Two-Way Token Passing Channels

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***Abstract*—An interactive communication channel model in which information is exchanged through the action of tossing one or more tokens, potentially of multiple types, is proposed. Two scenarios—point-to-point communication with token feedback and two-way communication—are studied. The capacity of the token passing channel with token feedback and the capacity region for the two-way token passing channel are characterized.**

I. Introduction

We propose a two-way token passing channel (TWTPC) model. In this model, there are finitely many tokens of different types and two agents that aim to send a sequence of symbols to one another via the action of tossing tokens through the channel.

Notation: Throughout this paper, we use boldface letters (e.g., $\mathbf{V}$) to denote vector-valued variables. A sequence of vectors $(\mathbf{V}_1, \ldots, \mathbf{V}_n)$ is denoted as $\mathbf{V}^n$. We write $\mathbf{V} \preceq \mathbf{W}$ if every component of $\mathbf{W} - \mathbf{V}$ is nonnegative.

Token: Suppose that there are L types of tokens and, for each type $\ell \in [L]$, there are b_ℓ tokens. It is assumed that tokens of the same type are indistinguishable. Thus, these tokens form a multiset, which is denoted by the multiplicity vector $\mathbf{b} = (b_1, b_2, \ldots, b_\ell)$. The vector $\mathbf{b}$ is fixed and known to both agents.

Channel description: The communication is assumed to be performed in a time-slotted manner. For $j = 1$, 2, we denote the tokens held by agent j at *the beginning of* the ith timeslot as $\mathbf{S}_i^{(j)}$. We refer to the variable pair $(\mathbf{S}_i^{(1)}, \mathbf{S}_i^{(2)})$ as the state of the channel at the ith timeslot. Before the communication starts, all the tokens are assumed to be possessed by agent 1, i.e., $(\mathbf{S}_1^{(1)}, \mathbf{S}_1^{(2)}) = (\mathbf{b}, \mathbf{0})$.

Each timeslot has two phases—the transmission phase and the reception phase. In the transmission phase, agent j selects up to η tokens from the ones it has as channel input $\mathbf{X}_i^{(j)}$ and pushes it towards the other agent. Next, in the reception phase, agent $j+1$ receives $\mathbf{Y}_i^{(j+1)} = \mathbf{X}_i^{(j)}$ as channel output and stores the tokens it gained[1]. The channel states, the channel inputs and the channel outputs are related through the state transition equation

$$\mathbf{S}_{i+1}^{(j)} = \mathbf{S}_i^{(j)} - \mathbf{X}_i^{(j)} + \mathbf{Y}_i^{(j)}, \tag{1}$$

for $j = 1$, 2. Note that from (1) we have

$$\mathbf{S}_i^{(1)} + \mathbf{S}_i^{(2)} = \mathbf{b} \qquad \text{for all } i \in \mathbb{N}. \tag{2}$$

Therefore, both agents are clearly aware of the channel state.

[1]For convenience, indicies with the same parity refer to the same agent.

Related work and contribution: Some other channel models that consider information exchange via the transfer of discrete physical resources have been proposed in the literature. For example, Tutuncuoglu et al. [1–3] had extensive research on point-to-point communication over the binary energy harvesting channel (BEHC), in which the transmitter harvests discretized energy stochastically from an exogenous source to sustain power needed for transmission of binary data[2]. They developed an upper bound on the capacity of the BEHC and a lower bound which is asymptotically capacity-achieving at low energy harvesting rate.

For multi-user channels, Popovski et al. [4] investigated the two-way binary communication with discrete energy exchange. In contrast to the BEHC, where the energy is assumed to be harvested from an external source independent of the communication process, the model in [4] is a closed system in which the agents can only harvest energy from the received signal. Popovski et al. derived an inner and an outer bound on the capacity region of this two-way binary energy exchange channel (TWBEEC). These bounds, though seem to be close from the numerical results, do not match when the system has more than one unit of energy. Our TWTPC is essentially a generalization in this line of work. When having only one single token type $L = 1$ and input constraint $\eta = 1$, the TWTPC boils down to the TWBEEC. In this paper, we characterize the capacity region of the TWTPC, and therefore the capacity region of the TWBEEC as a special case. In addition, we investigate the scenario in which agent 2 simply provides feedback to help the point-to-point communication from agent 1 to agent 2. We refer to this point-to-point channel as the token passing channel with token feedback (TPC-TF) and characterize its capacity.

We note that in [4], Popovski et al. also considered a noisy version of the TWBEEC, where loss and replenishment is assumed to occur randomly during energy transfer. They proposed an inner bound and an outer bound on the capacity region for this model, and the inner bound was later improved by Huang and Lee [5]. The characterization of the capacity region of the noisy TWBEEC still remains an open problem.

Organization: We first study the point-to-point communication over the TPC-TF in Section II and then the two-way communication over the TWTPC in Section III. Finally, Section IV concludes the paper.

