Feedback Capacity of Stationary Gaussian Channels

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***Abstract*—The feedback capacity of additive stationary Gaussian noise channels is characterized as the solution to a variational problem in the noise power spectral density. When specialized to the first-order autoregressive moving-average noise spectrum, this variational characterization yields a closed-form expression for the feedback capacity. In particular, this result shows that the celebrated Schalkwijk–Kailath coding achieves the feedback capacity for the first-order autoregressive moving-average Gaussian channel, positively answering a long-standing open problem studied by Butman, Tiernan–Schalkwijk, Wolfowitz, Ozarow, Ordentlich, Yang–Kavčić–Tatikonda, and others. More generally, it is shown that a k-dimensional generalization of the Schalkwijk–Kailath coding achieves the feedback capacity for any autoregressive moving-average noise spectrum of order k. Simply put, the optimal transmitter iteratively refines the receiver's knowledge of the intended message. This development reveals intriguing connections between estimation, control, and feedback communication.**

***Index Terms*—Additive Gaussian noise channel, autoregressive moving-average spectrum, channel capacity, feedback, iterative refinement, linear coding, stationary Gaussian process.**

I. Introduction

We consider a communication scenario in which one wishes to communicate a message index $W \in [1:2^{nR}] := \{1,2,\ldots,2^{nR}\}$ over the additive Gaussian noise channel

$$Y_i = X_i + Z_i, \quad i = 1,2,\ldots$$

where the additive Gaussian noise process $\{Z_i\}_{i=1}^{\infty}$ is stationary with $Z^n := (Z_1,\ldots,Z_n) \sim N_n(0, K_{Z,n})$ for each $n = 1,2,\ldots$. For block length n, we specify a $(2^{nR}, n)$ feedback code with the encoding maps

$$X_i : [1:2^{nR}] \times \mathbb{R}^{i-1} \to \mathbb{R}, \quad i = 1,2,\ldots,n$$

that result in codewords (or more precisely, code functions)

$$X^n(W, Y^{n-1}) = (X_1(W), X_2(W, Y_1), \ldots, X_n(W, Y^{n-1}))$$

satisfying the average power constraint

$$\frac{1}{n}\sum_{i=1}^{n} EX_i^2(W, Y^{i-1}) \le P \tag{1}$$

and the decoding map

$$\hat{W}_n : \mathbb{R}^n \to [1:2^{nR}].$$

The probability of error $P_e^{(n)}$ is defined as

$$\begin{aligned} P_e^{(n)} &:= \frac{1}{2^{nR}}\sum_{w=1}^{2^{nR}} \Pr\left\{\hat{W}_n(Y^n) \ne w \mid W = w\right\} \\ &= \Pr\left\{\hat{W}_n(Y^n) \ne W\right\} \end{aligned}$$

where the message W is uniformly distributed over $[1:2^{nR}]$ and is independent of Z^n. We say that the rate R is achievable if there exists a sequence of $(2^{nR}, n)$ codes with $P_e^{(n)} \to 0$ as $n \to \infty$. The feedback capacity $C_{\rm FB}$ is defined as the supremum of all achievable rates.

In comparison, when there is no feedback, the codewords $X^n(W) = (X_1(W), \ldots, X_n(W))$ are independent of the previous channel outputs. We define the nonfeedback capacity C, or the capacity in short, in a manner similar to the feedback case.

It is well known that the nonfeedback capacity is characterized by water-filling on the noise spectrum, which is arguably one of the most beautiful results in information theory. More specifically, the capacity of the additive Gaussian noise channel $Y_i = X_i + Z_i, i = 1,2,\ldots$, under the power constraint P, is

$$C = \int_{-\pi}^{\pi} \frac{1}{2}\log\frac{\max\left\{S_Z(e^{i\theta}), \lambda\right\}}{S_Z(e^{i\theta})}\,\frac{d\theta}{2\pi} \tag{2}$$

where $S_Z(e^{i\theta})$ is the power spectral density of the stationary noise process $\{Z_i\}_{i=1}^{\infty}$, i.e., the Radon–Nikodym derivative of the spectral distribution $\mu_Z(e^{i\theta})$ of $\{Z_i\}_{i=1}^{\infty}$ (with respect to the Lebesgue measure), and the water level λ is chosen to satisfy

$$P = \int_{-\pi}^{\pi} \max\left\{0, \lambda - S_Z(e^{i\theta})\right\}\,\frac{d\theta}{2\pi}. \tag{3}$$

Although (2) and (3) give only a parametric characterization of the capacity $C(\lambda)$ under the power constraint $P(\lambda)$ for each parameter $\lambda \ge 0$, this solution is considered simple and elegant enough to be called *closed-form*. Just like many other fundamental developments in information theory, the idea of water-filling comes from Shannon [62], although it is sometimes attributed to Holsinger [23] or Ebert [13].

For the case of feedback, no such elegant solution exists. Most notably, Cover and Pombra [8] characterized the feedback capacity through the "n-block feedback capacity"

$$C_{{\rm FB},n} = \max_{K_V, B} \frac{1}{2}\log\frac{\det(K_V + (B+I)K_{Z,n}(B+I)')^{1/n}}{\det(K_{Z,n})^{1/n}} \tag{4}$$

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where the maximum is taken over all positive semidefinite matrices $K_V = K_{V,n}$ and all strictly lower triangular $B = B_n$ of sizes $n \times n$ satisfying $\operatorname{tr}(K_V + BK_{Z,n}B') \le nP$. Using the asymptotic equipartition property (AEP) for arbitrary (nonstationary nonergodic) Gaussian processes, a coding theorem can be then proved to characterize the feedback capacity as a limiting expression

$$C_{\rm FB} = \lim_{n\to\infty} C_{{\rm FB},n}. \tag{5}$$

Despite its generality, the Cover–Pombra formulation of the feedback capacity falls short of what we can call a closed-form solution. Indeed, the infinite-letter expression (5) in itself does not provide a computable characterization of the feedback capacity. Hence, much effort has been made to further understand and simplify the Cover–Pombra formulation. Apparently from (4), the most natural strategy towards a closed-form characterization seems to be first finding the optimal solution $\left(K^\star_{V,n}, B^\star_n\right)$ for each n, and then analyzing its limiting behavior.

Roughly speaking, there have been four distinct approaches in the literature that follow this strategy:

First, in a less publicized conference paper, Ordentlich [50] studied the properties of $\left(K^\star_{V,n}, B^\star_n\right)$ by a *fixed-point characterization*. This result shows the optimality of a filter matrix $B^\star_n$ that sends fresh information each time (i.e., transmitting an input signal orthogonal to the previously received output signals), the property of which is sometimes referred to as the optimality of a Kalman filter $B^\star_n$. Ordentlich's method also finds that the optimal covariance matrix $K^\star_{V,n}$ is of rank at most k when the noise process has the kth-order moving-average noise spectrum.

Second, Boyd and Ordentlich (circa 1994) found that the maximization problem for $C_{{\rm FB},n}$ is an instance of a well-known convex optimization problem called the matrix determinant maximization (max-det) problem [71, eq. (2.16)]]. For the case of the nonfeedback capacity C_n corresponding to $B \equiv 0$, Shannon's water-filling solution arises from the KKT condition for the max-det problem and Szegő's limit theorem in a straightforward manner. While the KKT condition for the feedback case is less manageable, one can compute $C_{{\rm FB},n}$ numerically with reasonable complexity in n, as demonstrated by Zahedi [79]. See [34, Ch. 4] for analytic implications of the Boyd–Ordentlich formulation.

Third, Yang–Kavčić–Tatikonda [77] considered Massey's directed information [44] and found the structure of the input distribution maximizing the directed information, or equivalently (see [8, eq. (53)]), found the structure of the optimal $\left(K^\star_{V,n}, B^\star_n\right)$ attaining $C_{{\rm FB},n}$. Using *dynamic programming* on the state–space representation of the noise process, a structural result similar to Ordentlich [50] was obtained, which shows the optimality of a kth-order Kalman filter $B^\star$ for the kth-order autoregressive moving-average (ARMA) noise spectrum. This approach, *inter alia,* provides yet another numerical technique for calculating $C_{{\rm FB},n}$ by dynamic programming (linear complexity in n) that solves a sequentially identical $O(k^2)$-dimensional suboptimization problem in each recursion step.

Finally, in [33] the optimal $\left(K^\star_{V,n}, B^\star_n\right)$ for the first-order moving-average noise spectrum was found analytically by a brute-force maximization method under per-symbol power constraints, and then power allocation (over time) was optimized in asymptotics, establishing the closed-form feedback capacity formula for the first time.

However, the major stumbling block for all these approaches of directly attacking the n-block capacity is that it is extremely difficult, if not impossible, to obtain an analytic expression for the optimal $\left(K^\star_{V,n}, B^\star_n\right)$ and the corresponding $C_{{\rm FB},n}$ for each n. Indeed, the approach taken in [33] is feasible only thanks to the simple tridiagonal covariance structure of the first-order moving-average noise spectrum. And even in this case, nontrivial tricks had to be developed because it does not seem tractable to find the optimal $\left(K^\star_{V,n}, B^\star_n\right)$ under the given power constraint (1).

In this paper we take an approach that is quite different from the approached discussed above. Instead of considering $C_{{\rm FB},n}$ individually for each n, we attack its limit directly by characterizing the feedback capacity in the following variational form (Theorem 3.2):

$$C_{\rm FB} = \sup_{S_V, B} \int_{-\pi}^{\pi} \frac{1}{2} \log \frac{S_V(e^{i\theta}) + |1 + B(e^{i\theta})|^2 S_Z(e^{i\theta})}{S_Z(e^{i\theta})} \, \frac{d\theta}{2\pi} \tag{6}$$

where $S_Z(e^{i\theta})$ is the power spectral density of the noise process $\{Z_i\}_{i=1}^\infty$ and the supremum is taken over all power spectral densities $S_V(e^{i\theta}) \ge 0$ and all strictly causal filters $B(e^{i\theta}) = \sum_{k=1}^\infty b_k e^{ik\theta}$ satisfying the power constraint

$$\int_{-\pi}^{\pi} (S_V(e^{i\theta}) + |B(e^{i\theta})|^2 S_Z(e^{i\theta})) \, \frac{d\theta}{2\pi} \le P.$$

Thus, a single infinite-dimensional optimization problem has the same answer as an infinite sequence of finite-dimensional optimization problems. At an intuitive level, this characterization can be viewed as a justification for the interchange of limit and maximum in (4) and (5), implicating the asymptotic optimality of a stationary (Toeplitz) solution $\left(K^\star_{V,n}, B^\star_n\right)$. Refer to [34, Ch. 3] for a parallel development for the nonfeedback capacity.

Once equipped with this variational characterization, the remaining challenge becomes finding the optimal solution $\left(S^\star_V(e^{i\theta}), B^\star(e^{i\theta})\right)$ that achieves $C_{\rm FB}$. Based on techniques from control and estimation theory, convex optimization, functional analysis, and information theory, we find the structure of the optimal solution $(S^\star_V, B^\star)$ in Theorem 4.1, Proposition 5.1, and Lemma 6.1, which is reminiscent of the structural results by Ordentlich [50] and Yang–Kavčić–Tatikonda [77] for the finite-dimensional case. This result, when specialized to the first-order autoregressive noise spectrum $S_Z(e^{i\theta}) = |1 + \beta e^{i\theta}|^{-2}$, $-1 < \beta < 1$, yields a closed-form expression for the feedback capacity as

$$C_{\rm FB} = -\log x_0$$

where x_0 is the unique positive root of the fourth-order polynomial

$$P\,x^2 = \frac{(1 - x^2)}{(1 + |\beta| x)^2}$$

establishing the long-standing conjecture by Butman [5], [6], Tiernan–Schalkwijk [70], [69], and Wolfowitz [73]. In fact, we will obtain an explicit feedback capacity formula for the first-order ARMA noise spectrum in Theorem 5.3, which generalizes the result in [33] and confirms positively a recent conjecture by Yang, Kavčić, and Tatikonda [78]. As we will see later, this result shows that the celebrated Schalkwijk–Kailath coding [60], [61] achieves the feedback capacity.

More generally, we will show in Theorem 6.1 that a k-dimensional generalization of the Schalkwijk–Kailath coding achieves the feedback capacity for any ARMA noise spectrum of order k.

The literature on Gaussian feedback capacity is vast. Instead of trying to be complete, we sample the results that are most relevant to our discussion. A more complete survey can be found in [33]. The standard literature on the Gaussian feedback channel and associated simple feedback coding techniques traces back to Elias's 1956 paper [16] and its sequels [17], [21]. Schalkwijk and Kailath [60], [61] made a major breakthrough by showing that simple linear feedback coding achieves the feedback capacity of the additive white Gaussian noise channel with doubly exponentially decreasing probability of decoding error. More specifically, the transmitter sends a real-valued information bearing signal at the beginning of communication and subsequently refines the receiver's knowledge by sending the error of the receiver's estimate of the message. This simple and constructive coding, or no coding in a sense, achieves the capacity of the Gaussian channel and the resulting error probability of the maximum likelihood decoding decays doubly exponentially in the duration of the communication.

