On the Capacity for Distributed Index Coding

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Abstract—The distributed index coding problem is studied, whereby multiple messages are stored at different servers to be broadcast to receivers with side information. First, the existing composite coding scheme is enhanced for the centralized (single-server) index coding problem, which is then merged with fractional partitioning of servers to yield a new coding scheme for distributed index coding. New outer bounds on the capacity region are also established. For all distributed index coding problems with $n \leq 4$ messages and equal server link capacities, the achievable sum-rate of the proposed distributed composite coding scheme match the outer bounds, thus establishing the sum-capacity for these problems.

I. INTRODUCTION

Introduced by Birk and Kol [1], the index coding problem studies the optimal broadcast rate from a server to multiple receivers with some side information about the messages. This paper considers the distributed index coding problem in which, unlike the aforementioned single-server index coding problem, the messages are distributed over multiple servers. The distributed index coding problem was first studied by Ong, Ho, and Lim [2], where lower and upper bounds on the (optimal) broadcast rate were derived in the special case in which each receiver has a distinct message as side information and it is shown that the bounds match if no two servers have any messages in common. Thapa, Ong, and Johnson [3] considered the distributed index coding problem with two servers each having an arbitrary subset of messages and extended some of the existing schemes for the centralized index coding to the two-server distributed case.

The main objective of this paper is to study a general distributed index coding problem and to establish new inner and outer bounds on the capacity region. For the inner bounds, we propose distributed composite coding schemes that extend those in our earlier work [4]. This is done through first enhancing the centralized composite coding scheme, which was originally proposed in [5]. The enhancement is then combined with fractional partitioning of servers to yield a new coding scheme for distributed index coding. We also establish a more general polymatroidal outer bound on the capacity region that can strictly improve the existing polymatroidal outer bound derived in [4]. Using this outer bound, as well as the fd-separation principle for some special cases, we have been able to establish the sum capacity of all 218 non-isomorphic distributed index coding problems with four messages and equal server link capacities. Note that in [4], the capacity region for all distributed index coding problems with up to three messages and equal server link capacities was already established.
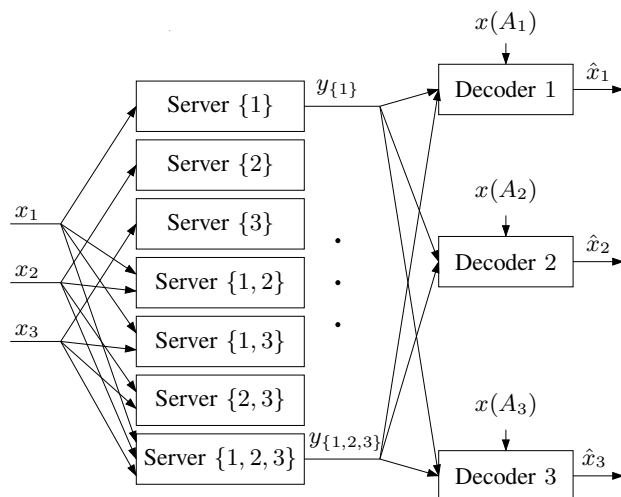

Figure 1. The distributed index coding problem with $n = 3$.

Throughout the paper, $[n]$ denotes the set $\{1, 2, \ldots, n\}$ and $N \doteq \{J \subseteq [n]: J \neq \emptyset\}$ denotes the set of all nonempty subsets of $[n]$. Given a tuple $(x_1, \ldots, x_n)$ and $A \subseteq [n]$, $x(A)$ denotes the subtuple $(x_i : i \in A)$.

II. SYSTEM MODEL AND PROBLEM SETUP

Consider the general distributed index coding problem with n messages $(x_i \in \{0, 1\}^{t_i}: i \in [n])$, as depicted in Fig. 1. There are $2^n - 1$ servers, where server $J \in N$ has access to messages $x(J)$. There are n receivers, where receiver $i \in [n]$ wishes to obtain x_i and knows $x(A_i)$ as side information for some $A_i \subseteq [n] \setminus \{i\}$. Server J is connected to all receivers via a noiseless broadcast link of finite capacity C_J. This model allows for all possible message availabilities on different servers. If $C_J = 1$ only for $J = [n]$ and is zero everywhere else, we recover the centralized index coding problem. The question is to find the maximum amount of information that can be communicated to the receivers and the optimal coding scheme that achieves this maximum. To answer this question formally, we define a $(\mathbf{t}, \mathbf{r}) = ((t_i, i \in [n]), (r_J, J \in N))$ *distributed index code* by