[2]In the BEHC model, it is assumed that sending a symbol 1 costs one unit of energy, whereas sending symbol 0 does not require any energy expenditure.

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II. Token Passing Channels with Token Feedback

Consider the point-to-point communication from agent 1 to agent 2 with the following feedback mechanism. At the transmission phase of the ith timeslot, agent 2 returns all the tokens it received in the $(i-1)$st timeslot to agent 1, i.e.,

$$\mathbf{X}_i^{(2)} = \mathbf{Y}_{i-1}^{(2)}. \tag{3}$$

We note that the objective of the feedback signal $\mathbf{X}_i^{(2)}$ here is not providing information back from the receiver to the transmitter but giving the transmitter the best possible flexibility for the subsequent transmission. In this section, we study the capacity of the TPC-TF.

A. Communication over the TPC-TF

Suppose that agent 1 wishes to convey a message $M \sim \mathrm{Unif}([1:2^{nR}])$ to agent 2 through a TPC-TF. A $(2^{nR}, n)$ code for the TPC-TF consists of the following:

- A message set $[1:2^{nR}]$,
- An encoder that assigns a symbol $\mathbf{x}_i^{(1)}(m, \mathbf{s}^{(1),i})$ to each pair $(m, \mathbf{s}^{(1),i})$ for $i \in [n]$, and
- A decoder that assigns an estimate $\widehat{m}(\mathbf{y}^{(2),n})$ to each $\mathbf{y}^{(2),n}$.

The average error probability is defined as

$$P_e^{(n)} = \mathsf{P}\left(M \neq \widehat{M}\right). \tag{4}$$

A rate R is said to be achievable if there exists a sequence of $(2^{nR}, n)$ codes such that

$$\lim_{n\to\infty} P_e^{(n)} = 0. \tag{5}$$

The capacity $\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b}, \eta)$ of the TPC-TF is the closure of the set of achievable rates. When there is no input constraint, we simply denote the capacity as $\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b})$.

B. Capacity of the TPC-TF

Having established the operational definition of the capacity for the TPC-TF, we now give a information theoretical characterization of $\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b}, \eta)$. Note that due to (1), (3) and the assumption $\mathbf{X}_i^{(2)} = \mathbf{Y}_i^{(1)}$, it can be deduced by induction that

$$\mathbf{S}_i^{(1)} = \mathbf{b} - \mathbf{X}_{i-1}^{(1)}. \tag{6}$$

for all $i \in [n]$. Let

$$\mathcal{X}(\mathbf{s}) = \left\{ \mathbf{x} : \mathbf{0} \preceq \mathbf{x} \preceq \mathbf{s}, \sum_{\ell=1}^{L} x_\ell \leq \eta \right\} \tag{7}$$

denote the set of available symbols at the encoder at the beginning of timeslot i given that the local state $\mathbf{S}_i^{(1)} = \mathbf{s}$. We have the following result.

Theorem 1. *The capacity $\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b}, \eta)$ of the TPC-TF with token set $\mathbf{b}$ and input constraint η is equal to*

$$\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b}, \eta) = \limsup_{n\to\infty} \frac{\log |\mathcal{C}_n|}{n}, \tag{8}$$

where

$$\mathcal{C}_n = \{\mathbf{x}^n : \mathbf{x}_1 \in \mathcal{X}(\mathbf{b}), \mathbf{x}_i \in \mathcal{X}(\mathbf{b} - \mathbf{x}_{i-1}) \text{ for } i \in [2:n]\}$$

is the set of all valid length-n input sequences.

Proof: Let $\mathbf{X}^n \sim \mathrm{Unif}(\mathcal{C}_n)$. Since $\mathcal{C}_n$ is the set of all valid input sequences, the standard converse proof technique yields the capacity upper bound $\limsup_{n\to\infty} \frac{H(\mathbf{X}^n)}{n} = \limsup_{n\to\infty} \frac{\log|\mathcal{C}_n|}{n}$. In addition, the rate $\frac{\log|\mathcal{C}_n|}{n}$ is achieved by the TPC-TF code, where the encoder maps bijectively each message $m \in [1:|\mathcal{C}_n|]$ to a sequence in $\mathcal{C}_n$ and the decoder decodes $\mathbf{y}^n = \mathbf{x}^n$ by finding the unique message m that corresponds to $\mathbf{x}^n$. ■

Remark 1. When there is no input constraint, we can decouple the TPC-TF into L parallel TPC-TFs, each of which possesses one single type of tokens. Thus, we have

$$\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b}) = \sum_{\ell=1}^{L} \mathscr{C}_{\mathsf{TPC-TF}}(b_\ell). \tag{9}$$

C. Algebraic Characterization of the Capacity

We present an algebraic method to compute the capacity $\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b}, \eta)$. Consider the directed graph $G = (V, E)$ on the state space $V = \mathcal{X}(\mathbf{b})$ with edge set

$$E = \{(\mathbf{s}, \mathbf{s}') : \mathbf{b} - \mathbf{s}' \in \mathcal{X}(\mathbf{s})\}. \tag{10}$$