This fascinating result has been extended in many directions, including the feedback communication over the additive *nonwhite* Gaussian noise channel. Butman [5], [6] extended the Schalkwijk–Kailath coding to autoregressive noise channels. Subsequently, Tiernan and Schalkwijk [70], [69], Wolfowitz [73], and Ozarow [51], [52] studied the feedback capacity of finite-order autoregressive moving-average additive Gaussian noise channels and obtained many interesting upper and lower bounds. As mentioned above, Yang, Kavčić, and Tatikonda [78] (see also Yang's thesis [76]) recently revived the control-theoretic approach (cf. Omura [49], Tiernan and Schalkwijk [70]) to the finite-order autoregressive moving-average Gaussian feedback capacity problem and brought up several interesting results. After reformulating the feedback capacity problem as a stochastic control problem, Yang *et al.* used dynamic programming for the numerical computation of $C_{\mathrm{FB},n}$ and offered a conjecture that C_{FB} can be achieved by a stationary policy for each iteration in the dynamic programming. Our Theorem 6.1 confirms their conjecture in a somewhat stronger form.

In a more general line of attack, Cover and Pombra [8] proved the coding theorem for the arbitrary nonwhite Gaussian channel with or without feedback, using an AEP theorem for nonstationary nonergodic Gaussian processes. They also showed that feedback does not increase the capacity much; namely, feedback at most doubles the capacity (a result obtained by Pinsker [57] and Ebert [14]), and feedback increases the capacity at most by half a bit. The extensions and refinements of the Cover–Pombra result abound. Ihara obtained a coding theorem for continuous-time Gaussian channels with feedback [26], [28] and showed that the factor-of-two bound on the feedback capacity is tight by considering cleverly constructed nonstationary channels for both discrete [27] and continuous [25] cases. Dembo [10] studied the upper bounds on $C_{\mathrm{FB},n}$ and showed that feedback does not increase the capacity at very low signal-to-noise ratio or very high signal-to-noise ratio. (See Ozarow [51] for a minor technical condition on the result for very low signal-to-noise ratio.) In the aforementioned work [50], Ordentlich examined the properties of the optimal solution $(K_{V,n}, B_n)$ for $C_{\mathrm{FB},n}$ in (4) for a fixed n and derived the optimality of a k-dimensional Kalman filter for the kth-order moving-average noise spectrum. In [33], an alternative method was developed for the first-order moving-average noise spectrum and the maximization problem in (4) was solved analytically under the modified power constraint on each input signal X_i, $i = 1, 2, \ldots$. Then, a fixed-point theorem exploiting the convexity of the problem is deployed to show the asymptotic optimality of the uniform power allocation over time, establishing the feedback capacity for this special case.

As is hinted by the similarity between the n-block capacity (4) and the variational characterization (6), the Cover–Pombra characterization is again the very starting point of our development. However, the variational formula (6) certainly has the flavor of spectral analysis for control and estimation problems, in the context of which we will find the optimal solution $\big(S_V^\star(e^{i\theta}), B^\star(e^{i\theta})\big)$. This optimal solution will be then linked to the asymptotic behavior of the linear coding by Schalkwijk and Kailath. Thus in a sense our development goes in a full circle through the literature cited above.

The rest of the paper is organized as follows. We will first establish the variational characterization (6) of the feedback capacity in Section III. Subsequently, the resulting variational problem will be solved in several steps. Section IV finds the sufficient and necessary condition for the optimal $(S_V^\star, B^\star)$ under a general noise spectrum. This optimality condition is then specialized to the finite-order autoregressive moving-average (i.e., rational) noise spectrum in Section V. In particular, for the first-order ARMA noise spectrum, we find a closed-form expression for the feedback capacity and show the optimality of the Schalkwijk–Kailath coding. The state-space approach provides a richer set of tools for analyzing the ARMA noise spectrum. Section VI is devoted to this approach and characterizes the feedback capacity as the maximum achievable rate of a multidimensional variant of the Schalkwijk–Kailath coding. Section VII concludes the paper with potential connections to other mathematical problems. Section II recalls necessary results from various branches of mathematics.

II. Mathematical Preliminaries

A. Toeplitz Matrices, Szegő's Limit Theorem, and Entropy Rate

We first review a few important results on spectral properties of stationary Gaussian processes, which we will use heavily for the variational characterization of feedback capacity.

Let $R(k) = R(-k) = E(Z_1 Z_{k+1})$, $k = 0, 1, 2, \ldots$, be the covariance sequence of a stationary Gaussian process $\{Z_i\}_{i=1}^{\infty}$. Then, as the elegant answer to the classical trigonometric moment problem shows (see, for example, Akhiezer [1] and Landau [42]), there exists a positive measure μ on $[-\pi, \pi)$, sometimes called the power spectral distribution of the process $\{Z_i\}_{i=1}^{\infty}$, such that

$$R(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta} d\mu(\theta)$$

for all k. From the Lebesgue decomposition theorem, we can write μ as a sum $\mu = \mu_{ac} + \mu_s$, where μ_{ac} is absolutely continuous with respect to the Lebesgue measure and μ_s is singular. The Radon–Nikodym derivative of μ_{ac} (with respect to the Lebesgue measure), called the power spectral density of $\{Z_i\}_{i=1}^{\infty}$, exists almost everywhere and can be written as a function of $e^{i\theta}$, or more specifically, we have $d\mu_{ac} = S(e^{i\theta})d\theta = \mathrm{Re}\{F(e^{i\theta})\}d\theta$ for some function $F(z)$ analytic on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with $F(0) > 0$ and $\mathrm{Re}\{F(z)\} > 0$ on $\mathbb{D}$.

Conversely, given a nontrivial (i.e., supported by infinitely many points) positive measure $d\mu = S(e^{i\theta})d\theta + d\mu_s$, the Toeplitz matrix K_n of size $n \times n$ given by

$$K_n(j,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(k-j)\theta} d\mu(\theta), \quad 1 \le j,k \le n$$

is positive definite Hermitian. Hence, K_n has n positive eigenvalues $\lambda_1(K_n), \ldots, \lambda_n(K_n)$, counting multiplicity. In his famous limit theorem [66], [67], Szegő proved an elegant relationship between the asymptotic behavior of the eigenvalues of K_n and the associated spectral distribution μ. This result lies at the heart of many different fields, including operator theory, time-series analysis, quantum mechanics, approximation theory, and, of course, information theory. Here we recall a fairly general version of Szegő's limit theorem, which can be found in Simon [64, Theorem 2.7.13].

Lemma 2.1 (Szegő's Limit Theorem): Let f be a continuous function on $[0, \infty)$ such that

$$\lim_{x\to\infty} \frac{f(x)}{x} = c < \infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(\lambda_i(K_n)) = \int_{-\pi}^{\pi} f(S(e^{i\theta}))\,\frac{d\theta}{2\pi} + \frac{c}{2\pi}\int_{-\pi}^{\pi} d\mu_s(\theta).$$

The above limit theorem is sometimes called the first Szegő theorem, in order to be distinguished from the second-order asymptotics often called the strong Szegő theorem and obtained by Szegő himself after a 38-year gap [68]. Refer to Grenander and Szegő [22, Ch. 5], Böttcher and Silbermann [3, Ch. 5], Gray [20], and Simon's recent two-part tome on orthogonal polynomials on the unit circle [64] for different flavors of Szegő's theorem under different levels of generality.

As a canonical application of Szegő's limit theorem, the following variational statement, attributed to Szegő, Kolmogorov [36], and Krein [39], [40], connects the entropy rate, the spectral distribution, and the minimum mean-square prediction error of a stationary Gaussian process.

Lemma 2.2 (Szegő–Kolmogorov–Krein Theorem): Let $\{Z_i\}_{i=-\infty}^{\infty}$ be a stationary Gaussian process with a nontrivial spectral distribution $d\mu = S(e^{i\theta})d\theta + d\mu_s$. Then the minimum mean-squared prediction error $E_\infty = E\left(Z_0 - E\left(Z_0|Z_{-\infty}^{-1}\right)\right)^2$ of Z_0 from the entire past Z_k, $k < 0$, is given by

$$\begin{aligned} E_\infty &= \inf_{\{a_k\}} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1 - \sum_{k=1}^{\infty} a_k e^{ik\theta}\right|^2 d\mu(\theta) \\ &= \exp\left(\int_{-\pi}^{\pi} \log S(e^{i\theta})\,\frac{d\theta}{2\pi}\right) \\ &= \frac{1}{2\pi e} e^{2h(\mathcal{Z})} \end{aligned}$$

where $h(\mathcal{Z}) = \lim_{n\to\infty} n^{-1}h(Z_1,\ldots,Z_n)$ denotes the differential entropy rate of the process $\{Z_i\}$.

The proof of this result follows almost immediately from Szegő's limit theorem with $f(x) = \log x$. Note that the prediction error depends only on the absolutely continuous part of the spectral measure, which follows because $\lim_{x\to\infty}(\log x)/x = 0$. (The fact that the prediction error is independent of the singular part of the spectral distribution can be also proved from somewhat deeper results on shift operators and Wold–Kolmogorov decomposition. See, for example, Nikolski [48] and the references therein.) We stress the relationship between the entropy rate of a stationary Gaussian process $\{Z_i\}$ and its spectral density $S(e^{i\theta})$ in the following familiar expression:

$$h(\mathcal{Z}) = \int_{-\pi}^{\pi} \frac{1}{2}\log(2\pi e S(e^{i\theta}))\,\frac{d\theta}{2\pi}. \tag{7}$$

Throughout this paper, in order to exclude the trivial case of unbounded capacity, we will assume that the power spectral distribution μ of the additive Gaussian noise process $\{Z_i\}_{i=1}^{\infty}$ is nontrivial (equivalently, K_n is positive definite for all n), and that the power spectral density $S_Z(e^{i\theta})$ satisfies the so-called *Paley–Wiener condition*

$$\int_{-\pi}^{\pi} |\log S_Z(e^{i\theta})|\,\frac{d\theta}{2\pi} < \infty \tag{8}$$

which is equivalent to having prediction error $E_\infty > 0$.

B. Hardy Spaces, Causality, and Spectral Factorization

We review some elementary results on Hardy spaces (see, for example, Duren [12], Koosis [37], Rudin [58, Ch. 17]) that are needed for analysis of optimal feedback filters. Our exposition loosely follows two monographs by Partington [53], [54].

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic function on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We say that $f(z)$ belongs to the class $\mathcal{H}_p$, $1 \le p < \infty$, if

$$\|f\|_{\mathcal{H}_p} = \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\,\frac{d\theta}{2\pi}\right)^{1/p}$$

is bounded for all $r < 1$. Similarly we say that $f(z)$ belongs to the class $\mathcal{H}_\infty$ if

$$\|f\|_{\mathcal{H}_\infty} = \sup_{|z|<1} |f(z)|$$

is bounded. We can easily check that $\mathcal{H}_p$ is a Banach space for $1 \le p \le \infty$.

It is well-known that $f \in \mathcal{H}_p$ can be extended to $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ by taking the pointwise radial limit

$$\tilde{f}(e^{i\theta}) = \lim_{r\uparrow 1} f(re^{i\theta})$$

which exists for almost all θ. The extended function $\tilde{f}$ belongs to the standard Lebesgue space $\mathcal{L}_p$ on $\mathbb{R}/[-\pi,\pi) \simeq \mathbb{T}$ with the same norm $\|\tilde{f}\|_p = \|f\|_{\mathcal{H}_p}$, so that we can consider $\mathcal{H}_p$ as a closed (and thus complete) subspace of $\mathcal{L}_p$. Therefore, we will identify $f \in \mathcal{H}_p$ with its radial extension $\tilde{f} \in \mathcal{L}_p$ and use the same symbol f for both f and $\tilde{f}$ throughout. More specifically, when we say that a function $f(e^{i\theta})$ for $\theta \in [-\pi,\pi)$ belongs to $\mathcal{H}_p$, we implicitly mean that $f(z)$ is also well-defined and analytic on $\mathbb{D}$. Also we will use $f(z)$ and $f(e^{i\theta})$ interchangeably if the context is clear. Recall the following relationship among important classes of functions on $\mathbb{T}$:

$$\mathcal{H}_p \subset \mathcal{L}_p, \qquad 1 \le p \le \infty$$
$$\mathcal{H}_\infty \subset \mathcal{H}_2 \subset \mathcal{H}_1$$

and

$$\mathcal{L}_\infty \subset \mathcal{L}_2 \subset \mathcal{L}_1.$$

Let $f \in \mathcal{L}_p, 1 \le p \le \infty$. We say that f is *causal* if its Fourier coefficients

$$c_n = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}, \quad n = 0, \pm 1, \pm 2, \dots$$

satisfy $c_n = 0$ for $n < 0$. We also say that f is *strictly causal* if $c_n = 0$ for $n \le 0$, or equivalently, $f(z) = zg(z)$ for some causal $g \in \mathcal{L}_p$. By reversing the direction of the time index, we also define *anticausality* and *strict anticausality* in a similar way.

If $f(z) \in \mathcal{H}_p$, then $f(e^{i\theta})$ (or more precisely, the extension of $f(z)$ in $\mathcal{L}_p$) can be easily shown to be causal; see Lemma 2.3 below. Conversely, if $f(e^{i\theta}) \in \mathcal{L}_p$ is causal, then $\sup_n |c_n| < \infty$ so that $f(z)$ is analytic on $\mathbb{D}$, where

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{9}$$

and the series on the right-hand side converges pointwise on $\mathbb{D}$. Therefore, we can identify the class $\mathcal{H}_p$ with the class of causal $\mathcal{L}_p$ functions, which gives an alternative definition of the $\mathcal{H}_p$ space.