- $2^n - 1$ encoders, one for each server $J \in N$, such that $\phi_J : \prod_{j \in J}\{0, 1\}^{t_j} \rightarrow \{0, 1\}^{r_J}$ maps the messages in server J, $x(J)$, to an r_J-bit sequence y_J, and
- n decoders $\psi_i : \prod_{J \in N}\{0, 1\}^{r_J} \times \prod_{k \in A_i}\{0, 1\}^{t_k} \rightarrow \{0, 1\}^{t_i}$ that maps the received sequences $\phi_J(x_j, j \in J)$ and the side information $x(A_i)$ to $\hat{x}_i$ for $i \in [n]$.

Let $X_1, \ldots, X_n$ be random messages, uniformly distributed and independent of each other, and $\hat{X}_1, \ldots, \hat{X}_n$ be

corresponding message estimates. Also let Y_J be the random variable corresponding to the server output y_J, $\forall J \in N$.

We say that a rate–capacity tuple $(\mathbf{R}, \mathbf{C}) = ((R_i, i \in [n]), (C_J, J \in N))$ is achievable if for every $\epsilon > 0$, there exist a $(\mathbf{t}, \mathbf{r})$ code and r such that

$$R_i \leq \frac{t_i}{r}, \quad i \in [n], \qquad C_J \geq \frac{r_J}{r}, \quad J \in N, \quad (1)$$

and the probability of error

$$\mathsf{P}\{(\hat{X}_1, \ldots, \hat{X}_n) \neq (X_1, \ldots, X_n)\} \leq \epsilon. \quad (2)$$

For a given $\mathbf{C}$, the capacity region $\mathscr{C}$ of this index coding problem is the closure of the set of all $\mathbf{R}$ such that $(\mathbf{R}, \mathbf{C})$ is achievable. Unlike the centralized case in which the capacity region is equal to the zero-error capacity region [6], it is not known whether these two capacity regions are equal for distributed index coding. We will compactly represent a distributed index coding instance (for a given $\mathbf{C}$) by a sequence $(i|j \in A_i)$, $i \in [n]$. For example, for $A_1 = \emptyset$, $A_2 = \{3\}$, and $A_3 = \{2\}$, we write $(1|-)$, $(2|3)$, $(3|2)$.

III. COMPOSITE CODING FOR THE CENTRALIZED CASE

We present a new coding scheme for the centralized problem that extends the composite coding scheme in [5].

A. Existing Centralized Composite Coding Scheme

First, we briefly review the original scheme in [5]. Our slightly modified presentation here will lead to better understanding of both centralized and distributed composite coding schemes that are developed in this paper. There is a single server containing all messages in $[n]$, which is connected to the receivers via a noiseless broadcast channel of capacity C. To each non-empty subset $K \subseteq [n]$ (or $K \in N$) of the messages, we associate a virtual encoder with composite coding rate S_K. In the first step of composite coding, the virtual encoder K maps $x(K)$ into a single composite index w_K, which is generated randomly and independently as a Bern(1/2) sequence of length rS_K bits. In the second step, the server uses flat coding [5] to encode the composite indices $(w_K, K \in N)$ into a single sequence $y \in \{0,1\}^r$.

As with encoding, decoding also has two steps. Each receiver first uses its side information to recover all composite indices $(w_K, K \in N)$. This is successful with vanishing probability of error (for $\mathbf{t}$, $\mathbf{r}$, and r sufficiently large) if

$$\sum_{K: K \nsubseteq A_i} S_K < C, \quad i \in [n]. \quad (3)$$

Each receiver then recovers the desired message (and a subset of other messages) from the composite indices and the side information. Let D_i be the set of the messages that receiver i recovers ($i \in D_i$) and $\Delta = \prod_{i=1}^{n} \mathcal{D}_i$ be the set of all possible decoding set tuples across all receivers, where $\mathcal{D}_i = \{D_i | D_i \subseteq [n] \setminus A_i : i \in D_i\}$ is the set of all possible decoding sets for receiver i. Assume $\mathbf{D} \in \Delta$ is the chosen decoding set tuple. Then, receiver i can successfully recover messages in D_i with vanishing probability of error if