In other words, there is an edge pointing $\mathbf{s}'$ from $\mathbf{s}$ if and only if $\mathbf{s}'$ is a valid next state given that $\mathbf{s}$ is the current local state at agent 1. Since G is a graph with no parallel edges (i.e., two edges that share the same initial and terminal vertices), every length-n path in G is uniquely determined by a length-$(n+1)$ sequence of vertices. In addition, due to (6), every valid length-n input sequence $\mathbf{x}^n$ is uniquely generated by a length-$(n+1)$ state sequence $\mathbf{s}^{n+1}$ that starts with $\mathbf{s}_1 = \mathbf{b}$. Thus, the cardinality of $\mathcal{C}_n$ is equal to the number of length-n paths in G that starts from the vertex $\mathbf{b}$.

Let $G_{\mathbf{b}}$ be the largest irreducible component of G that contains $\mathbf{b}$. It can be shown that the capacity is determined by the spectral radius of the adjacency matrix of $G_{\mathbf{b}}$.

Theorem 2. *The capacity $\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b}, \eta)$ of the TPC-TF with token set $\mathbf{b}$ and input constraint η is equal to*

$$\mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b}, \eta) = \log \lambda(\mathsf{A}), \tag{11}$$

where A is the adjacency matrix of $G_{\mathbf{b}}$, and $\lambda(\mathsf{A})$ is the spectrum radius of A (also known as the Perron-Frobenius eigenvalue of A).

Theorem 2 follows from the constraint coding theory, by which it can be shown that the limsup in (8) can be replaced by a proper limit. For a detailed proof, we refer an interested reader to the handbook [6]. Here we demonstrate this spectral graph theoretic approach through the following two examples.

Example 1. Consider the simplest case that the TPC-TF has exactly one token and no input constraint. For this channel, we have $G = G_{\mathbf{b}}$. The adjacency matrix A of $G_{\mathbf{b}}$ is equal to

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The Perron-Frobenius eigenvalue λ of A is the largest root of the characteristic polynomial $p_A(t) = t^2 - t - 1$ of A, which is equal to the golden ratio $\phi = \frac{\sqrt{5}+1}{2}$. Therefore, the capacity $\mathscr{C}_{\mathsf{TPC-TF}}(1) = \log \phi$.

Example 2. Consider the case that $\mathbf{b} = (1, 1)$ and $\eta = 1$. Different from the previous example, the TPC-TF now has the input constraint $\eta = 1$. The input sequence generator graph G is depicted in Figure 1. For this channel, the largest component

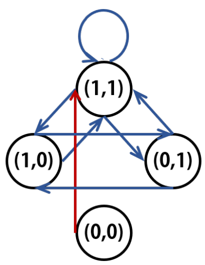

Fig. 1: Input sequence generator graph for the MBPC with token set $(1, 1)$ and input constraint $\eta = 1$.

$G_{\mathbf{b}}$ of G that contains $\mathbf{b} = (1, 1)$ is the subgraph induced by the blue edges. The adjacency matrix A of $G_{\mathbf{b}}$ is given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The Perron-Frobenius eigenvalue λ of A is the largest root of the characteristic polynomial $p_A(t) = t^3 - t^2 - 3t + 1$ of A. The capacity $\mathscr{C}_{\mathsf{TPC-TF}}((1, 1), 1)$ is equal to $\log(\lambda) \approx 1.2716$.

III. Two-Way Token Passing Channels

In this section, we characterize the capacity region of the TWTPC.

A. Communication over the TWTPC

First we define the channel code for the TWTPC. A $(2^{nR_1}, 2^{nR_2}, n)$ code for the TWTPC consists of the following:

- Two message sets $[1 : 2^{nR_1}]$ and $[1 : 2^{nR_2}]$,
- Two encoders, where, for $j = 1, 2$, encoder j assigns a symbol $\mathbf{x}_i^{(j)}(m_j, \mathbf{s}^{(j),i}, \mathbf{y}^{(j),i-1})$ to each tuple $(m_j, \mathbf{s}^{(j),i}, \mathbf{y}^{(j),i-1})$ for $i \in [n]$, and
- Two decoders, where, for $j = 1, 2$, decoder j assigns an estimate $\widehat{m}_{j+1}(m_j, \mathbf{s}^{(j),n}, \mathbf{y}^{(j),n})$ to each tuple $(m_j, \mathbf{s}^{(j),n}, \mathbf{y}^{(j),n})$.

Suppose that the messages (M_1, M_2) are uniformly distributed over $[1 : 2^{nR_1}] \times [1 : 2^{nR_2}]$. The average error probability is defined as

$$P_e^{(n)} = \mathsf{P}\left((M_1, M_2) \neq (\widehat{M}_1, \widehat{M}_2)\right). \quad (12)$$

A rate pair (R_1, R_2) is said to be achievable if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$. The capacity region $\mathscr{C}_{\mathsf{TWTPC}}(\mathbf{b}, \eta)$ of the TWTPC is the closure of the set of the achievable rate pairs (R_1, R_2).