When $f \in \mathcal{H}_\infty$, we have the pointwise convergence of the infinite series (9) on $\mathbb{T} = \{e^{i\theta} : \theta \in [-\pi,\pi)\}$ for almost all θ. Hence, $f \in \mathcal{H}_\infty$ preserves the causality when acting on $\mathcal{L}_1$ by multiplication. For later use, we stress this simple fact in the following statement, the proof of which easily follows from the dominated convergence theorem.

Lemma 2.3: Let $f \in \mathcal{H}_\infty$ and let $g \in \mathcal{L}_1$ be causal. Then, $fg \in \mathcal{L}_1$ is causal. If, in addition, f is strictly causal, then $fg \in \mathcal{L}_1$ is strictly causal and

$$\int_{-\pi}^{\pi} f(e^{i\theta}) g(e^{i\theta}) \frac{d\theta}{2\pi} = 0.$$

We recall a few important factorization theorems. The first set of results deals with the factorization of $\mathcal{H}_p$ functions. Suppose $f \in \mathcal{H}_p$, $1 \le p \le \infty$, is not identically zero. Then, f has a factorization $f(z) = g(z)u(z)$ that is unique up to a constant of modulus 1, where $g(z)$ is an inner function (i.e., $g(z)$ is an $\mathcal{H}_\infty$ function with $g(e^{i\theta}) = 1$ almost everywhere) and $u(z)$ is an $\mathcal{H}_p$ outer function given by

$$u(z) = \exp\left(\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \right).$$

Consequently, the zeros of f in $\mathbb{D}$ coincide with the zeros of g, and $\|f\|_p = \|u\|_p$.

We define the (infinite) Blaschke product $b(z)$ formed with the zeros of $f(z)$ as

$$b(z) = z^k \prod_{|z_n| \ne 0} \frac{\overline{z}_n}{|z_n|} \frac{z_n - z}{1 - \overline{z}_n z},$$

where $\{z_n\}$ are the zeros of f in $\mathbb{D}$, listed according to their multiplicity, k of them being at 0. It is easy to check that $b(z)$ is well-defined in the sense that $b(z)$ converges uniformly on compact sets to an $\mathcal{H}_\infty$ function. Also, $b(z) \le 1$ and $|b(e^{i\theta})| = 1$ almost everywhere. As a refinement of the above inner–outer factorization theorem, Riesz showed that f has a factorization $f(z) = b(z)s(z)u(z)$ that is unique up to a constant of modulus 1, where b is the Blaschke product of the zeros of f, s is a singular inner function (without zeros), and u is an outer function. Again $\|f\|_p = \|u\|_p$.

For our purposes, it is more convenient to introduce a normalized variant of the Blaschke product as

$$\hat{b}(z) = z^k \prod_{|z_n| \ne 0} \frac{1 - z_n^{-1} z}{1 - z_n z}.$$

Then, $|\hat{b}(e^{i\theta})| = \prod_{|z_n| \ne 0} (1/|z_n|)$ almost everywhere. This normalized Blaschke product is often called an *all-pass filter* in the signal processing literature if $\{z_n\}$ is finite and $k = 0$.

If $f \in \mathcal{H}_2$ and $f(0) = 1$, then f has the unique factorization $f(z) = \hat{b}(z)\hat{u}(z)$, where $\hat{b}(z)$ is the normalized Blaschke product formed with zeros z_n of f in $\mathbb{D}$ and $\hat{u}(z)$ does not have any zero in $\mathbb{D}$. In particular, $\hat{b}(0) = \hat{u}(0) = 1$. Now Jensen's formula [58, Theorem 15.18] states that, if $g(z) \in \mathcal{H}_2$ with $g(0) = 1$, then

$$\int_{-\pi}^{\pi} \log|g(re^{i\theta})| \frac{d\theta}{2\pi} = \log \prod_{k=1}^{m} \frac{r}{|\alpha_k|}$$

where $\alpha_1, \dots, \alpha_m$ denote the zeros of $g(z)$ within the circle of radius $r < 1$. Therefore

$$\begin{aligned} &\int_{-\pi}^{\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \\ &= \int_{-\pi}^{\pi} \log|\hat{b}(e^{i\theta})| \frac{d\theta}{2\pi} + \int_{-\pi}^{\pi} \log|\hat{u}(e^{i\theta})| \frac{d\theta}{2\pi} \\ &= \log \prod \frac{1}{|z_n|}. \end{aligned} \tag{10}$$

As a trivial corollary, if f is rational of the form

$$f(z)=\frac{P(z)}{Q(z)}=\frac{1+\sum_{n=1}^{k}p_nz^n}{1+\sum_{n=1}^{k}q_nz^n}=\frac{\prod\left(1-\beta_n^{-1}z\right)}{\prod\left(1-\gamma_n^{-1}z\right)}$$

with all zeros γ_n of $Q(z)$ strictly outside the unit circle, then

$$\int_{-\pi}^{\pi}\log|f(e^{i\theta})|\,\frac{d\theta}{2\pi}=\log\prod_{j=1}^{m}\frac{1}{|\beta_j|}$$

where $\beta_1,\ldots,\beta_m$ denote the zeros of $P(z)$ in $\mathbb{D}$.

Our last factorization theorem is concerned with the factorization of positive $\mathcal{L}_1$ functions and is usually called the canonical factorization theorem. Suppose $f(e^{i\theta})\in\mathcal{L}_1$. Then, $f(e^{i\theta})=|g(e^{i\theta})|^2$ for some $g(e^{i\theta})\in\mathcal{H}_2$ if and only if $f(e^{i\theta})\geq 0$ almost everywhere and the Paley–Wiener condition (8) is satisfied. In light of the aforementioned factorization theorem due to F. Riesz, we can always take the canonical factor g with no zeros inside the unit circle and $g(0)>0$.

C. Discrete Algebraic Riccati Equations

Discrete algebraic Riccati equations (DAREs) often play a crucial role in many estimation and control problems. Our problem is no exception, especially in the state-space approach for the ARMA(k) feedback capacity in Section VI.

Here we focus on a very special class of Riccati equations and review a few properties of them. Since the necessary results are somewhat scattered in the literature, we also provide short proofs along with probabilistic interpretations; some of these might be new. Whenever possible, however, we will refer to standard references. For a more general treatment, refer to Kailath, Sayed, and Hassibi [32] and Lancaster and Rodman [41].

Given matrices $F\in\mathbb{R}^{k\times k}$ and $H\in\mathbb{R}^{1\times k}$, we study the following discrete algebraic Riccati equation:

$$\Sigma=F\Sigma F'-\frac{(F\Sigma H')(F\Sigma H')'}{1+H\Sigma H'}. \tag{11}$$

For each $k\times k$ Hermitian matrix Σ, define

$$\Gamma=\Gamma(\Sigma)=\frac{F\Sigma H'}{1+H\Sigma H'}.$$

We are concerned with solutions of (11), especially the ones with stable $F-\Gamma H$.

Lemma 2.4 (DARE): Suppose F has no unit-circle eigenvalue and (F,H) is detectable, that is, there exists $G\in\mathbb{R}^{1\times k}$ such that $F-GH$ is stable (i.e., every eigenvalue of $F-GH$ lies inside the unit circle). Then, the following statements hold.

i) $\Sigma\equiv 0$ is a solution to (11).

ii) There is a unique solution $\Sigma=\Sigma_+$ to (11) such that $F-\Gamma H$ is stable. Furthermore, $\Sigma_+\succeq\Sigma$ for any other Σ satisfying (11). In particular, Σ_+ is positive semidefinite.

iii) If F is invertible, then $F-\Gamma H$ is invertible for each solution Σ and

$$1+H\Sigma H'=\frac{\det(F)}{\det(F-\Gamma H)}.$$

iv) Let $\Gamma_+=\Gamma(\Sigma_+)$. If F has eigenvalues $\lambda_1,\ldots,\lambda_k$ with $|\lambda_1|\geq\cdots\geq|\lambda_j|>1>|\lambda_{j+1}|\geq\cdots\geq|\lambda_k|$, then $F-\Gamma_+H$ has eigenvalues $1/\lambda_1,\ldots,1/\lambda_j,\lambda_{j+1},\ldots,\lambda_k$.

v) If every eigenvalue of F lies inside the unit circle, then the stabilizing solution Σ_+ is identically zero. Thus, $\Sigma_+=0$ is the unique positive semidefinite solution to the DARE (11).

vi) If every eigenvalue of F lies outside the unit circle, then $\Sigma_+\succ 0$.

vii) More generally, suppose F has j eigenvalues outside the unit circle and $k-j$ eigenvalues inside the unit circle. Then, $\operatorname{rank}(\Sigma_+)=j$.

Proof:

i) Trivial.

ii) Refer to [32, Theorem E.5.1].

iii) Note that $\det(1+H\Sigma H')=\det(I+\Sigma H'H)$. Now simple algebra reveals that $(F-\Gamma H)(I+\Sigma H'H)=F$.

iv) For simplicity, we assume that F is invertible. We can easily check that

$$\begin{bmatrix}F^{-1} & 0\\ -H'HF^{-1} & F'\end{bmatrix}=\begin{bmatrix}I & \Sigma\\ 0 & I\end{bmatrix}\begin{bmatrix}(F-\Gamma H)^{-1} & 0\\ -H'HF^{-1} & (F-\Gamma H)'\end{bmatrix}\begin{bmatrix}I & \Sigma\\ 0 & I\end{bmatrix}^{-1}$$

for any solution Σ, which implies that the eigenvalues of $\{(F-\Gamma H)',(F-\Gamma H)^{-1}\}$ coincides with those of $\{F',F^{-1}\}$. Now the desired result follows from the fact that $F-\Gamma_+H$ is stable.

v) Refer to [32, Theorem E.6.1].

vi) Refer to [32, Theorem E.6.2].

vii) For simplicity, suppose F can be diagonalized; the general case can be proved by using the generalized eigenvectors associated with the Jordan canonical form of F. Take each eigenvalue–(right) eigenvector pair (λ,v) of F with $|\lambda|>1$. Suppose $v\Sigma_+=0$. Then, we can easily check that $v(F-\Gamma_+H)=vF=\lambda v$, which violates the stability of $F-\Gamma_+H$. Thus, $v\Sigma_+\neq 0$, which implies $\operatorname{rank}(\Sigma_+)\geq j$. On the other hand, take each eigenvalue–eigenvector pair (λ,v) of F with $|\lambda|<1$. From (11), we have

$$v\Sigma_+v'=|\lambda|^2v\Sigma_+v'-\frac{F\Sigma_+H'H\Sigma_+F'}{1+H\Sigma_+H'}$$

or equivalently

$$(1-|\lambda|^2)v\Sigma_+v'+\frac{F\Sigma_+H'H\Sigma_+F'}{1+H\Sigma_+H'}=0.$$

Since both terms of the above sum are nonnegative, we must have $v\Sigma_+=0$, which implies $\operatorname{rank}(\Sigma_+)\leq j$. $\square$

Algebraic Riccati equations naturally arise from asymptotic behaviors of recursive filters (e.g., Kalman filters). In the following lemma, we collect a few results on the convergence of the Riccati recursion.

Lemma 2.5 (Discrete Riccati Recursion): Under the same assumption on $\{F, H\}$ as in Lemma 2.4, suppose $\{\Sigma_n\}_{n=1}^{\infty}$ is defined as

$$\Sigma_{n+1} = F\Sigma_n F' - \frac{(F\Sigma_n H')(F\Sigma_n H')'}{1 + H\Sigma_n H'} \tag{12}$$

for some Σ_0. Then, the following statements hold:

i) If $\Sigma_0 = 0$, then $\Sigma_n = 0$ for all n.
ii) If $\Sigma_0 \succeq 0$, then $\Sigma_n \succeq 0$ for all n.
iii) If $\Sigma_0 \succeq \tilde{\Sigma}_0 \succeq 0$, then $\Sigma_n \succeq \tilde{\Sigma}_n \succeq 0$ for all n.
iv) If $\Sigma_0 \succ 0$, then $\Sigma_n \to \Sigma_+$, where $\Sigma_+ \succeq 0$ is the unique stabilizing solution to the DARE (11).

Proof:

i) Trivial.
ii) Let $\Gamma_n = \Gamma(\Sigma_n) = F\Sigma_n H'/(1 + H\Sigma_n H')$ and write (12) as

$$\Sigma_{n+1} = (F - \Gamma_n H)\Sigma_n (F - \Gamma_n H)' + \Gamma_n \Gamma_n'.$$

iii) Refer to Caines [7, Theorem 3.5.1].
iv) Let $\Pi \succeq 0$ be the unique solution of the Lyapunov equation

$$\Pi = (F - \Gamma_+ H)'\Pi(F - \Gamma_+ H) + \frac{H'H}{1 + H\Sigma_+ H'}. \tag{13}$$

(Lemma 2.4 guarantees the stability of $F - \Gamma_+ H$ and hence there exists a unique positive semidefinite Π satisfying (13).) Take any $\epsilon > 0$ such that $\Sigma_0 \succeq \epsilon I$ and $I + (\epsilon I - \Sigma_+)\Pi$ is nonsingular. Now from [32, Lemma 14.5.7], we have

$$\begin{aligned} I &+ \left(\Pi^{1/2}\right)'(\Sigma_0 - \Sigma_+)\Pi^{1/2} \\ &\succ I + \left(\Pi^{1/2}\right)'(\epsilon I - \Sigma_+)\Pi^{1/2} \\ &\succ 0 \end{aligned}$$

which implies the exponential convergence of Σ_n to Σ_+ by [32, Theorem 14.5.2].