$$\sum_{j \in L} R_j < \sum_{K \subseteq (D_i \cup A_i): K \cap L \neq \emptyset} S_K \quad (4)$$

for all $L \subseteq D_i$. Now let $(\mathbf{R}(\mathbf{D}), \mathbf{S})$ be the set of index coding rate tuples and composite rate tuples that satisfy (3) and (4) for a given decoding choice $\mathbf{D} \in \Delta$, where $\mathbf{R}(\mathbf{D}) = (R_i, i \in [n])$ and $\mathbf{S} = (S_K, K \in N)$. The overall achievable rate region $\mathscr{R}_{\mathrm{CC}}$ can be written as

$$\mathscr{R}_{\mathrm{CC}} = \mathrm{Proj}\left[\mathrm{co}\left[\bigcup_{\mathbf{D} \in \Delta} (\mathbf{R}(\mathbf{D}), \mathbf{S})\right]\right], \quad (5)$$

where "co" denotes the convex hull and "Proj" denotes projecting $(\mathbf{R}, \mathbf{S})$ into $\mathbf{R}$ coordinates. It can be shown that $\mathscr{R}_{\mathrm{CC}}$ can equivalently be computed as

$$\mathscr{R}_{\mathrm{CC}} = \mathrm{co}\left[\bigcup_{\mathbf{D} \in \Delta} \mathrm{Proj}(\mathbf{R}(\mathbf{D}), \mathbf{S})\right]. \quad (6)$$

B. Enhanced Composite Coding

The main idea behind this new method is to allow the composite coding rates S_K to depend on the decoding choices of the receivers. Effectively, composite coding rates can be individually tailored to different decoding choices, as long as they collectively satisfy the conditions of (3). Splitting the rate of each message, we represent the message x_i by independent parts $x_i(\mathbf{D})$ at rate $R_i(\mathbf{D})$. Thus,

$$R_i = \sum_{\mathbf{D} \in \Delta} R_i(\mathbf{D}), \quad i \in [n]. \quad (7)$$

To send $(x_i(\mathbf{D}), i \in [n])$, we generate composite messages $w_K(\mathbf{D})$ at composite coding rate $S_K(\mathbf{D})$, $K \in N$. Each receiver first recovers all composite indices $(w_K(\mathbf{D}), K \in N, \mathbf{D} \in \Delta)$. This is successful with vanishing probability of error if

$$\sum_{\mathbf{D} \in \Delta} \sum_{K: K \nsubseteq A_i} S_K(\mathbf{D}) < C, \quad i \in [n]. \quad (8)$$

For a given $\mathbf{D} \in \Delta$, receiver i can successfully recover messages in D_i with vanishing probability of error if

$$\sum_{j \in L} R_j(\mathbf{D}) < \sum_{K \subseteq (D_i \cup A_i): K \cap L \neq \emptyset} S_K(\mathbf{D}), \quad (9)$$

for all $L \subseteq D_i$. The enhanced achievable rate region $\mathscr{R}_{\mathrm{CC}}^{(e)}$ is obtained by projecting out $(S_K(\mathbf{D}), K \in N, \mathbf{D} \in \Delta)$ and $(R_i(\mathbf{D}), i \in [n], \mathbf{D} \in \Delta)$ through Fourier-Motzkin elimination [7, Appendix D]. The enhanced composite coding inner bound is no smaller than the original composite coding inner bound due to a larger degrees of freedom in choosing the composite rates for different decoding choices $\mathbf{D}$. A possible disadvantage of this method is its computational complexity due to the increase in the number of composite coding rate variables and the necessity to perform a single Fourier-Motzkin elimination operation on all $(S_K(\mathbf{D}))$ and $(R_i(\mathbf{D}))$. To overcome this, one can either apply the technique to a subset of decoding choices in Δ (possibly at the expense of some reduction in the rate region) or use linear programming (LP) to solve for a desired weighted sum-rate or equal rates subject to (7)–(9).