B. Capacity Region of the TWTPC

Our next theorem provides a single-letter characterization for the capacity region of the TWTPC. Here we use the shorthand notations $\mathbf{S} = (\mathbf{S}^{(1)}, \mathbf{S}^{(2)})$ and $\mathbf{X} = (\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$. Also, we use

$$\mathcal{S} = \{\mathbf{s} = (\mathbf{s}^{(1)}, \mathbf{s}^{(2)}) : \mathbf{0} \preceq \mathbf{s}^{(1)} \preceq \mathbf{b}, \mathbf{s}^{(2)} = \mathbf{b} - \mathbf{s}^{(1)}\}$$

to denote the range of $\mathbf{S}$.

Theorem 3. *The capacity region $\mathscr{C}_{\mathsf{TWTPC}}(\mathbf{b}, \eta)$ is the set of rate pairs (R_1, R_2) that satisfy*

$$R_1 \leq H(\mathbf{X}^{(1)}|\mathbf{S}, Q) \quad (13a)$$
$$R_2 \leq H(\mathbf{X}^{(2)}|\mathbf{S}, Q) \quad (13b)$$

for some pmf $p(\mathbf{s}, q)p(\mathbf{x}^{(1)}|\mathbf{s}, q)p(\mathbf{x}^{(2)}|\mathbf{s}, q)$ with $|\mathcal{Q}| \leq 2 + \sum_{\mathbf{s}\in\mathcal{S}} |\mathcal{X}(\mathbf{s}^{(1)})||\mathcal{X}(\mathbf{s}^{(2)})|$ such that

$$\mathsf{P}(\mathbf{S} = \mathbf{s}') = \sum_{(\mathbf{s},\mathbf{x})\in\mathcal{Z}(\mathbf{s}')} \mathsf{P}(\mathbf{S} = \mathbf{s}, \mathbf{X} = \mathbf{x}) \quad (14)$$

for all $\mathbf{s}' = (\mathbf{s}'^{(1)}, \mathbf{s}'^{(2)}) \in \mathcal{S}$. Here the conditional pmfs $p(\mathbf{x}^{(j)}|q), j = 1, 2$, are supported on the set $\mathcal{X}(\mathbf{s}^{(j)})$ and the steady-state equation (14) is defined over the set

$$\mathcal{Z}(\mathbf{s}') = \left\{ (\mathbf{s}, \mathbf{x}) : \begin{array}{c} \mathbf{x}^{(1)} \in \mathcal{X}(\mathbf{s}^{(1)}) \\ \mathbf{x}^{(2)} \in \mathcal{X}(\mathbf{s}^{(2)}) \\ \mathbf{s}'^{(1)} = \mathbf{s}^{(1)} - \mathbf{x}^{(1)} + \mathbf{x}^{(2)} \\ \mathbf{s}'^{(2)} = \mathbf{s}^{(2)} - \mathbf{x}^{(2)} + \mathbf{x}^{(1)} \end{array} \right\}. \quad (15)$$

The achievability part of Theorem 3 is proved by random coding and random time sharing, whereby a codeword is generated for each tuple $(m_j, \mathbf{s}, q)$ with identically independently distributed (i.i.d.) entries according to the conditional pmf $p(\mathbf{x}^{(j)}|\mathbf{s}, q)$. At the ith timeslot, the encoder transmits the ith symbol from the codeword associated to the current state and a randomly drawn time sharing variable. On the other hand, the key of the converse proof is to establish the conditional independence of $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ given $(\mathbf{S}, Q)$. To this end, we adopted the proof technique similar to what Hekstra and Willems used to determine the capacity region of class $\mathscr{D}_1$ channels [7, Corollary 2]. A complete proof is given in Appendix A.

Remark 2. For TWTPCs with one token type $L = 1$ and input constraint $\eta = 1$, (13) coincides with the inner bound region given in [4, Proposition 1], showing that the bound is in fact tight for the TWBEEC.

Remark 3. Consider the individual capacity, which is defined as

$$C_1(\mathbf{b}, \eta) = \max_{(R_1,R_2)\in\mathscr{C}_{\mathsf{TWTPC}}(\mathbf{b},\eta)} R_1. \quad (16)$$

To maximize R_1, the optimal strategy for agent 2 is to return all the token it received in the previous timeslot to agent 1 since it will maximize the number of available symbols at agent 1 at the beginning of the next timeslot. As a result, we have $C_1(\mathbf{b},\eta) = \mathscr{C}_{\mathsf{TPC-TF}}(\mathbf{b},\eta)$.