□

Although our approach so far has been mostly algebraic, we can give probabilistic interpretations of the above results in the context of linear stochastic systems. Since (F, H) is detectable, we will take some G such that $F - GH$ is stable. Consider the following state–space representation (see, for example, Kailath [31]) of a stationary Gaussian process $\{Y_n\}_{n=-\infty}^{\infty}$:

$$\begin{aligned} S_{n+1} &= (F - GH)S_n - GU_n \\ Y_n &= HS_n + U_n \end{aligned} \tag{14}$$

where $\{U_n\}_{n=-\infty}^{\infty}$ are independent and identically distributed zero-mean unit-variance Gaussian random variables, and the *state* S_n is independent of U_n for each n. It is easy to see that $\{Y_n\}_{n=-\infty}^{\infty}$ corresponds to the filter output of the input process $\{U_n\}_{n=-\infty}^{\infty}$ through a linear-time invariant filter with transfer function

$$f(z) = \frac{\det(I - zF)}{\det(I - z(F - GH))}. \tag{15}$$

Consider the state–space representation for the innovations $\tilde{Y}_n = Y_n - E(Y_n | Y_{-\infty}^{n-1})$. Write $\tilde{S}_n = S_n - E\left(S_n | Y_{-\infty}^{n-1}\right)$ and $\Sigma_+ = \text{Cov}\left(S_n | Y_{-\infty}^{n-1}\right) = \text{Cov}(\tilde{S}_n)$. Define $\Gamma_+ = \Gamma(\Sigma_+)$ as before. Then, we can check through a little algebra that

$$\begin{aligned} \tilde{S}_{n+1} &= (F - \Gamma_+ H)\tilde{S}_n - \Gamma_+ U_n \\ \tilde{Y}_n &= H\tilde{S}_n + U_n \end{aligned}$$

which implies that

$$\begin{aligned} \Sigma_+ &= (F - \Gamma_+ H)\Sigma_+(F - \Gamma_+ H)' + \Gamma_+\Gamma_+' \\ &= F\Sigma_+ F' - \frac{(F\Sigma_+ H')(F\Sigma_+ H')'}{1 + H\Sigma_+ H'}. \end{aligned}$$

Clearly, there must be a unique solution Σ_+ to the above equation that makes the above state–space representation well-defined; this implies Lemma 2.4 ii).

Note that $\{\tilde{Y}_n\}_{n=-\infty}^{\infty}$ is the output of $\{U_n\}_{n=-\infty}^{\infty}$ via the filter

$$g(z) = \frac{\det(I - zF)}{\det(I - z(F - \Gamma_+ H))}.$$

On the other hand, the innovations process $\{\tilde{Y}_n\}_{n=-\infty}^{\infty}$ is white. Therefore, $g(z)$ should be a normalized Blaschke product (all-pass filter), which implies Lemma 2.4 iv). Furthermore, since $\text{var}(\tilde{Y}_n) = 1 + H\Sigma_+ H'$, applying Jensen's formula, we have a stronger version of Lemma 2.4 iii). The rank condition on Σ_+ in Lemma 2.4 vii) can be viewed as how many "modes" of the state can be causally determined by observing the output. Our development also gives a special case of the Szegő–Kolmogorov–Krein theorem. For example, if F is invertible

$$\begin{aligned} h(\mathcal{Y}) &= \frac{1}{2}\log(2\pi e(1 + H\Sigma_+ H')) \\ &= \frac{1}{2}\log\left(2\pi e\left(\frac{\det(F)}{\det(F - \Gamma_+ H)}\right)\right) \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\log\left(2\pi e\left|\frac{\det(I - e^{i\theta}F)}{\det(I - e^{i\theta}(F - GH))}\right|^2\right)\frac{d\theta}{2\pi} \end{aligned}$$

where the last equality can be justified by the canonical factorization theorem and Jensen's formula.

Now we consider a slightly nonstationary Gaussian process $\{Y_n'\}_{n=1}^{\infty}$, recursively defined by the same state–space (14), but under the different initial condition $S_0 = 0$ and $U_0 = 0$. Let T_n denote the linear transformation from $(U_1, \ldots, U_n)$ to $(Y_1', \ldots, Y_n')$ that corresponds to this state–space model. It is easy to see that T_n is Toeplitz (with respect to the natural basis on $(U_1, \ldots, U_n)$) and, in fact

$$T_n(j, k) = \int_{-\pi}^{\pi} f(e^{i\theta})e^{-i(j-k)\theta}\,\frac{d\theta}{2\pi} \tag{16}$$

where $f(z)$ is the very transfer function in (15). Since T_n is lower triangular with diagonal entries equal to 1 and thus $\det(T_n) = 1$ for all n, the entropy rate of $\{Y'_n\}$ is given as

$$h(\mathcal{Y}') = \lim_{n\to\infty} \frac{h(Y'_1,\ldots,Y'_n)}{n} = \lim_{n\to\infty} \frac{h(U_1,\ldots,U_n)}{n} = h(\mathcal{U}) = \frac{1}{2}\log(2\pi e) \quad (17)$$

which is strictly less than the entropy rate

$$h(\mathcal{Y}) = \frac{1}{2}\log\left(2\pi e\left(1 + H\Sigma_+ H'\right)\right) \quad (18)$$

of the stationary process $\{Y_n\}$ under the same state–space representation (14), provided that F has an eigenvalue outside the unit circle.

The nonzero gap between the entropy rate $h(\mathcal{Y})$ of the stationary process $\{Y_n\}_{n=1}^{\infty}$ and the entropy rate $h(\mathcal{Y}')$ of its nonstationary version $\{Y'_n\}_{n=1}^{\infty}$ can be understood from a beautiful result on Toeplitz operators by Widom; see Böttcher and Silbermann [3, Prop. 1.12, Prop. 2.12, and Example 5.1]. We use the notation $T(f)$ to denote the Toeplitz operator associated with *symbol* f as in (16) and $T_n(f) \in \mathbb{R}^{n\times n}$ to denote the finite truncation of $T(f)$. Since the power spectral density of the stationary process $\{Y_n\}$ is $|f(e^{i\theta})|^2$, our previous discussion on Toeplitz matrices and the trigonometric moment problem shows that the covariance matrix of $(Y_1,\ldots,Y_n)$ is simply $T_n(|f|^2)$. On the other hand, from our construction of the nonstationary process $\{Y'_n\}$, the covariance matrix of $(Y'_1,\ldots,Y'_n)$ is given as $T_n(f)(T_n(f))'$. Now Widom's theorem states that

$$T(|f|^2) = T(f)(T(f))' + (H(f))^2$$

where $H = H(f)$ is the Hankel operator associated with symbol f and is given by

$$H(j,k) = \int_{-\pi}^{\pi} f(e^{i\theta})e^{-i(j+k-1)\theta}\,\frac{d\theta}{2\pi}.$$

(This result should not be confused with the Wiener–Hopf factorization $T(|f|^2) = (T(f))'T(f)$; see [3, Section 1.5].) Thus, the Hankel adjustment term $H^2(f)$ contributes to the strict gap between the entropy rates in (17) and (18). We can represent $Y_n = Y'_n + V_n$ for some nonstationary process $\{V_n\}$ with infinite covariance matrix $H^2(f)$ such that $\sum_n EV_n^2 < \infty$. Roughly speaking, the perturbation process $\{V_n\}$ with bounded total power causes a strict boost in the entropy rate. (Although our f is rational, this phenomenon generalizes to any f in Krein algebra, in which case $H^2(f)$ is a trace class operator [3, Section 5.1].)

Finally we remark that our previous discussion on the Riccati recursion implies a much stronger result on the boost of the entropy rate due to small perturbation. Consider $Y''_n = Y'_n + V'_n$ where $(V'_1,\ldots,V'_k)$ has a positive definite covariance matrix and $V'_n \equiv 0$ for all $n > k$. Lemma 2.5 iv) shows that the entropy rate of $\{Y''_n\}$ is $\frac{1}{2}\log\left(2\pi e\left(1 + H\Sigma_+ H'\right)\right)$, and hence any tiny perturbation to the nonstationary process results in the entropy rate of the stationary version. Later, this phenomenon gives an alternative interpretation of the role of message-bearing signals in feedback communication.

The following example illustrates our point. Define a process $\{Y'_n\}_{n=1}^{\infty}$ as

$$Y'_1 = U_1$$
$$Y'_n = U_n + \alpha U_{n-1}, \quad n = 2, 3, \ldots$$

where α is a constant with $|\alpha| > 1$. Then, the entropy rate of the process $\{Y'_i\}_{i=1}^{\infty}$ is $\frac{1}{2}\log(2\pi e)$, although $\{Y_2, Y_3, \ldots\}$ is stationary with entropy rate $\frac{1}{2}\log(2\pi e\alpha^2)$. Now define $\{Y''_n\}_{n=1}^{\infty}$ as

$$Y''_1 = U_1 + \epsilon V$$
$$Y''_n = U_n + \alpha U_{n-1}, \quad n = 2, 3, \ldots$$

where $\epsilon > 0$ is an arbitrary constant and $V \sim N(0,1)$ is independent of $\{U_n\}_{n=1}^{\infty}$. Then, the entropy rate of the perturbed process can be easily shown to be $\frac{1}{2}\log(2\pi e\alpha^2)$. Evidently, the entropy rate is discontinuous at $\epsilon = 0$ and any tiny perturbation results in the same amount of boost in the entropy rate.

D. Matrix Inequalities

We recall the following facts on positive semidefinite Hermitian matrices. Proofs can be found in standard references on matrix analysis (see, for example, Gantmacher [19] and Horn and Johnson [24]) or can be derived easily from the related results therein.

Lemma 2.6: Suppose a Hermitian matrix K is partitioned as

$$K = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$$

where A and C are Hermitian. Further suppose C is positive definite. Then K is positive semidefinite if and only if $A - B'C^{-1}B$ is positive semidefinite.

Lemma 2.7: Suppose $K \in \mathbb{C}^{n\times n}$ is positive semidefinite Hermitian. Then

$$\log\det(K) \le \mathrm{tr}(K) - n$$

with equality if and only if $K = I_n$.

Lemma 2.8: Suppose K and $\tilde{K}$ are positive semidefinite Hermitian matrices of the same size. Then

$$\mathrm{tr}(K\tilde{K}) \ge 0.$$

Furthermore, the following statements are equivalent:

i) $\mathrm{tr}(K\tilde{K}) = 0$.
ii) $K\tilde{K} = 0$.
iii) There exist a unitary matrix Q and diagonal matrices D, $\tilde{D}$ such that $K = QDQ'$, $\tilde{K} = Q\tilde{D}Q'$, and $D\tilde{D} = 0$.

III. Variational Characterization of Gaussian Feedback Capacity

In this section, we present a variational characterization of the Gaussian feedback capacity C_{FB} as the solution to an infinite dimensional optimization problem. Our starting point is the following result by Cover and Pombra [8, Theorem 1], stated for stationary noise processes:

Lemma 3.1 (Cover–Pombra Theorem): Given a stationary Gaussian process $\{Z_n\}_{n=1}^{\infty}$ with $Z^n \sim N_n(0, K_{Z,n})$, let

$$C_{\mathrm{FB},n} = \max_{K_v, B} \frac{1}{2} \log \frac{\det(K_V + (I+B)K_{Z,n}(I+B)')^{1/n}}{\det(K_{Z,n})^{1/n}}$$

where the maximum is over all positive semidefinite $K_V = K_{V,n}$ and strictly lower triangular $B = B_n$ such that

$$\mathrm{tr}(K_V + BK_{Z,n}B') \le nP.$$

Then the feedback capacity of the stationary Gaussian channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under power constraint P, is

$$C_{\mathrm{FB}} = \lim_{n \to \infty} C_{\mathrm{FB},n}.$$

The following properties of the feedback capacity will be useful later, which are easy consequences of the Cover–Pombra theorem and are proved in [8, Theorem 3] and [75]:

Proposition 3.1: The feedback capacity $C_{\mathrm{FB}}(P)$ as a function of the power constraint P is concave and strictly increasing in P, and satisfies

$$C(P) \le C_{\mathrm{FB}}(P) \le 2C(P)$$

where $C(P)$ is the nonfeedback capacity under the same power constraint P.

While we do not repeat the proof of the Cover–Pombra theorem here (see [8, Sections VI and VII]), we examine closely the main ideas that are instrumental to the variational characterization in Theorem 3.2 later.

First note that the quantity $nC_{\mathrm{FB},n}$ corresponds to the maximum mutual information between the message index W and the channel output Y^n

$$\begin{aligned} I(W; Y^n) &= h(Y^n) - h(Y^n|W) \\ &= h(Y^n) - \sum_{i=1}^{n} h\left(Y_i|W, Y^{i-1}\right) \\ &= h(Y^n) - \sum_{i=1}^{n} h\left(Y_i|W, Y^{i-1}, X^i(W, Y^{i-1})\right) \\ &= h(Y^n) - \sum_{i=1}^{n} h\left(X_i + Z_i|W, Y^{i-1}, X^i, Z^{i-1}\right) \\ &= h(Y^n) - \sum_{i=1}^{n} h\left(Z_i|Z^{i-1}\right) \\ &= h(Y^n) - h(Z^n) \end{aligned} \tag{19}$$

which is maximized over all X^n of the form $X^n = V^n + B_n Z^n$, with strictly lower triangular B_n and multivariate Gaussian V^n, independent of Z^n, satisfying the power constraint $E\left(\sum_{i=1}^n X_i^2\right) \le nP$. Since $Y^n = V^n + (I + B_n)Z^n$, we have

$$\begin{aligned} I(V^n; Y^n) &= h(Y^n) - h(Y^n|V^n) \\ &= h(Y^n) - h\left(V^n + (I + B_n)Z^n|V^n\right) \\ &= h(Y^n) - h\left((I + B_n)Z^n\right) \\ &= h(Y^n) - h(Z^n). \end{aligned}$$

Thus $nC_{\mathrm{FB},n}$ can be viewed as the maximum mutual information between the channel output Y^n and the *auxiliary* channel input V^n over all strictly causal linear feedback

$$X^n = V^n + B_n Z^n \tag{20}$$

satisfying the power constraint P.