IV. COMPOSITE CODING FOR THE DISTRIBUTED CASE

We now present new composite coding schemes for distributed index coding. First, we apply the enhanced composite coding scheme to all servers as a single group, which is an extension of [4, Sec. IV-B]. Next, we generalize the scheme to fractional partitions of the servers, which is an extension of the partitioned distributed composite coding [4, Sec. IV-C].

A. All-server Distributed Composite Coding

Using rate splitting, similar to the centralized case, we have

$$R_i = \sum_{\mathbf{D} \in \Delta} R_i(\mathbf{D}), \quad i \in [n]. \tag{10}$$

For each non-empty subset $K \subseteq J$ at server $J \in N$ and each decoding choice $\mathbf{D} \in \Delta$, there is a virtual encoder at server J. In the first step of composite coding, virtual encoder K at server J maps $(x_i(\mathbf{D}), i \in K)$ into a composite index $w_{K,J}(\mathbf{D})$ with rate $S_{K,J}(\mathbf{D})$, which is generated randomly and independently as a Bern(1/2) sequence of length $r_J S_{K,J}(\mathbf{D})$ bits. In the second step, server J uses flat coding to encode the composite indices $(w_{K,J}(\mathbf{D}), K \subseteq J, \mathbf{D} \in \Delta)$ into $y_J \in \{0,1\}^{r_J}$.

Each receiver first recovers all composite indices $(w_{K,J}(\mathbf{D}))$, which is successful with vanishing probability of error if

$$\sum_{\mathbf{D} \in \Delta} \sum_{K:K \not\subseteq A_i} S_{K,J}(\mathbf{D}) \le C_J, \quad i \in [n], J \in N. \tag{11}$$

As the second step of decoding, for each decoding choice $\mathbf{D} \in \Delta$, receiver $i \in [n]$ recovers $(x_j(\mathbf{D}), j \in D_i)$ (which includes the desired message $x_i(\mathbf{D})$) using the composite indices and its side information. This is successful with vanishing probability of error if

$$\sum_{j \in L} R_j(\mathbf{D}) < \sum_{K \subseteq (D_i \cup A_i): K \cap L \ne \emptyset} \sum_{J:K \subseteq J} S_{K,J}(\mathbf{D}), \tag{12}$$

for all $L \subseteq D_i$. The second summation on the right hand side of the above ensures that all servers that contain the message subset K are taken into account.

The computational complexity of the enhanced composite coding for distributed index coding is even higher than its centralized counterpart, since the number of composite coding rates rapidly grows with the number of servers. For each decoding set tuple $\mathbf{D} \in \Delta$, there are $\sum_{k=1}^{n} \binom{n}{k}(2^k - 1)$ composite coding rates $S_{K,J}(\mathbf{D})$ and n message rates $R_i(\mathbf{D})$. Hence, even for $n = 4$ and $|\Delta| = 2$, the number of variables to eliminate is $2 \times (65 + 4)$. However, an LP can be solved subject to (10)-(12).

Note that in the scheme proposed in [4, Section IV-B], the rates of the composite messages $S_{K,J}, K \subseteq J, J \in N$ did not depend on the choice of the decoding set tuple $\mathbf{D}$.

B. Fractional Distributed Composite Coding

In the previous scheme, all servers participated in a single group to perform composite coding. However, it is also possible that servers form different groups and participate in group-based composite coding. For each group, they allocate a fraction of their server capacity, hence the name "fractional".

Let Π be the collection of non-empty subsets of N. For each $P \in \Pi$, let $I(P) = \{i | \exists J \in P : i \in J\} \subseteq [n]$ be the union of all messages held by at least one server in P. For every $i \in I(P)$, let $A_i(P) = A_i \cap I(P)$, $\mathcal{D}_i(P) = \{D_i(P) | D_i(P) \subseteq I(P) \setminus A_i(P) : i \in D_i(P)\}$, and $\Delta(P) = \mathcal{D}_1(P) \times \cdots \times \mathcal{D}_n(P)$.