IV. Conclusion

In this paper, we studied point-to-point communication over the TPC-TF and multiuser communication over the TWTPC. For the TPC-TF, we provided a multi-letter characterization of the capacity and showed how it can be computed from the Perron-Frobenius eigenvalue of the adjacency matrix of the state transition graph. Next, for the TWTPC, we characterized its capacity region. As a special case, our result for the TWTPC characterizes the capacity region of the TWBEEC in the literature.

Appendix A
Proof of Theorem 3

A. Proof of the Converse

Suppose that a rate pair (R_1, R_2) is achievable, which means that there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$. By Fano's inequality and the data processing inequality,

$$H(M_1|M_2, \mathbf{S}^{(2),n}, \mathbf{Y}^{(2),n}) \le H(M_1|\widehat{M}_1) \le n\epsilon_n, \tag{17}$$

where $\epsilon_n \to 0$ as $n \to \infty$. Now consider

$$\begin{aligned}
nR_1 &= H(M_1)\\
&\overset{(a)}{=} H(M_1|M_2)\\
&= I(M_1; \mathbf{S}^{(2),n}, \mathbf{Y}^{(2),n}|M_2) + H(M_1|M_2, \mathbf{S}^{(2),n}, \mathbf{Y}^{(2),n})\\
&\le I(M_1; \mathbf{S}^{(2),n}, \mathbf{Y}^{(2),n}|M_2) + n\epsilon_n\\
&\le H(\mathbf{S}^{(2),n}, \mathbf{Y}^{(2),n}|M_2) + n\epsilon_n\\
&= \sum_{i=1}^n H(\mathbf{S}_i^{(2)}, \mathbf{Y}_i^{(2)}|M_2, \mathbf{S}^{(2),i-1}, \mathbf{Y}^{(2),i-1}) + n\epsilon_n\\
&\overset{(b)}{=} \sum_{i=1}^n H(\mathbf{Y}_i^{(2)}|M_2, \mathbf{S}^{i}, \mathbf{X}^{(2),i-1}, \mathbf{Y}^{(2),i-1}) + n\epsilon_n\\
&\overset{(c)}{=} \sum_{i=1}^n H(\mathbf{X}_i^{(1)}|M_2, \mathbf{S}^{i}, \mathbf{X}^{(1),i-1}, \mathbf{X}^{(2),i-1}) + n\epsilon_n\\
&\overset{(d)}{\le} \sum_{i=1}^n H(\mathbf{X}_i^{(1)}|\mathbf{S}^{i}, \mathbf{X}^{(1),i-1}, \mathbf{X}^{(2),i-1}) + n\epsilon_n\\
&\overset{(e)}{=} nH(\mathbf{X}_T^{(1)}|\mathbf{S}^{T}, \mathbf{X}^{(1),T-1}, \mathbf{X}^{(2),T-1}, T) + n\epsilon_n,
\end{aligned} \tag{18}$$

where (a) follows since M_2 is independent of M_1; (b) follows from the encoding functions, (1) and (2); (c) follows since $\mathbf{X}_i^{(1)} = \mathbf{Y}_i^{(2)}$; (d) follows since conditioning reduces entropy and (e) follows by letting $T \sim \mathrm{Unif}[n]$ independent of all the other variables.

Let $\mathbf{S} = \mathbf{S}_T$, $\mathbf{X}^{(1)} = \mathbf{X}_T^{(1)}$, $\mathbf{X}^{(2)} = \mathbf{X}_T^{(2)}$, and $Q = (T, \mathbf{S}^{T-1}, \mathbf{X}^{(1),T-1}, \mathbf{X}^{(2),T-1})$. We can write (18) as

$$R_1 \le H(\mathbf{X}^{(1)}|\mathbf{S}, Q) + \epsilon_n. \tag{19a}$$

Similarly, for R_2 we have

$$R_2 \le H(\mathbf{X}^{(2)}|\mathbf{S}, Q) + \epsilon_n. \tag{19b}$$

Letting $n \to \infty$, we have shown that the rate pair (R_1, R_2) must satisfy

$$R_1 \le H(\mathbf{X}^{(1)}|\mathbf{S}, Q) \tag{20a}$$

$$R_2 \le H(\mathbf{X}^{(2)}|\mathbf{S}, Q) \tag{20b}$$

for some joint pmf $p(q, \mathbf{s}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)})$.