The achievability of C_{FB} over the sequence of channels $Y^n = V^n + (I + B_n)Z^n$, $n = 1, 2, \ldots$, follows from the asymptotic equipartition property for arbitrary nonstationary nonergodic Gaussian processes [8, Theorem 5], [55]. Note that the usual stationary ergodic AEP (the Shannon–McMillan–Breiman theorem) is not applicable since the optimal V^n (or the corresponding Y^n) is not necessarily stationary ergodic. (Alternatively, we can prove the achievability by recasting (19) as Massey's directed information $I(X^n \to Y^n)$ [44] and applying the coding theorem proved in [35].)

For the proof of the converse, it can be shown that the maximum of (19) over all causal feedback encoding functions $X_i(W, Y^{i-1})$, or equivalently, $X_i(W, Z^{i-1})$, can be attained over all *Gaussian linear* feedback input distributions of the form (20). This is a consequence of the usual maximum entropy argument for $h(Y^n)$ under the given covariance constraint for (X^n, Y^n, Z^n) and of the following little lemma that is hidden in the original proof in [8].

Lemma 3.2: Suppose that zero-mean random vectors $\xi_1 \to \xi_2 \to \xi_3$ form a Markov chain and

$$\xi_3 = L\xi_2 + \zeta \tag{21}$$

for some linear function (matrix) L and random vector ζ independent of (ξ_1, ξ_2). Let $(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)$ be jointly Gaussian zero-mean random vectors with the same covariance as (ξ_1, ξ_2, ξ_3). Then, $\tilde{\xi}_1 \to \tilde{\xi}_2 \to \tilde{\xi}_3$ also form a Markov chain.

Proof: Let $\hat{\xi}_1(\xi_2, \xi_3) = L_2^\star \xi_2 + L_3^\star \xi_3 = (L_2^\star + L_3^\star L)\,\xi_2 + L_3^\star \zeta$ be the best linear estimator of ξ_1 given (ξ_2, ξ_3) minimizing the mean square error $E(\xi_1 - \hat{\xi}_1)^2$. Since $\hat{\xi}_1$ is also the best linear estimator of ξ_1 given (ξ_2, ζ), and ξ_1 and ζ are uncorrelated, $L_3^\star$ should be zero and $\hat{\xi}_1(\xi_2, \xi_3) = L_2^\star \xi_2$.

Now focusing on the Gaussian version $(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)$ with the same covariance matrix, we note that the best linear estimator depends only on the covariance matrix, that the best nonlinear estimator (conditional expectation) is linear for jointly Gaussian random vectors, and that the conditional distribution of $\tilde{\xi}_1$ given $(\tilde{\xi}_2, \tilde{\xi}_3)$ is also Gaussian. Hence, $E(\tilde{\xi}_1|\tilde{\xi}_2, \tilde{\xi}_3) = L_2^\star \tilde{\xi}_2$, which proves the desired Markovity $\tilde{\xi}_1 \to \tilde{\xi}_2 \to \tilde{\xi}_3$. □

In other words, the Markovity is a second-moment property for a random triple $\xi_1 \to \xi_2 \to \xi_3$ satisfying (21). (The condition (21) is crucial; in general, the Gaussian version of a Markov chain under the same covariance structure is not necessarily Markov.)

As a corollary of this lemma, consider any feedback encoding function $X_i(W, Y^{i-1})$, $i = 1, 2, \ldots, n$. For each i, we have $X_i \to Z^{i-1} \to Z_i^n$. Also Z_i^n can be represented as $Z_i^n = L(Z^{i-1}) + \zeta_i$, where ζ_i is independent of (X^n, Z^{i-1}). Thus, from Lemma 3.2 we can construct a jointly Gaussian version $(\tilde{X}^n, Z^n)$ with the same covariance structure that satisfies the desired Markovity $\tilde{X}_i \to Z^{i-1} \to Z_i^n$ for each i, i.e., $\tilde{X}_i = \sum_{j=1}^{i-1} b_{ij} Z_j + V_i$, with V^n independent of Z^n. The corresponding Gaussian $\tilde{Y}^n = \tilde{X}^n + Z^n$ maximizes the entropy in (19).

For each n, the optimization problem for $C_{\mathrm{FB},n}$ can be reformulated by the change of variable $K_Y = K_V + (I+B)K_Z(I+B)'$), which results in the following convex optimization problem [71, eq. (2.16)] due to Boyd and Ordentlich (circa 1994):

$$\begin{aligned} &\text{maximize} && \log\det(K_Y) \\ &\text{subject to} && \begin{bmatrix} K_Y & I+B \\ (I+B)' & K_Z^{-1} \end{bmatrix} \succeq 0 \\ &&& \mathrm{tr}(K_Y - BK_Z - K_Z B' - K_Z) \le nP \\ &&& B \text{ strictly lower triangular.} \end{aligned} \tag{22}$$

(Here the linear matrix inequality constraint follows from Lemma 2.6.)

This optimization problem is an instance of matrix determinant maximization under linear matrix inequality constraints [71] and the optimal solution $(K_Y^\star, B^\star)$ (or equivalently, $(K_V^\star, B^\star)$) can be characterized via convex duality. In particular, we can show that the optimal $K_V^\star$ water-fills the new noise spectrum $(I+B^\star)K_Z(I+B^\star)'$ (so as to maximize $I(V^n;Y^n)$) and that the optimal filter $B^\star$ makes the input signal orthogonal to the past output, i.e., $K_V^\star + B^\star K_Z(I+B^\star)'$ should be upper-triangular (why waste power to convey something that is already known to the receiver?). When specialized to the kth-order ARMA noise spectrum, this observation implies that $K_V^\star$ should be of rank k, which in turn implies an important structural property of $B^\star$ from the orthogonality [50], [77], [34, Ch. 4].

From a numerical point of view, the duality result developed above gives the "solution" to the n-block feedback capacity problem, since there is a polynomial-time algorithm for the determinant maximization problem (22), based on the interior-point method. (See Nesterov and Nemirovskii [47] and Vandenberghe *et al.* [71].) In fact, Zahedi [79] developed a numerical solver that can handle arbitrary covariance matrices of size, say, $n = 200$, with moderate computing power.

As for the (infinite-block) feedback capacity, however, there is still much to be done, even numerically. First, the above duality theory is for a finite block size n, however large it may be. Since the limit $C_{\mathrm{FB}} = \lim_{n\to\infty} C_{\mathrm{FB},n}$ is approached from below without any convergent sequence of upper bounds, even the exact computation of $C_{\mathrm{FB},n}$ for very large n does not provide a computable characterization of C_{FB}. Second, the sequence of optimal $\left(K_{V,n}^\star, B_n^\star\right)$ is not necessarily consistent, that is, $\left(K_{V,n}^\star, B_n^\star\right)$ is not necessarily a subblock of $\left(K_{V,n+1}^\star, B_{n+1}^\star\right)$. Hence, *a priori* it is unclear whether the optimal solution is (asymptotically) stationary or Toeplitz. The question of the asymptotic optimality of stationarity becomes more challenging when we limit our attention to solutions that satisfy the optimality condition—in general, there is no stationary $\left(K_{V,n}^\star, B_n^\star\right)$ that satisfy water-filling condition and orthogonality. Finally, the optimality condition for $(K_V^\star, B^\star)$ has both temporal and spectral components (water-filling and orthogonality), and consequently, it seems very difficult, if not impossible, to derive an analytic solution for $\left(K_{V,n}^\star, B_n^\star\right)$ even for small n; cf. [9, Section 9.5] for the nonfeedback case.

Thus motivated, we move on to the main theme of this section—the variational characterization of the feedback capacity.

Theorem 3.2: The feedback capacity of the additive Gaussian noise channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P, is

$$C_{\mathrm{FB}} = \sup_{S_V, B} \int_{-\pi}^{\pi} \frac{1}{2} \log \frac{S_V(e^{i\theta}) + |1+B(e^{i\theta})|^2 S_Z(e^{i\theta})}{S_Z(e^{i\theta})} \frac{d\theta}{2\pi}$$

where $S_Z(e^{i\theta})$ is the power spectral density of $\{Z_i\}_{i=1}^\infty$, and the supremum is taken over all $S_V(e^{i\theta}) \ge 0$ and all strictly causal $B(e^{i\theta})$ satisfying the power constraint

$$\int_{-\pi}^{\pi} \left(S_V(e^{i\theta}) + |B(e^{i\theta})|^2 S_Z(e^{i\theta})\right) \frac{d\theta}{2\pi} \le P.$$

Proof: Let $\tilde{C}_{\mathrm{FB}}$ denote the information capacity

$$\sup_{S_V, B} \int_{-\pi}^{\pi} \frac{1}{2} \log \frac{S_V + |1+B|^2 S_Z}{S_Z} \frac{d\theta}{2\pi}.$$

In light of the Szegő–Kolmogorov–Krein theorem, we can express $\tilde{C}_{\mathrm{FB}}$ also as

$$\tilde{C}_{\mathrm{FB}} = \sup_{\{X_i\}} h(\mathcal{Y}) - h(\mathcal{Z})$$

where the supremum is taken over all stationary Gaussian processes $\{X_i\}_{i=-\infty}^\infty$ of the form $X_i = V_i + \sum_k b_k Z_{i-k}$ where $\{V_i\}_{i=-\infty}^\infty$ is stationary and independent of $\{Z_i\}_{i=-\infty}^\infty$ such that $EX_i^2 \le P$.

We first show that

$$C_{\mathrm{FB},n} \le \tilde{C}_{\mathrm{FB}} \tag{23}$$

for all n. Fix n and let $\left(K_{V,n}^\star, B_n^\star\right)$ achieve $C_{\mathrm{FB},n}$. Consider a process $\{V_i\}_{i=-\infty}^\infty$ that is independent of $\{Z_i\}_{i=-\infty}^\infty$ and blockwise white with $V_{kn+1}^{(k+1)n}$, $-\infty < k < \infty$, independent and identically distributed (i.i.d.) $\sim N_n(0, K_{V,n}^\star)$. Define a process $\{X_i\}_{i=-\infty}^\infty$ as $X_{kn+1}^{(k+1)n} = V_{kn+1}^{(k+1)n} + B_n^\star Z_{kn+1}^{(k+1)n}$ for all k. Let $Y_i = X_i + Z_i$, $-\infty < i < \infty$, be the corresponding output process through the stationary Gaussian channel; hence $Y_{kn+1}^{(k+1)n} = V_{kn+1}^{(k+1)n} + (I+B_n^\star)\, Z_{kn+1}^{(k+1)n}$ for all k. Then

$$\begin{aligned} 2C_{\mathrm{FB},n} &= I\left(V_1^n; Y_1^n\right) + I\left(V_{n+1}^{2n}; Y_{n+1}^{2n}\right) \\ &= h\left(V_1^n\right) + h(V_{n+1}^{2n}) - h(V_1^n|Y_1^n) - h(V_{n+1}^{2n}|Y_{n+1}^{2n}) \\ &\le h\left(V_1^{2n}\right) - h\left(V_1^{2n}|Y_1^{2n}\right) \\ &= I\left(V_1^{2n}; Y_1^{2n}\right) \\ &= h\left(Y_1^{2n}\right) - h\left(Z_1^{2n}\right). \end{aligned}$$

(In case $K_{V,n}^*$ is singular and $h(V_1^n)$ is ill-defined, we can consider a sequence of nonsingular $K_{V,n}^*$ that achieves $C_{\mathrm{FB},n}$ in the limit, as in the proof of the Cover–Pombra theorem [8, Theorem 1].)

By repeating the same argument, we have

$$C_{\mathrm{FB},n} \le \frac{1}{kn} I\left(V_1^{kn}; Y_1^{kn}\right) = \frac{1}{kn}\left(h\left(Y_1^{kn}\right) - h\left(Z_1^{kn}\right)\right)$$

for all k, which implies that

$$C_{\mathrm{FB},n} \le \frac{1}{m}\left(h\left(Y_1^m\right) - h\left(Z_1^m\right)\right) + \epsilon_m \tag{24}$$

where $\epsilon_m := (n/m)C_{\mathrm{FB},n}$ absorbs the edge effect.

For each $t = 0, 1, \ldots, n-1$, define the time-shifted process $\{V_i(t)\}_{i=-\infty}^{\infty}$ as $V_i(t) = V_{t+i}$ for all i, and similarly define $\{X_i(t)\}_{i=-\infty}^{\infty}$, $\{Y_i(t)\}_{i=-\infty}^{\infty}$, and $\{Z_i(t)\}_{i=-\infty}^{\infty}$. Then using a similar argument as in (24), we have

$$C_{\mathrm{FB},n} \le \frac{1}{m}\left(h\left(Y_1^m(t)\right) - h\left(Z_1^m(t)\right)\right) + 2\epsilon_m$$

for all $m = 1, 2, \ldots$, and each $t = 0, \ldots, n-1$.