Essentially, we now apply the all-server distributed composite coding methodology to each server group. Then, each receiver is able to decode its desired message with vanishing probability of error if the following sets of inequalities are satisfied. For all $P \in \Pi$ and $J \in P$, we have

$$\sum_{\mathbf{D} \in \Delta(P)} \sum_{K:K \not\subseteq A_i(P)} S_{K,J}(P, \mathbf{D}) \le C_J(P), \quad i \in I(P). \tag{13}$$

For all $P \in \Pi$, $i \in I(P)$ and a given $\mathbf{D} \in \Delta(P)$, (12) is modified as

$$\sum_{j \in L} R_j(P, \mathbf{D}) < \sum_{K \subseteq (D_i \cup A_i(P)): K \cap L \ne \emptyset} \sum_{J:J \in P, K \subseteq J} S_{K,J}(P, \mathbf{D}), \tag{14}$$

for all $L \subseteq D_i$. Finally, we have the following conditions on message rates and server capacities

$$R_i = \sum_{P \in \Pi: i \in I(P)} \sum_{\mathbf{D} \in \Delta(P)} R_i(P, \mathbf{D}), \qquad i \in [n], \tag{15}$$

$$C_J \ge \sum_{P \in \Pi: J \in P} C_J(P), \qquad J \in N. \tag{16}$$

The achievable rate region is characterized by eliminating the variables $(C_J(P), P \in \Pi, J \in P)$, $(R_i(P, \mathbf{D}), P \in \Pi, i \in I(P), \mathbf{D} \in \Delta(P))$, and $(S_{K,J}(P, \mathbf{D}), P \in \Pi, J \in P, K \subseteq J, \mathbf{D} \in \Delta(P))$.

A possible advantage of this technique is that each receiver only recovers composite indices of the groups that hold its desired message. Consider the index coding problem

$$(1|2,3), (2|1,3), (3|1,2), (4|-).$$

Without server grouping, for receiver $i = 4$ and server $J = \{1,2\}$ we have

$$\sum_{\mathbf{D}} \sum_{K:K \not\subseteq A_4} S_{K,J}(\mathbf{D}) = \sum_{\mathbf{D}} \left[S_{\{1\},\{1,2\}}(\mathbf{D}) + S_{\{2\},\{1,2\}}(\mathbf{D}) + S_{\{1,2\},\{1,2\}}(\mathbf{D}) \right] \le C_J. \tag{17}$$

With partitioning, let us consider the server group $P = \{\{1,2\}, \{2,3\}\}$. Since $i = 4$ does not belong to this group, we do not have the above constraint for that group. A disadvantage of this technique is its computational complexity. For n messages, the number of possible non-empty server subsets is $|\Pi| = 2^{2^n - 1} - 1$, which is doubly exponential in n. As a result, Fourier-Motzkin elimination is not practical. Linear programs can be solved by excluding some servers from the problem (for example through dealing with trivial singleton servers, $|J| = 1$, in N separately), by considering only a subset of Π (at the possible expense of some reduction in the rate region), or when $\max |\Delta|$ is not very large. Clearly the fractional distributed composite coding scheme reduces to

the all-server scheme of Section IV-A when all the servers are put into a single group. At this point, however, we do not know any example for which the inner bound corresponding to the fractional distributed composite coding is strictly larger.

Remark 1: The idea of fractional partitioning of servers as an improvement of [4, Sec. IV-C], is independently developed in a recent paper [8]. The coding scheme in [8] is then further enhanced by using common messages in the servers in a cooperative manner. This enhancement by cooperation and the enhancement introduced in (13) in the current paper are not comparable. It seems quite natural to combine the two methods into "enhanced cooperative fractional composite coding," the investigation of which is left for future work.

V. Outer Bounds

For any set $S \subseteq [n]$, we define $X_S \doteq (X_i : i \in S)$. Similarly, for any set $P \subseteq N$, we define $Y_P \doteq (Y_J : J \in P)$. The complement of set S with respect to its ground set (being $[n]$ unless otherwise stated) is denoted by S^c. The decoding condition at receiver i stipulates $H(X_i|Y_N, X_{A_i}) = \delta(\epsilon)$ with $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$ by Fano's inequality. For simplicity, we assume exact decoding and $\delta(\epsilon) \equiv 0$.

We now restate an outer bound on the capacity region [4].

