I(U_1; Y_1), I(U_1; Y_2)\} \\ &\quad + \sum_{j=2}^{s} \min\{I(U_j; Y_1|U^{j-1}, V^{j-1}), I(U_j; Y_2|U^{j-1})\} \\ &\leq I(U_1; Y_1) + I(X_1; Y_2|U_1), \quad (13) \\ R_2 &< \sum_{j=1}^{s-1} \min\{I(V_{2j}, Y_1|V_1^{j-1}, V_2^{j-1}), I(V_{2j}; Y_2|X_1, V_2^{j-1})\} \\ &\qquad + I(V_{2s}; Y_2|X_1, V_2^{s-1}) \\ &\leq \min\{I(V_1, Y_1|U_1), I(V_1; Y_2|X_1)\} + I(X_2; Y_2|X_1, V_1). \quad (14) \end{aligned}$$

Note that (13) and (14) are of the same form as (7) and (10), respectively. Therefore, the suboptimality follows from the same arguments as when $s = 2$.

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