We now introduce a random variable T, uniformly distributed over $\{0, 1, \ldots, n-1\}$ and independent of everything else. It is easy to check the following.

i) $\{X_i(T), Y_i(T), Z_i(T)\}_{i=-\infty}^{\infty}$ is stationary with $Y_i(T) = X_i(T) + Z_i(T)$.

ii) $\{Z_i(T)\}_{i=-\infty}^{\infty}$ has the same distribution as $\{Z_i\}_{i=-\infty}^{\infty}$.

iii) $\{X_i(T)\}_{i=-\infty}^{\infty}$ has the average power upper bounded by

$$\begin{aligned} E\left[X_i^2(T)\right] &= E\left[E\left(X_1^2(T)|T\right)\right] \\ &= \frac{1}{n}\sum_{j=1}^{n} E\left(X_j^2\right) \\ &= \frac{1}{n}\operatorname{tr}\left(K_{V,n}^{\star} + B_n^{\star} K_{Z,n} (B_n^{\star})'\right) \le P. \end{aligned}$$

iv) $X_i(T) \to Z_{i-1-n}^{i-1}(T) \to Z_j(T)$ form a Markov chain for all i, j, because

$$\begin{aligned} & I\left(X_i(T); Z_j(T) | Z_{i-1-n}^{i-1}(T)\right) \\ &\quad \le I\left(X_i(T); Z_j(T) | Z_{i-1-n}^{i-1}(T), T\right) = 0 \end{aligned}$$

where the inequality follows since $\{Z_i\}_{i=-\infty}^{\infty}$ is stationary and hence

$$\begin{aligned} h\left(Z_j | Z_{i-1-n}^{i-1}\right) &= h\left(Z_j(T) | Z_{i-1-n}^{i-1}(T)\right) \\ &= h\left(Z_j(T) | Z_{i-1-n}^{i-1}(T), T\right). \end{aligned}$$

Finally, define $\{\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i\}_{i=-\infty}^{\infty}$ to be a jointly Gaussian process with the same mean and autocorrelation as the stationary process $\{X_i(T), Y_i(T), Z_i(T)\}_{i=-\infty}^{\infty}$. It is easy to check that $\{\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i\}$ also satisfies the properties i)–iv); in particular, the Markovity iv) follows from Lemma 3.2. Therefore, from these properties and the Gaussianity of $\{\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i\}$, there must exist a stationary Gaussian process $\{\tilde{V}_i\}$ and $b_0, \ldots, b_{n-1} \in \mathbb{R}$ such that $\tilde{X}_i = \tilde{V}_i + \sum_{k=1}^{n} b_k \tilde{Z}_{i-k}$. Thus,

$$\begin{aligned} C_{\mathrm{FB},n} &\le \frac{1}{m}(h(Y_1^m(T)|T) - h(Z_1^m)) + 2\epsilon_m \\ &\le \frac{1}{m}(h(Y_1^m(T)) - h(Z_1^m)) + 2\epsilon_m \\ &\le \frac{1}{m}(h(\tilde{Y}_1^m) - h(Z_1^m)) + 2\epsilon_m. \end{aligned}$$

Since the singular part of the power spectral distribution of the process $\{\tilde{V}_i\}$ would not contribute to $h(\tilde{Y}_1^m)$ while consuming the power, we can assume without loss of generality that the process $\{\tilde{V}_i\}$ has an absolutely continuous power spectral distribution. Letting $m \to \infty$ and using the definition of $\tilde{C}_{\mathrm{FB}}$, we get

$$C_{\mathrm{FB},n} \le h(\tilde{\mathcal{Y}}) - h(\mathcal{Z}) \le \tilde{C}_{\mathrm{FB}}.$$

For the other direction of the inequality, we use the notation $\tilde{C}_{\mathrm{FB}}(P)$ and $C_{\mathrm{FB},n}(P)$ to stress the dependence of feedback capacity on the power constraint P. Given $\epsilon > 0$, let $\{\tilde{X}_i = V_i + \sum_{k=1}^{m} b_k Z_{i-k}\}_{i=-\infty}^{\infty}$ achieve $\tilde{C}_{\mathrm{FB}}(P) - \epsilon$ under the power constraint P. The corresponding channel output is

$$\tilde{Y}_i = V_i + Z_i + \sum_{k=1}^{m} b_k Z_{i-k}.$$

Now, we define a single-sided nonstationary random process $\{X_i\}_{i=1}^{\infty}$ as

$$X_i = \begin{cases} U_i + V_i + \sum_{k=1}^{i-1} b_k Z_{i-k}, & i \le m \\ U_i + V_i + \sum_{k=1}^{m} b_k Z_{i-k}, & i > m \end{cases}$$

where $U_1, U_2, \ldots$ are i.i.d. $\sim N(0, \epsilon)$, independent of $\{Z_i\}$ and $\{V_i\}$. Thus, X_i depends causally on Z_1^{i-1} for all i. Let $\{Y_i\}_{i=1}^{\infty}$ be the corresponding channel output $Y_i = X_i + Z_i$. Since $EX_i^2 < \infty$ for $i \le m$ and

$$EX_i^2 = E\tilde{X}_i^2 + EU_i^2 = P + \epsilon$$

for $i > m$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} EX_i^2 = P + \epsilon.$$

Also, since $h\left(Y_1^m | Y_{m+1}^n\right) \ge h\left(U_1^m | Y_{m+1}^n\right) = h\left(U_1^m\right) > -\infty$ and $Y_i = \tilde{Y}_i + U_i$ for $i > m$

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} h\left(Y_1^n\right) &= \lim_{n\to\infty} \frac{1}{n} h(Y_{m+1}^n) \\ &\ge h(\tilde{\mathcal{Y}}) = h(\mathcal{Z}) + \tilde{C}_{\mathrm{FB}}(P) - \epsilon. \end{aligned}$$

Consequently, for n sufficiently large

$$\frac{1}{n}\sum_{i=1}^{n} EX_i^2 \le (P + 2\epsilon)$$

and

$$\frac{1}{n} h\left(Y_1^n\right) - h\left(Z_1^n\right) \ge \tilde{C}_{\mathrm{FB}}(P) - 2\epsilon.$$

Therefore, we can conclude that

$$C_{\mathrm{FB},n}(P + 2\epsilon) \ge \tilde{C}_{\mathrm{FB}}(P) - 2\epsilon$$

for n sufficiently large, whence

$$\liminf_{n\to\infty} C_{\mathrm{FB},n}(P + 2\epsilon) \ge \tilde{C}_{\mathrm{FB}}(P) - 2\epsilon.$$

Taking $\epsilon \to 0$ and using Proposition 3.1, we get

$$\liminf_{n\to\infty} C_{\mathrm{FB},n}(P) \ge \tilde{C}_{\mathrm{FB}}(P),$$

which, combined with (23), implies the desired result

$$\lim_{n\to\infty} C_{\mathrm{FB},n}(P) = \tilde{C}_{\mathrm{FB}}(P). \qquad \square$$

One implication of the above proof is that we can limit our attention to the class of stationary solutions $(K_{V,n}, B_n)$ or $(K_{Y,n}, B_n)$. (Note that in general K_Y and K_V cannot be simultaneously Toeplitz under a Toeplitz matrix B.) The cost of this simplification is, however, that the optimality condition for $C_{\mathrm{FB},n}$ becomes no longer relevant after this stationarization technique. For example, the optimality of rank-k $K_V^\star$ does not necessarily imply the asymptotic optimality of a rank-k Toeplitz $K_{V,n}$. Furthermore, adding the stationarity constraint does not lead to a simpler analytic characterization of $(K_V^\star, B^\star)$. Therefore, it is inevitable that we should set aside the previously known results on the finite-block feedback capacity [50], [77] and attack the variational problem in Theorem 3.2 rather directly. Compared to the finite-optimization problem for $C_{\mathrm{FB},n}$, this infinite-dimensional problem poses several technical challenges. Thus, we characterize the optimal $(S_V^\star, B^\star)$ over Sections IV–VI at different levels of generality.

IV. Optimal Feedback Coding

Equipped with Theorem 3.2, our next goal is to solve the resulting infinite-dimensional optimization problem:

$$\begin{aligned} \text{maximize} \quad & \int_{-\pi}^{\pi} \log\big(S_V(e^{i\theta}) + |1+B(e^{i\theta})|^2 S_Z(e^{i\theta})\big)\,\frac{d\theta}{2\pi} \\ \text{subject to} \quad & S_V(e^{i\theta}) \ge 0 \\ & B(e^{i\theta}) = \sum_{k=1}^{m} b_k e^{ik\theta} \ \text{ strictly causal} \\ & \int_{-\pi}^{\pi} S_V(e^{i\theta}) + |B(e^{i\theta})|^2 S_Z(e^{i\theta})\,\frac{d\theta}{2\pi} \le P. \end{aligned} \tag{25}$$

While characterizing the optimal solution $(S_V^\star, B^\star)$ in a completely analytic expression seems out of reach, we show that one can take $S_V^\star \equiv 0$ without loss of generality and find a sufficient and necessary condition for the optimal solution that can be easily verified. In particular, we will prove the following theorem.

Theorem 4.1: Suppose the power spectral density $S_Z(e^{i\theta})$ of the Gaussian noise process $\{Z_i\}_{i=1}^{\infty}$ is bounded away from zero, i.e., $\operatorname{ess\,inf}_\theta S_Z(e^{i\theta}) > 0$, and has a canonical spectral factorization $S_Z(e^{i\theta}) = |H_Z(e^{i\theta})|^2$. Then, the feedback capacity of the additive Gaussian noise channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P, is

$$C_{\mathrm{FB}} = \max_b \int_{-\pi}^{\pi} \frac{1}{2} \log|1+B(e^{i\theta})|^2 \,\frac{d\theta}{2\pi} \tag{26}$$

where the maximum is taken over all strictly causal $B(e^{i\theta})$ satisfying the power constraint

$$\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta})\,\frac{d\theta}{2\pi} \le P.$$

Furthermore, $B(e^{i\theta})$ attains the maximum in (26) if and only if

i) *Power:*

$$\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta})\,\frac{d\theta}{2\pi} = P.$$

ii) *Output spectrum:*

$$\eta := \operatorname*{ess\,inf}_{\theta\in[-\pi,\pi)} |1+B(e^{i\theta})|^2 S_Z(e^{i\theta}) > 0.$$

iii) *Strong orthogonality:* For some $0 < \lambda \le \eta$

$$\frac{\lambda}{(1+B(z))H_Z(z)} - B(z^{-1})H_Z(z^{-1})$$

is causal.

One important aspect of the theorem is the existence of the optimal solution $B^\star$ and its characterization via conditions i)–iii). The condition iii) deals with the canonical factorization of H_Z and is reminiscent of the Wiener–Hopf solution for causal filtering. The stipulation that $S_Z(e^{i\theta})$ is bounded away from zero should not incur much loss of generality, for we can always perturb the noise spectrum by a small amount of power.

The proof of Theorem 4.1 requires several auxiliary propositions, which are interesting and useful on their own. To simplify the notation, we will omit the argument $e^{i\theta}$ from functions on $\mathbb{T} = \{e^{i\theta} : \theta \in [-\pi, \pi)\}$. We use $\overline{f}$ to denote the complex conjugate of $f(e^{i\theta})$. Since we only consider $f(z) \in \mathcal{H}_p$, $1 \le p \le \infty$, with real Fourier coefficients, $\overline{f} = f(e^{-i\theta})$. In the same vein, we use $\overline{f}$ as a shorthand notation for $f(z^{-1})$.

Recall that the optimization problem (25) is equivalent to the maximization of the entropy rate of the stationary process $\{Y_i\}_{i=-\infty}^{\infty}$ given by $Y_i = X_i + Z_i$, over all stationary processes $\{X_i\}_{i=-\infty}^{\infty}$ of the form $X_i = V_i + \sum_{k=1}^{\infty} b_k Z_{i-k}$. Whenever necessary, our discussion will resort to the context of stationary processes and corresponding entropy rates.

We start by studying the properties of an optimal solution $(S_V^\star, B^\star)$ to the optimization problem (25).

Proposition 4.2 (Necessary Condition): Suppose $(S_V^\star, B^\star)$ attains the maximum for the optimization problem (25). Then the following must hold.

i) *Power:* $\int_{-\pi}^{\pi} S_V^\star + |B^\star|^2 S_Z \frac{d\theta}{2\pi} = P$.

ii) *Output spectrum:* $\eta^\star := \operatorname{ess\,inf}_{\theta\in[-\pi,\pi)} S_Y^\star > 0$, where $S_Y^\star = S_V^\star + |1+B^\star|^2 S_Z$.

iii) *Water-filling:* $S_V^\star$ water-fills the modified noise spectrum $|1+B^\star|^2 S_Z$, that is, $S_V^\star\,(S_Y^\star - \eta^\star) = 0$ a.e.

iv) *Weak orthogonality:* The current input X_n is independent of the past output $\{Y_i\}_{-\infty}^{n-1}$. Equivalently

$$S_V^\star(z) + B^\star(z^{-1}) S_Z(z)(1+B^\star(z))$$

is causal.

Furthermore, if S_Z is bounded away from zero, then there exist $S_V^\star$ and $B^\star = \sum_{k=1}^{\infty} b_k e^{ik\theta}$ attaining the maximum of (25).