Theorem 2: Let $B_i = [n] \setminus (A_i \cup \{i\})$. If $(\mathbf{R}, \mathbf{C})$ is achievable, then for every $T \in N$ and every $i \in T$,

$$R_i \leq f_T(B_i \cup \{i\}) - f_T(B_i) \tag{18}$$

for some set function $f_T(S)$, $S \subseteq T$, such that

1) $f_T(\emptyset) = 0$,
2) $f_T(T) = \sum_{J:J \cap T \neq \emptyset} C_J$,
3) $f_T(A) \leq f_T(B)$, $\forall A \subseteq B \subseteq T$, and
4) $f_T(A \cup B) + f_T(A \cap B) \leq f_T(A) + f_T(B), \forall A, B \subseteq T$.

The above theorem utilizes the polymatroidal axioms, yet is not powerful enough and does not give tight results in general. Now we introduce a new outer bound, also based on the polymatroidal axioms, which is strictly tighter than Theorem 2. The following structure on servers will be useful.

Definition 3: We say that set J "touches" set T if $J \cap T \neq \emptyset$ and does not touch set T if $J \cap T = \emptyset$. Based on this, we define the following:

- $J_T \doteq (J \subseteq [n], J \cap T \neq \emptyset)$,
- $J_{\overline{T}} \doteq (J \subseteq [n], J \cap T = \emptyset) = (J \subseteq [n], J \subseteq T^c)$,
- $J_{T,S} \doteq (J \subseteq [n], J \cap T \neq \emptyset, J \cap S \neq \emptyset)$,
- $J_{T,\overline{S}} \doteq (J \subseteq [n], J \cap T \neq \emptyset, J \cap S = \emptyset)$, and so on which can be also extended to three or more sets.
- Note that $J_{T,S} = J_S$ for all $S \subseteq T$. Due to symmetry, $J_{T,S} = J_{S,T}$.

We use $Y_{J_T} = (Y_J : J \cap T \neq \emptyset)$ to denote the output of servers that have at least one message from T. For example, $Y_{J_{[n]}}$, means the outputs from all servers. When the context is clear, we use the shorthand notation J_T to refer to Y_{J_T}.

Theorem 4: If $(\mathbf{R}, \mathbf{C})$ is achievable, then for every $i \in [n]$,

$$R_i \leq g([n], B_i \cup \{i\}) - g([n], B_i), \tag{19}$$

for some $g(T, S)$, $T, S \subseteq [n]$, such that

1) $g(T, S) = g(S, S)$, if $S \subseteq T$,

2) $g(T, S) = 0$, if $T = \emptyset$ or $S = \emptyset$,
3) $g(T, S) \leq \sum_{J:J \cap T \neq \emptyset, J \cap S \neq \emptyset} C_J$, $\forall T, S$,
4) $g(T, S) \leq g(T', S')$, if $S \subseteq S'$ and $T \subseteq T'$,
5) $g(T \cup T', S \cap S') + g(T \cap T', S \cup S') \leq g(T, S) + g(T', S')$, $\forall T, T', S, S'$.

Proof: For all $i \in [n]$, due to the decoding conditions as well as source independence, we have

$$rR_i \leq t_i = H(X_i) = H(X_i|X_{A_i}) - H(X_i|J_{[n]}, X_{A_i})$$
$$= H(J_{[n]}|X_{A_i}) - H(J_{[n]}|X_{A_i \cup \{i\}}).$$

Now, for $T, S \subseteq [n]$ define

$$g(T, S) \doteq \frac{1}{r} H(J_T | X_{S^c}). \tag{20}$$

Then for every $i \in [n]$ we have

$$R_i \leq g([n], B_i \cup \{i\}) - g([n], B_i). \tag{21}$$

Due to space limitations, we only provide the proofs of $g(T, S)$ satisfying axioms 1 and 5. The other proofs are relatively straightforward and thus omitted here. For axiom 1, by breaking J_T into two sets $J_{T,S}$ and $J_{T,\overline{S}}$ we obtain

$$H(J_T | X_{S^c}) = H(J_{T,S}, J_{T,\overline{S}} | X_{S^c}) \tag{22}$$
$$= H(J_{T,S} | X_{S^c}) + H(J_{T,\overline{S}} | X_{S^c}, J_{T,S}). \tag{23}$$