Next, we establish the conditional independence of $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ given $(\mathbf{S}, Q)$. In what follows, the notation $I(A;B;C) := I(A;B) - I(A;B|C)$ is the interaction mutual information for three random variables. Since

$$\begin{aligned}
0 &\le I(M_1; M_2|\mathbf{S}^{n+1}, \mathbf{X}^n)\\
&\overset{(a)}{=} I(M_1; M_2|\mathbf{S}^{n+1}, \mathbf{X}^n) - I(M_1; M_2)\\
&= -I(M_1; M_2; \mathbf{S}^{n+1}, \mathbf{X}^n)\\
&= -\sum_{i=1}^n I(M_1; M_2; \mathbf{S}_{i+1}, \mathbf{X}_i|\mathbf{S}^i, \mathbf{X}^{i-1})\\
&= \sum_{i=1}^n \Big[-H(\mathbf{S}_{i+1}, \mathbf{X}_i|\mathbf{S}^i, \mathbf{X}^{i-1}) + H(\mathbf{S}_{i+1}, \mathbf{X}_i|M_1, \mathbf{S}^i, \mathbf{X}^{i-1})\\
&\qquad + H(\mathbf{S}_{i+1}, \mathbf{X}_i|M_2, \mathbf{S}^i, \mathbf{X}^{i-1})\\
&\qquad - H(\mathbf{S}_{i+1}, \mathbf{X}_i|M_1, M_2, \mathbf{S}^i, \mathbf{X}^{i-1})\Big]\\
&\overset{(b)}{=} \sum_{i=1}^n \Big[-H(\mathbf{S}_{i+1}, \mathbf{X}_i|\mathbf{S}^i, \mathbf{X}^{i-1})\\
&\qquad + H(\mathbf{S}_{i+1}, \mathbf{X}_i|M_1, \mathbf{S}^i, \mathbf{X}^{i-1}, \mathbf{X}_i^{(1)})\\
&\qquad + H(\mathbf{S}_{i+1}, \mathbf{X}_i|M_2, \mathbf{S}^i, \mathbf{X}^{i-1}, \mathbf{X}_i^{(2)})\\
&\qquad - H(\mathbf{S}_{i+1}, \mathbf{X}_i|M_1, M_2, \mathbf{S}^i, \mathbf{X}^{i-1}, \mathbf{X}_i^{(1)}, \mathbf{X}_i^{(2)})\Big]\\
&\overset{(c)}{\le} \sum_{i=1}^n \Big[-H(\mathbf{S}_{i+1}, \mathbf{X}_i|\mathbf{S}^i, \mathbf{X}^{(1),i-1})\\
&\qquad + H(\mathbf{S}_{i+1}, \mathbf{X}_i|\mathbf{S}^i, \mathbf{X}^{i-1}, \mathbf{X}_i^{(1)})\\
&\qquad + H(\mathbf{S}_{i+1}, \mathbf{X}_i|\mathbf{S}^i, \mathbf{X}^{i-1}, \mathbf{X}_i^{(2)})\\
&\qquad - H(\mathbf{S}_{i+1}, \mathbf{X}_i|\mathbf{S}^i, \mathbf{X}^{i-1}, \mathbf{X}_i^{(1)}, \mathbf{X}_i^{(2)})\Big]\\
&= -\sum_{i=1}^n I(\mathbf{X}_i^{(1)}; \mathbf{X}_i^{(2)}; \mathbf{S}_{i+1}, \mathbf{X}_i|\mathbf{S}^i, \mathbf{X}^{i-1})\\
&= -\sum_{i=1}^n \Big[I(\mathbf{X}_i^{(1)}; \mathbf{X}_i^{(2)}|\mathbf{S}^i, \mathbf{X}^{i-1}) - I(\mathbf{X}_i^{(1)}; \mathbf{X}_i^{(2)}|\mathbf{S}^{i+1}, \mathbf{X}^{i})\Big]\\
&= -\sum_{i=1}^n I(\mathbf{X}_i^{(1)}; \mathbf{X}_i^{(2)}|\mathbf{S}^i, \mathbf{X}^{i-1})\\
&= -nI(\mathbf{X}_T^{(1)}; \mathbf{X}_T^{(2)}|\mathbf{S}^T, \mathbf{X}^{T-1}, T)\\
&= -nI(\mathbf{X}^{(1)}; \mathbf{X}^{(2)}|\mathbf{S}, Q),
\end{aligned} \tag{21}$$

where (a) follows since M_1 and M_2 are independent; (b) follows from the encoding functions; (c) follows from that conditioning reduces entropy and the Markov chain relation (M_1, M_2)—$(\mathbf{S}_i, \mathbf{X}_i)$—$\mathbf{S}_{i+1}$, we obtain that $\mathbf{X}^{(1)}$—$(\mathbf{S}, Q)$—$\mathbf{X}^{(2)}$. Thus, the input distribution must have the form

$$p(q, \mathbf{s}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = p(\mathbf{s}, q)p(\mathbf{x}^{(1)}|\mathbf{s}, q)p(\mathbf{x}^{(2)}|\mathbf{s}, q).$$