Proof: Necessity of i) and iii) is obvious; since each fixed B gives a nonfeedback channel $|1+B(e^{i\theta})|^2 S_Z(e^{i\theta})$ with the input spectrum $S_V(e^{i\theta})$, the optimality conditions for the nonfeedback capacity apply. See, for example, [34, Ch. 3].

For the condition ii), suppose $\eta^\star = 0$. Then for each n, there exists $T_n \subseteq \mathbb{T}$ such that $S_Y^\star \le 1/n$ on T_n. Let $S_V = \delta$ on T_n and 0 elsewhere. Now consider a feasible solution $(S_V^\star + S_V, B^\star)$ with corresponding output spectrum $S_Y^{\star\star}$ and power constraint $P+\delta|T_n|/(2\pi)$, where $|T_n|$ is the Lebesgue measure of T_n. It is easy to see that

$$\int_{-\pi}^{\pi} \frac{1}{2}\log S_Y^{\star\star}\,\frac{d\theta}{2\pi} - \int_{-\pi}^{\pi} \frac{1}{2}\log S_Y^{\star}\,\frac{d\theta}{2\pi} \geq \frac{|T_n|}{2\pi}\log(1+n\delta).$$

But dividing both sides by the power increase $\delta|T_n|/(2\pi)$ and taking $\delta\to 0$, we see that $C'_{\rm FB}(P)$ is unbounded, which contradicts the facts that $C_{\rm FB}(P)$ is concave and hence that $C'(P)<\infty$ for $P>0$.

The orthogonality condition iv) is also intuitively clear; there is no reason to spend power to send something that is already in the linear span of previous output symbols. To verify this intuition rigorously, we use the following perturbation method similar to Ordentlich's fixed-point argument [50, Presentation slides] developed to find the optimality condition for the finite-dimensional case $C_{{\rm FB},n}$. Suppose $S_V^\star+\overline{B^\star}(1+B^\star)S_Z$ is *not* causal. Then

$$\int_{-\pi}^{\pi} \left(S_V+\overline{B^\star}(1+B^\star)S_Z\right)e^{-in\theta}\,\frac{d\theta}{2\pi}=\gamma\neq 0$$

for some $n\geq 1$. Let $A(e^{i\theta})=xe^{in\theta}$ with $|x|<1$. Consider $(S_V^{\star\star},B^{\star\star})=\left(|1+A|^2S_V^\star,(1+A)(1+B^\star)-1\right)$. Since the corresponding output spectrum

$$\begin{aligned}S_Y^{\star\star}&=S_V^{\star\star}+|1+B^{\star\star}|^2S_Z\\&=|1+A|^2S_V^\star+|1+A|^2|1+B^\star|^2S_Z\\&=|1+A|^2S_Y^\star\end{aligned}$$

the entropy rate stays the same for $S_Y^{\star\star}$ by Jensen's formula (10). On the other hand, the power usage becomes

$$\begin{aligned}P^{\star\star}(x)&=\int_{-\pi}^{\pi} S_V^{\star\star}+|B^\star|^2S_Z\,\frac{d\theta}{2\pi}\\&=\int_{-\pi}^{\pi}|1+A|^2S_V^\star+|A(1+B^\star)+B^\star|^2S_Z\,\frac{d\theta}{2\pi}\\&=\int_{-\pi}^{\pi}\left(S_V^\star+|B^\star|^2S_Z\right)\frac{d\theta}{2\pi}\\&\quad+2\int_{-\pi}^{\pi}A\left(S_V^\star+\overline{B^\star}(1+B^\star)S_Z\right)\frac{d\theta}{2\pi}\\&\quad+\int_{-\pi}^{\pi}|A|^2\left(S_V^\star+|1+B^\star|^2S_Z\right)\frac{d\theta}{2\pi}\\&=P+2\gamma x+P_Yx^2\end{aligned}$$

where $P_Y=\int_{-\pi}^{\pi}S_Y^\star\frac{d\theta}{2\pi}$ is the original output power. Since $P^{\star\star}(x)$ is quadratic in x with the leading coefficient $P_Y>0$, we can choose x small with appropriate sign so that $P^{\star\star}(x)<P$. But this implies that $(S_V^{\star\star},B^{\star\star})$ achieves the same entropy rate as the original $(S_V^\star,B^\star)$ using strictly less power. This contradicts the optimality of $(S_V^\star,B^\star)$ and hence we have the causality of $S_V^\star+\overline{B^\star}(1+B^\star)S_Z$.

The proof of the existence of the optimal $(S_V^\star,B^\star)$ is rather technical, so it will be given in Appendix A. □

Although the conditions i)–iv) are not sufficient and fall short of tighter conditions in Theorem 4.1, we can deduce several interesting observations from them.

Corollary 4.3: Feedback does not increase the capacity if and only if S_Z is constant (i.e., the noise spectrum is white).

Proof: Shannon's 1956 paper shows that feedback does not increase the capacity for memoryless channels, taking care of the sufficiency. (See also Kadota, Zakai, and Ziv [29], [30].)

For the necessity, we assume that S_Z is bounded away from zero without loss of generality. Indeed, we can use a small amount of power to water-fill the spectrum first, then use the remaining power to code with or without feedback. If the stated claim is true, then feedback increases the capacity for the modified channel and hence for the original channel. (For the nonfeedback coding, there is no loss of optimality in dividing the power into two parts and water-filling successively.)

Proceeding on to the proof of the necessity, suppose nonfeedback capacity is the same as the feedback capacity and is achieved by $S_X^\star$. Then $(S_V^\star,B^\star)=(S_X^\star,0)$ achieves the feedback capacity. But, from the condition iv) of Proposition 4.2,

$$S_V^\star+\overline{B^\star}S_Z(e^{i\theta})(1+B^\star)=S_X^\star$$

is causal and hence is white. Therefore, S_Z, whose water-filling spectrum is white. □

Corollary 4.4: Suppose $(S_V^\star,B^\star)$ attains the maximum for the optimization problem (25). Then, there exists $B^{\star\star}$ such that

$$S_Y^\star=S_V^\star+|1+B^\star|^2S_Z=|1+B^{\star\star}|^2S_Z$$

and

$$\int_{-\pi}^{\pi}S_V^\star+|B^\star|^2S_Z\,\frac{d\theta}{2\pi}=\int_{-\pi}^{\pi}|B^{\star\star}|^2S_Z\,\frac{d\theta}{2\pi}.$$

In particular, $(0,B^{\star\star})$ is also an optimal solution to (25).

In order to prove Corollary 4.4, we need the following simple result, which essentially establishes the optimality of the original Schalkwijk–Kailath coding for the additive white Gaussian noise channel.

Lemma 4.1: Suppose the noise spectrum is white with $S_Z\equiv N$. Then, the choice of $S_V^\star\equiv 0$ and

$$B^\star(e^{i\theta})=\frac{1-a^{-1}e^{i\theta}}{1-ae^{i\theta}}-1 \qquad (27)$$

with $a=\sqrt{\frac{N}{P+N}}$ achieves the feedback capacity $C_{\rm FB}=C=\frac{1}{2}\log\left(1+\frac{P}{N}\right)$ under the power constraint P. Furthermore, the resulting output spectrum is given by

$$S_Y^\star\equiv P+N.$$

Proof: We first check that

$$\begin{aligned}S_Y^\star(z)&=N(1+B^\star(z))(1+B^\star(z^{-1}))\\&=N\cdot\frac{1-a^{-1}z}{1-az}\cdot\frac{1-a^{-1}z^{-1}}{1-az^{-1}}\\&=Na^{-2}=P+N.\end{aligned}$$

On the other hand, since

$$B^\star(e^{i\theta})=(1-a^{-2})\sum_{k=1}^{\infty}a^ke^{ik\theta}$$

we have

$$\begin{aligned}\int_{-\pi}^{\pi} |B^\star|^2 S_Z \frac{d\theta}{2\pi} &= N(1-a^{-2})^2 \int_{-\pi}^{\pi} \left|\sum_{k=1}^{\infty} a^k e^{ik\theta}\right|^2 \frac{d\theta}{2\pi} \\ &= N(1-a^{-2})^2 \cdot \frac{a^2}{1-a^2} \\ &= a^{-2} - 1 \\ &= P.\end{aligned}$$

Clearly, we have achieved $C_{\mathrm{FB}}(P) = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$. □

The choice of the feedback filter (27) is far from unique; for example, we can use (check!) any causal filter derived from the normalized Blaschke product as

$$B(z) = \prod_{k=1}^{\infty} \frac{1 - a_k^{-1} z^{j_k}}{1 - a_k z^{j_k}} - 1 \tag{28}$$

where $\{j_k\}_{k=1}^{\infty}$ is an arbitrary sequence of positive integers and $\{a_k\}_{k=1}^{\infty}$ is a sequence of real numbers such that $|a_k| < 1$ for all k and $\prod_{k=1}^{\infty} a_k^2 = N/(P+N)$.

Now we move on to the proof of Corollary 4.4.

Proof of Corollary 4.4: Suppose

$$\int_{-\pi}^{\pi} S_V^\star \frac{d\theta}{2\pi} = P_1$$

and

$$\int_{-\pi}^{\pi} |B^\star|^2 S_Z \frac{d\theta}{2\pi} = P - P_1.$$

We assume $P_1 > 0$; otherwise, there is nothing to prove.

We argue that $S := |1+B^\star|^2 S_Z$ must be white. Assume the contrary and consider the Gaussian feedback channel with the noise spectrum S under the power constraint P_1. But from Corollary 4.3, $(S_V^\star, 0)$ is strictly dominated by some (S_V, B) with nonzero B. Hence, for the original channel, we have a two-stage solution $(S_V, (1+B^\star)(1+B)-1)$ with the corresponding output entropy higher than that of the original $S_Y^\star$, which contradicts the optimality of $(S_V^\star, B^\star)$.

Now suppose the white spectrum $S = |1+B^\star|^2 S_Z$ has the power, say, N_1. From the water-filling condition iii) in Proposition 4.2, $S_V^\star \equiv P_1$ and the resulting output spectrum $S_Y^\star \equiv P_1 + N_1$. On the other hand, from Lemma 4.1, we can achieve the feedback capacity $\frac{1}{2}\log\left(1+\frac{P_1}{N_1}\right)$ for the new channel S by using $B = (1-a^{-1}e^{i\theta})/(1-ae^{i\theta})$, $a = \sqrt{N_1/(P_1+N_1)}$. Consequently, we can achieve the feedback capacity of the original channel S_Z through a two-stage strategy: first transform the channel into S using $B^\star$, and then use B for the white spectrum S. The corresponding combined filter is given by

$$B^{\star\star} = (1+B^\star)(1+B) - 1$$

and $(0, B^{\star\star})$ achieves the feedback capacity with the same output spectrum $S_Y^\star$. □

Remark 4.5: We can in fact make a stronger statement—if S_Z is nonwhite, then $S_V^\star$ must be zero. To see this, first note from the above proof that, if $S_V^\star$ is nonzero, then $|1+B^\star|^2 S_Z$ and $S_V^\star$, as well as $S_Y^\star$ should be white. Now from the orthogonality condition iv) in Proposition 4.2, $S_V^\star + \overline{B^\star} S_Z(1+B^\star) = S_Y^\star - S_Z(1+B^\star)$ is causal, or equivalently, $S_Z(1+B^\star)$ is causal, which is true only if S_Z is white.

The essential content of Corollary 4.4 is that we can restrict attention to the solutions of the form $(0, B)$, even in the case the maximum in (25) is not attainable. Indeed, we can easily modify the proof of Corollary 4.4 to show that for any solution (S_V, B), there exists another solution $(0, \tilde{B})$ such that the corresponding output entropy rate is no less than the original output entropy rate under the same power usage. This observation yields the following simplification of Theorem 3.2.

Theorem 4.6: The feedback capacity of the additive Gaussian noise channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P, is

$$C_{\mathrm{FB}} = \sup_B \int_{-\pi}^{\pi} \frac{1}{2}\log|1+B(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

where $S_Z(e^{i\theta})$ is the power spectral density of $\{Z_i\}_{i=1}^{\infty}$ and the supremum is taken over all strictly causal $B(e^{i\theta})$ satisfying the power constraint

$$\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} \le P.$$

Equipped with Corollary 4.4, we are ready to show that the strong orthogonality condition is necessary, which completes the proof of the necessity part of Theorem 4.1.

Proposition 4.7 (Necessity of the Strong Orthogonality): Suppose $S_Z = |H_Z|^2$ is bounded away from zero and $(S_V^\star \equiv 0, B^\star)$ attains the maximum for the optimization problem (25). Let $\eta^\star := \operatorname{ess\,inf}_{\theta\in[-\pi,\pi)} |1+B^\star|^2 S_Z$. Then there exists a $\lambda \in (0, \eta^\star]$ such that

$$\frac{\lambda}{(1+B^\star(z))H_Z(z)} - B^\star(z^{-1})H_Z(z^{-1})$$

is causal.