By the definition of distributed index code, $H(J_{\overline{S}}|X_{S^c}) = H(J \subseteq S^c|X_{S^c}) = 0$. Since $J_{T,\overline{S}} \subseteq S^c$, we conclude that

$$H(J_{T,\overline{S}} | X_{S^c}, J_{T,S}) = 0.$$

Now if $S \subseteq T$, we further have $J_{T,S} = J_S$. We now focus on proving axiom 5, which states $\forall T, T', S, S'$,

$$g(T \cup T', S \cap S') + g(T \cap T', S \cup S')$$
$$\leq g(T, S) + g(T', S'). \tag{24}$$

We first define three disjoint sets $S_1 = S^c \setminus S'^c$, $S_2 = S'^c \setminus S^c$ and $S_0 = S^c \cap S'^c$, so that $S^c \cup S'^c = S_0 \cup S_1 \cup S_2$, $S^c = S_0 \cup S_1$ and $S'^c = S_0 \cup S_2$. Similarly, we define three disjoint sets $T_1 = T \setminus T'$, $T_2 = T' \setminus T$ and $T_0 = T \cap T'$, so that $T \cup T' = T_0 \cup T_1 \cup T_2$, $T = T_0 \cup T_1$ and $T' = T_0 \cup T_2$.

Add $H_1 = H(X_{S_0 \cup S_1 \cup S_2}) + H(X_{S_0})$ to the LHS and $H_2 = H(X_{S_0 \cup S_1}) + H(X_{S_0 \cup S_2})$ to the RHS of (24), where $H_1 = H_2$ due to source independence and disjointness of S_0, S_1, S_2. We must prove $\forall T, T', S, S'$, the following holds

$$H(J_{T_0 \cup T_1 \cup T_2}, X_{S_0 \cup S_1 \cup S_2}) + H(J_{T_0}, X_{S_0})$$
$$\leq H(J_{T_0 \cup T_1}, X_{S_0 \cup S_1}) + H(J_{T_0 \cup T_2}, X_{S_0 \cup S_2}). \tag{25}$$

Denote the LHS and RHS of the above inequality by L and R, respectively. Define the following three sets:

- $J_1 = \{J_{T_0, T_1, \overline{T_2}}, J_{T_0, \overline{T_1}, T_2}, J_{\overline{T_0}, T_1, \overline{T_2}}\}$,
- $J_2 = \{J_{T_0, \overline{T_1}, T_2}, J_{T_0, \overline{T_1}, \overline{T_2}}, J_{\overline{T_0}, \overline{T_1}, T_2}\}$,
- $J_0 = \{J_{T_0, T_1, T_2}, J_{T_0, \overline{T_1}, T_2}, J_{\overline{T_0}, T_1, T_2}, J_{T_0, T_1, \overline{T_2}}, J_{T_0, \overline{T_1}, \overline{T_2}}\}$.

It can be verified that $J_{T_0 \cup T_1 \cup T_2} = J_0 \cup J_1 \cup J_2$, $J_{T_0 \cup T_1} = J_0 \cup J_1$, $J_{T_0 \cup T_2} = J_0 \cup J_2$ and $J_{T_0} \subseteq J_0$. Hence, we can show that

$$L \leq H(J_0 \cup J_1 \cup J_2, X_{S_0 \cup S_1 \cup S_2}) + H(J_0, X_{S_0}) \tag{26}$$
$$\leq H(J_0 \cup J_1, X_{S_0 \cup S_1}) + H(J_0 \cup J_2, X_{S_0 \cup S_2}) \tag{27}$$
$$= R, \tag{28}$$

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Table I

ACHIEVABLE SUM-RATE OF ALL DISTRIBUTED INDEX CODING PROBLEMS WITH $n = 4$ MESSAGES AND $C_J = 1$, $\forall J \in N$ IN INCREASING ORDER. FOR ALL SUCH 218 NON-ISOMORPHIC PROBLEMS, THE ACHIEVABLE SUM-RATE MATCH THE OUTER BOUNDS, THUS ESTABLISHING THE SUM-CAPACITY.