Finally, we impose restriction on the input distribution $p(q, \mathbf{s}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ according to the transition rule of the channel states. For all $\mathbf{s}' \in \mathcal{S}$,

$$\begin{aligned}\lim_{n\to\infty} \mathsf{P}(\mathbf{S} = \mathbf{s}') &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathsf{P}(\mathbf{S}_i = \mathbf{s}', T = i) \\ &\overset{(a)}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{i=2}^{n} \mathsf{P}(\mathbf{S}_i = \mathbf{s}') \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \sum_{(\mathbf{s},\mathbf{x})\in\mathcal{Z}(\mathbf{s}')} \mathsf{P}(\mathbf{S}_i = \mathbf{s}, \mathbf{X}_i = \mathbf{x}) \\ &\overset{(a)}{=} \lim_{n\to\infty} \sum_{(\mathbf{s},\mathbf{x})\in\mathcal{Z}(\mathbf{s}')} \mathsf{P}(\mathbf{S} = \mathbf{s}, \mathbf{X} = \mathbf{x}), \quad (22)\end{aligned}$$

where (a) follows since $T \sim \mathrm{Unif}[n]$ is independent of all the other random variables and dropping one bounded term in the summation does not affect the limit. Thus, for all $\epsilon > 0$,

$$\left| \mathsf{P}(\mathbf{S} = \mathbf{s}') - \sum_{(\mathbf{s},\mathbf{x})\in\mathcal{Z}(\mathbf{s}')} \mathsf{P}(\mathbf{S} = \mathbf{s}, \mathbf{X} = \mathbf{x}) \right| < \epsilon \qquad (23)$$

for all sufficiently large n. By the continuity of the entropy in probability, it follows that the steady-state equation (14) must hold. Note that we have from the support lemma [8, Lemma 3] the cardinality constraint $|\mathcal{Q}| \leq 2 + \sum_{\mathbf{s}\in\mathcal{S}} |\mathcal{X}(\mathbf{s}^{(1)})||\mathcal{X}(\mathbf{s}^{(2)})|$ on the time-sharing random variable Q.

B. Proof of Achievability

The achievability is proved by using random coding. Fix a pmf $p(q, \mathbf{s}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = p(\mathbf{s}, q)p(\mathbf{x}^{(1)}|\mathbf{s}, q)p(\mathbf{x}^{(2)}|\mathbf{s}, q)$ such that the steady-state equation (14) holds.

Codebook generation: For $j = 1, 2$, and for every $(\mathbf{s}, q) \in \mathcal{S} \times \mathcal{Q}$, randomly and independently generate 2^{nR_j} sequences $\mathbf{u}^{(j),n}_{(\mathbf{s},q)}(m_j)$, $m_j \in [1 : 2^{nR_j}]$, each according to $\prod_{i=1}^{n} p_{\mathbf{X}^{(j)}|(\mathbf{S},Q)=(\mathbf{s},q)}(\mathbf{u}^{(j)}_{(\mathbf{s},q),i})$. Then, for each $\mathbf{s} \in \mathcal{S}$, randomly and independently generate a sequence $q'^n(\mathbf{s})$ according to $\prod_{i=1}^{n} p_{Q|\mathbf{S}=\mathbf{s}}(q'_i)$.

Encoding: To send message m_j given the channel state $\mathbf{S}_i = \mathbf{s}_i$, encoder j first sets $q_i = q'_i(\mathbf{s}_i)$ and then transmits

$$\mathbf{x}^{(j)}_i(m_j) = \mathbf{u}^{(j)}_{(\mathbf{s}_i,q_i),i}(m_j). \qquad (24)$$

Let $\mathbf{U}_i = \{(\mathbf{U}^{(1)}_{(\mathbf{s},q),i}, \mathbf{U}^{(2)}_{(\mathbf{s},q),i})\}_{(\mathbf{s},q)\in\mathcal{S}\times\mathcal{Q}}$. If we let $n \to \infty$, the random codebook generation and the encoding process imply that $\{Q_i, \mathbf{S}_i, \mathbf{U}_i, \mathbf{Y}_i\}_{i=0}^{\infty}$ forms a Markov chain with the stationary distribution[3] $\pi(q, \mathbf{s}, \mathbf{u}, \mathbf{y})$. For $j = 1, 2$, the marginal distribution $\pi(q, \mathbf{s}, \mathbf{u}^{(j)}, \mathbf{y}^{(j+1)})$ has the form

$$\pi(q, \mathbf{s}, \mathbf{u}^{(j)}, \mathbf{y}^{(j+1)}) = \pi(\mathbf{s}, q)\pi(\mathbf{u}^{(j)})\pi(\mathbf{y}^{(j+1)}|\mathbf{s}, q, \mathbf{u}^{(j)}), \qquad (25)$$

where

$$\pi(\mathbf{s}, q) = p(\mathbf{s}, q), \qquad (26)$$

$$\pi(\mathbf{u}^{(j)}) = \prod_{(\mathbf{s},q)\in\mathcal{S}\times\mathcal{Q}} p_{\mathbf{X}^{(j)}|(\mathbf{S},Q)=(\mathbf{s},q)}(\mathbf{u}^{(j)}_{(\mathbf{s},q)}), \qquad (27)$$

and

$$\pi(\mathbf{y}^{(j+1)}|q, \mathbf{s}, \mathbf{u}^{(j)}) = \mathbb{1}\{\mathbf{y}^{(j+1)} = \mathbf{u}^{(j)}_{(\mathbf{s},q)}\}. \qquad (28)$$