$R_1 + R_2 + R_3 + R_4$	Problem Number
15	1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 15, 17, 19, 20, 22, 25, 26, 33, 35, 38, 39, 40, 41, 49, 63, 65, 67, 69, 70, 100
18.667	47
19	**4, 9, 18, 21, 23, 24, 34, 36, 48, 55, 64, 66, 68, 86, 95, 99, 138**, $\overline{16, 30, 60, 102}$
20	43, 78, 83, 85, 130, 132
21	**14, 27, 28, 29, 31, 32, 37, 50, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 87, 88, 89, 90, 91, 92, 94, 96, 97, 98, 101, 134, 136, 137, 139, 140, 141, 173**
22	42, 44, 45, 71, 72, 73, 74, 75, 76, 77, 79, 80, 82, 84, 103, 104, 105, 106, 107, 108, 109, 110, 111, 113, 116, 117, 118, 120, 122, 123, 124, 125, 126, 127, 128, 131, 133, 142, 143, 144, 145, 147, 151, 152, 153, 154, 158, 159, 161, 162, 163, 164, 165, 166, 167, 168, 169, 174, 177, 182, 183, 184, 185, 186, 187, 201
23.333	$\overline{46}$
23.5	$\overline{81, 112, 115, 119, 148}$
24	114, 121, 129, 146, 150, 155, 156, 157, 160, 170, 171, 175, 178, 180, 181, 188, 189, 190, 191, 192, 194, 195, 196, 197, 198, 202, 204, 206, 208, 210, 216
25	**93, 135, 172, 199**
26	207, **149, 176, 179, 200, 203, 212**
28	193, 205, 209, 211, 213, 214, 215, 217
32	218

where (26) is due to the fact that $J_{T_0} \subseteq J_0$ and (27) is due to the sub-modularity of the entropy function. Therefore, $g(T, S)$ satisfying axiom 5 has been proved. ∎

Theorem 4 is not weaker than Theorem 2. Indeed, whenever there exists a set function $g(T, S)$ satisfying the axioms of Theorem 4, there always exists a set function $f_T(S)$, for every $T \in N$, satisfying the axioms of Theorem 2. This can be easily proved by setting $f_T(S) = g(T, S), \forall S \subseteq T \subseteq [n]$. In summary, let $\mathscr{R}_2$ and $\mathscr{R}_4$ be the outer bounds given by Theorems 2 and 4, respectively. Then, $\mathscr{R}_4 \subseteq \mathscr{R}_2$. Similar to the inner bounds, Fourier-Motzkin elimination is not practical for computing the outer bound of Theorem 4 due to its computational complexity, while LP can be used instead.

VI. NUMERICAL RESULTS

We numerically evaluated inner and outer bounds on the sum-rate of all 218 non-isomorphic problems with $n = 4$ messages. We set $C_J = 1$ for all $J \in N$. The side information of these problems are listed in [9]. The inner bound is computed by applying LP on the composite coding scheme of Section IV-A and compares with the outer bound of Theorems 2 and 4 on the sum-rate as follows.

- For 145 out of 218 problems, the outer bound on the sum-rate due to Theorem 2 and the inner bound matched. These cases are shown normally in Table V.
- For 63 out of the remaining 73 cases, the outer bound of Theorem 4 gave a tighter result than Theorem 2 and matched the inner bound. These cases are shown in bold font in Table V.
- For the final 10 cases, the outer bounds of Theorems 2 or 4 did not match the inner bound, and we had to utilize fd-separation [10], [11] to obtain a tighter outer bound that matched the inner bound. The details are omitted due to space limitations. These cases are shown as overlined in Table V.

To conclude, we report that for 28 out of these 218 problems, the all-server composite coding scheme of [4, Section IV-B] gives a looser inner bound on the sum-rate than the enhanced all-server method of Section IV-A in this paper. For example, for problem 155: $(1|4), (2|3, 4), (3|1, 2), (4|2, 3)$, [4, Section IV-B] gives $\sum_{i \in [n]} R_i \leq 23$ (even after convexification over all decoding sets), whereas the sum-capacity 24 is achievable through the method of Section IV-A. In effect, server partitioning of [4, Section IV-C] is not necessary to achieve the sum-capacity for this problem. Our ongoing research is to establish whether and when the fractional method of Section IV-B offers strict improvement over the enhanced all-server method of Section IV-A.

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