Decoding: Decoder j declares that $\widehat{m}_{j+1} \in [1 : 2^{nR_{j+1}}]$ is sent if it is the unique message in the set $\mathcal{D}_{j+1} = \left\{ m \in [2^{nR_{j+1}}] : (q^n, \mathbf{s}^n, \mathbf{u}^{(j+1),n}(m), \mathbf{y}^{(j),n}) \in \mathcal{T}^{(n)}_{j,\epsilon} \right\}$, where $\mathcal{T}^{(n)}_{j,\epsilon}$ is the typical set with respect to the marginal stationary pmf $\pi(q, \mathbf{s}, \mathbf{u}^{(j+1)}, \mathbf{y}^{(j)})$. If $\mathcal{D}_{j+1}$ is not a singleton, then decoder j declares an error message e.

Error analysis: Without loss of generality, assume that $(M_1, M_2) = (1, 1)$ is sent. The error event $\mathcal{E} = \{(\widehat{M}_1, \widehat{M}_2) \neq (1, 1)\}$ can be expressed as $\mathcal{E} = \bigcup_{j=1}^{2} (\mathcal{E}_{j1} \cup \mathcal{E}_{j2})$, where, for $j = 1, 2$, $\mathcal{E}_{j1} = \{1 \notin \mathcal{D}_j\}$ and $\mathcal{E}_{j2} = \{m_j \in \mathcal{D}_j \text{ for some } m_j \neq 1\}$. By the union bound, $\mathsf{P}(\mathcal{E}) \leq \sum_{j=1}^{2} (\mathsf{P}(\mathcal{E}_{j1}) + \mathsf{P}(\mathcal{E}_{j2}))$. Since $\pi(q, \mathbf{s}, \mathbf{u}^{(j)}, \mathbf{y}^{(j+1)})$ is the marginal of the stationary distribution of the Markov chain $\{Q_i, \mathbf{S}_i, \mathbf{U}_i, \mathbf{Y}_i\}_{i=0}^{\infty}$, we have by Birkhoff's ergodic theorem [9, Theorem 7.2.1] that $\mathsf{P}(\mathcal{E}_{j1}) \to 0$ as $n \to \infty$ for $j = 1, 2$. Next, since (27) holds, by the packing lemma [10, Lemma 3.1]) we have, for $m_j \in [2 : 2^{nR_j}]$

$$\mathsf{P}(m_j \in \mathcal{D}_j) \leq 2^{-n(I(\mathbf{U}^{(j)};Q,\mathbf{S},\mathbf{Y}^{(j+1)})-\delta(\epsilon))}, \qquad (29)$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. By the union bound,

$$\mathsf{P}(\mathcal{E}_{j2}) \leq 2^{-n(I(\mathbf{U}^{(j)};Q,\mathbf{S},\mathbf{Y}^{(j+1)})-\delta(\epsilon))+nR_j}, \qquad (30)$$

which tends to 0 as $n \to \infty$ if $R_j < I(\mathbf{U}^{(j)}; Q, \mathbf{S}, \mathbf{Y}^{(j+1)}) - \delta(\epsilon)$. Since

$$\begin{aligned}&I(\mathbf{U}^{(j)}; Q, \mathbf{S}, \mathbf{Y}^{(j+1)}) \\ &\quad = I(\mathbf{U}^{(j)}; \mathbf{S}, Q) + I(\mathbf{U}^{(j)}; \mathbf{Y}^{(j+1)}|\mathbf{S}, Q) \\ &\quad \overset{(a)}{=} I(\mathbf{U}^{(j)}; \mathbf{Y}^{(j+1)}|\mathbf{S}, Q) \\ &\quad = H(\mathbf{Y}^{(j+1)}|\mathbf{S}, Q) - H(\mathbf{Y}^{(j+1)}|\mathbf{S}, Q, \mathbf{U}^{(j)}) \\ &\quad \overset{(b)}{=} H(\mathbf{Y}^{(j+1)}|\mathbf{S}, Q) \\ &\quad \overset{(c)}{=} H(\mathbf{X}^{(j)}|\mathbf{S}, Q), \qquad (31)\end{aligned}$$

where (a) holds since $\mathbf{U}^{(j)}$ and $(\mathbf{S}, Q)$ are independent; (b) follows from (28) and (c) holds due to (24), (26), (27) and (28), letting ϵ in equation (30) tend to 0 completes the proof.

[3]We assume that the conditional pmfs $p(\mathbf{x}^{(1)}|\mathbf{s}, q)$ and $p(\mathbf{x}^{(2)}|\mathbf{s}, q)$ are chosen so that the Markov chain is irreducible and aperiodic. If it is not the case, the Markov chain must be a deterministic process, yielding a zero communication rate.

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