Proof: Consider the Lagrangian

$$\begin{aligned}J(S_V, B, \lambda) \\ &= \lambda \int \log\left(S_V + |(1+B)H_Z|^2\right) - \int \left(S_V + |BH_Z|^2\right)\end{aligned}$$

for the optimization problem (25). Since $C_{\mathrm{FB}}(P)$ is concave in P (recall Proposition 3.1), if $(S_V^\star \equiv 0, B^\star)$ attains the maximum under the power constraint $P > 0$, then $J(S_V^\star, B^\star, \lambda) \ge J(S_V, B, \lambda)$ for all (S_V, B) for some $\lambda = \lambda(P) > 0$. In particular

$$J(0, B^\star, \lambda) \ge J(0, B^\star + xe^{in\theta}, \lambda)$$

for all x and $n = 1, 2, \ldots$. This implies that

$$\frac{d}{dx} J(B^\star + xe^{in\theta}) = 0$$

at $x = 0$, or equivalently

$$2\int_{-\pi}^{\pi} e^{-in\theta}\left[\frac{\lambda}{(1+B^\star)H_Z} - \overline{B^\star H_Z}\right] = 0.$$

(Here the interchange of the order of differentiation and integration is justified by the bounded convergence theorem.) Hence, $\lambda/(1+B^\star(z))H_Z(z) - B^\star(z^{-1})H_Z(z^{-1})$ is causal (and in $\mathcal{H}_2$).

To see that $\lambda \le \eta^\star$, we use another simple variational method. For each $\delta > 0$, let $T_\delta \subseteq \mathbb{T}$ be such that $S_Y^\star = |1+B^\star|^2 S_Z \le \eta^\star + \delta$ on T_δ. Let $S_V = \delta$ on T_δ and 0 elsewhere. Now from the optimality of $(S_V^\star \equiv 0, B^\star)$, we have

$$\lambda \int_{-\pi}^{\pi} \log(S_Y^\star + S_V) - \log(S_Y^\star) \frac{d\theta}{2\pi} \le \int_{-\pi}^{\pi} S_V \frac{d\theta}{2\pi}$$

which implies

$$\lambda \log\left(1 + \frac{\delta}{\eta^\star + \delta}\right) \le \delta.$$

Taking $\delta \to 0$, we get the desired result. □

Now we move on to the sufficiency part of Theorem 4.1. We reformulate the original infinite-dimensional optimization problem (25) as a convex optimization problem by the change of variable $S_Y = |1+B|^2 S_Z + S_V$

$$\begin{aligned} \text{maximize} \quad & \int_{-\pi}^{\pi} \log S_Y \frac{d\theta}{2\pi} \\ \text{subject to} \quad & S_Y \ge |1+B|^2 S_Z \\ & B \text{ strictly causal} \\ & \int_{-\pi}^{\pi} S_Y - (B + \overline{B} + 1) S_Z \frac{d\theta}{2\pi} \le P. \end{aligned} \tag{29}$$

Note that this optimization problem is an infinite-dimensional analogue of the matrix determinant maximization problem (22) for $C_{\mathrm{FB},n}$. However, it is often very difficult to establish the strong duality for infinite-dimensional optimization problems, even when the problem is convex. (See Ekeland and Temam [15].) Here we avoid using the general duality theory on topological vector spaces and take a rather elementary approach to duality, which turns out to be powerful enough to characterize the optimal solution $B^\star$ in a reasonable form.

Assume that $S_Z = |H_Z|^2$ is bounded away from zero with canonical factor $H_Z \in \mathcal{H}_2$. Then from Proposition 4.2, the maximum in (29) must be attained by some B and $S_Y = |1+B|^2 S_Z$.

Take any $\nu > 0$, $\phi, \psi_1 \in \mathcal{L}_\infty$, and $\psi_2, \psi_3 \in \mathcal{L}_1$ such that

$$\phi > 0 \tag{30}$$
$$\log\phi \in \mathcal{L}_1 \tag{31}$$
$$\overline{\psi_2} H_Z^{-1} \in \mathcal{L}_2 \tag{32}$$
$$\psi_1 = \nu - \phi \ge 0 \tag{33}$$
$$A := \overline{\psi_2} + \nu S_Z \in \mathcal{H}_1 \text{ is causal} \tag{34}$$

and

$$\begin{bmatrix} \psi_1 & \psi_2 \\ \overline{\psi_2} & \psi_3 \end{bmatrix} \succeq 0. \tag{35}$$

Since any feasible B and S_Y satisfy

$$\begin{bmatrix} S_Y & 1+B \\ 1+\overline{B} & S_Z^{-1} \end{bmatrix} \succeq 0$$

we have from Lemma 2.8 that

$$\begin{aligned} &\mathrm{tr}\left(\begin{bmatrix} S_Y & 1+B \\ 1+\overline{B} & S_Z^{-1} \end{bmatrix} \begin{bmatrix} \psi_1 & \psi_2 \\ \overline{\psi_2} & \psi_3 \end{bmatrix} \right) \\ &= \psi_1 S_Y + \psi_2(1+\overline{B}) + \overline{\psi_2}(1+B) + \psi_3 S_Z^{-1} \\ &\ge 0. \end{aligned}$$

Since $\log x \le x - 1$ for all $x > 0$, by taking $x = \phi S_Y$

$$\begin{aligned} \log S_Y &\le -\log\phi + \phi S_Y - 1 \\ &= -\log\phi + \nu S_Y - \psi_1 S_Y - 1 \\ &\le -\log\phi + \nu S_Y + \psi_2(1+\overline{B}) + \overline{\psi_2}(1+B) \\ &\quad + \psi_3 S_Z^{-1} - 1. \end{aligned} \tag{36}$$

Furthermore, since $A = \overline{\psi_2} + \nu S_Z$ is causal and B is strictly causal, $AB \in \mathcal{L}_1$ is strictly causal; recall Lemma 2.3. (Indeed, $\overline{\psi_2} B = \left(\overline{\psi_2} H_Z^{-1}\right) \cdot (H_Z B) \in \mathcal{L}_1$ since the first factor is $\mathcal{L}_2$ while the second factor is $\mathcal{H}_2$.) Hence

$$\int AB = \int \overline{AB} = 0. \tag{37}$$

By integrating both sides of (36), we get

$$\begin{aligned} &\int \log S_Y \\ &\le \int -\log\phi + \nu S_Y + \psi_2(1+\overline{B}) + \overline{\psi_2}(1+B) \\ &\quad + \psi_3 S_Z^{-1} - 1 \\ &\le \int -\log\phi + \nu\left((B+\overline{B}+1)S_Z + P\right) \\ &\quad + \psi_2(1+\overline{B}) + \overline{\psi_2}(1+B) + \psi_3 S_Z^{-1} - 1 \\ &= \int -\log\phi + \psi_2 + \overline{\psi_2} + \psi_3 S_Z^{-1} + \nu(S_Z + P) - 1 \\ &\quad + AB + \overline{AB} \\ &= \int -\log\phi + 2\psi_2 + \psi_3 S_Z^{-1} + \nu(S_Z+P) - 1 \end{aligned} \tag{38}$$

where the second inequality follows from the power constraint in (29) and the last equality follows from (37).

In summary, we have derived a general upper bound on the feedback capacity as follows.

Proposition 4.8: Suppose the noise power spectral density S_Z is bounded away from zero and has the canonical spectral factorization $S_Z = |H_Z|^2$. Then, the feedback capacity C_{FB} under the power constraint P is upper bounded by

$$C_{\mathrm{FB}} \le \frac{1}{2} \int -\log\left(\frac{\phi}{S_Z}\right) + 2\psi_2 + \left(\frac{\psi_3}{S_Z}\right) + \nu(S_Z+P) - 1$$

for any $\nu > 0$, $\phi, \psi_1 \in \mathcal{L}_\infty$, and $\psi_2, \psi_3 \in \mathcal{L}_1$ satisfying (30)–(35).

While this upper bound might be of potential use in itself, for now the major utility of the upper bound lies in the characterization of the optimal solution $B^\star$. Tracing the equality conditions in (38), we can establish the following sufficient condition for the optimality of a specific feedback filter B.

Proposition 4.9 (Sufficient Condition): Suppose the power spectral density S_Z is bounded away from zero and has a canonical spectral factorization $S_Z = |H_Z|^2$. Then $B^\star$, along with $S_Y^\star = |1+B^\star|^2 S_Z$, attains the maximum in (29) if all of the following hold:

i) *Power:* $\int_{-\pi}^{\pi} |B^\star|^2 S_Z \frac{d\theta}{2\pi} = P$.

ii) *Output spectrum:* $\eta^\star := \operatorname{ess\,inf}_{\theta\in[-\pi,\pi)} S_Y^\star > 0$.

iii) *Strong orthogonality:* For some $0 < \lambda \le \eta$

$$\frac{\lambda}{(1+B^\star(z))H_Z(z)} - B^\star(z^{-1})H_Z(z^{-1}) \in \mathcal{H}_2.$$

Proof: Let $S_Y = |1+B|^2 S_Z \ge \lambda > 0$. Let

$$\begin{aligned} \nu &= 1/\lambda > 0 \\ \phi &= S_Y^{-1} \in \mathcal{L}_\infty \\ \psi_1 &= \nu - \phi = \frac{1}{\lambda} - \frac{1}{S_Y} \in \mathcal{L}_\infty \\ \psi_2 &= -\psi_1(1+B)S_Z \in \mathcal{L}_1 \\ \psi_3 &= \psi_1|1+B|^2 S_Z^2 \in \mathcal{L}_1. \end{aligned} \tag{39}$$

It is straightforward to verify that ν, ϕ, ψ_1, ψ_2, and ψ_3 defined above satisfy the conditions (30)–(35). Moreover, from the strong orthogonality condition iii)

$$\begin{aligned} \overline{\psi_2} + \nu S_Z &= -\nu(1+\overline{B})S_Z + \frac{1}{1+B} + \nu S_Z \\ &= -\frac{1}{\lambda}\left(\frac{\lambda}{1+B} - \overline{B}S_Z\right) \in \mathcal{L}_1 \end{aligned}$$

is causal.

Now, it is easy to check that

$$\psi_1 S_Y + \psi_2(1+\overline{B}) = \overline{\psi_2}(1+B) + \psi_3 S_Z^{-1} = 0$$

which makes the second inequality of (36) an equality. On the other hand, (39) makes the first inequality of (36) an equality, while the power condition i) makes the second inequality in (38) an equality. Combining these three equality conditions, we have the equality in (38), and hence the optimality of $B^\star$ satisfying i)–iii). □

To conclude this section, we remark that although the conditions i)–iii) in Theorem 4.1 characterizes the optimal feedback filter $B^\star$, this characterization is rather implicit and still falls short of yielding what can be called a closed-form solution for the feedback capacity problem (26). In Sections V and VI, we find more explicit answers by narrowing our attention to special classes of noise models.

V. Optimal Feedback Coding for the ARMA Noise

In this section and Section VI, we focus on a rational power spectral density $S_Z(e^{i\theta})$, or equivalently, an autoregressive moving-average noise process $\{Z_i\}_{i=1}^\infty$ with finite order, say, k. More specifically, we assume that the noise power spectral density $S_Z(e^{i\theta})$ has the canonical spectral factorization $S_Z(e^{i\theta}) = |H_Z(e^{i\theta})|^2$, where

$$H_Z(z) = \frac{P(z)}{Q(z)} = \frac{1+\sum_{n=1}^k p_n z^n}{1+\sum_{n=1}^k q_n z^n} \tag{40}$$

such that at least one of the monic coprime polynomials $P(z)$ and $Q(z)$ has degree k and all zeros of $P(z)$ and $Q(z)$ lie strictly outside the unit circle (i.e., both $P(z)$ and $Q(z)$ are *stable*). In particular, $S_Z(e^{i\theta})$ is bounded away from zero. Note that rational power spectral densities with no unit circle zero are dense (in $\mathcal{L}_2$ sense) in the space of all power spectral densities, so the feedback capacity of any noise spectrum can be approximated arbitrarily close by the feedback capacity of some rational noise spectrum.

We first prove a proposition on the structure of the optimal output spectrum, which is a direct application of Theorem 4.1.

Proposition 5.1 (Optimal Feedback for the ARMA Noise): Suppose the noise power spectral density is $S_Z = |H_Z|^2$, where H_Z is defined as (40) and S_Z is not white (i.e., not identically 1). Then the feedback capacity C_{FB} in (26) of Theorem 3.2 is necessarily achieved by a filter B of the form

$$B(z) = b(z)\frac{R(z)}{P(z)} - 1 \tag{41}$$

where $R(z) = 1 + \sum_{n=1}^k r_n z^n$ is a stable polynomial of degree at most k and

$$b(z) = \frac{A(z)}{A^\#(z)} = \frac{\prod_n\left(1-\alpha_n^{-1}z\right)}{\prod_n(1-\alpha_n z)}$$

is a normalized Blaschke product of at most k zeros. In particular, the corresponding output spectrum is

$$S_Y(e^{i\theta}) = |b(e^{i\theta})|^2\frac{|R(e^{i\theta})|^2}{|Q(e^{i\theta})|^2} = \sigma_0^2\frac{|R(e^{i\theta})|^2}{|Q(e^{i\theta})|^2}.$$

where $\sigma_0 = 1/\left(\prod_n|\alpha_n|\right)$. Furthermore, a filter $B(z)$ of the form (41) is optimal if and only if the following hold:

i) *Power:*

$$\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta})\,\frac{d\theta}{2\pi} = P.$$

ii) *Output spectrum:* For all zeros α_n of $b(z)$

$$0 < S_Y(\alpha_n) = \lambda \le \eta$$

where $\eta = \min_{\theta\in[-\pi,\pi)} S_Y(e^{i\theta})$.

iii) *Factorization:*

$$P(z)A^\#(z) - R(z)A(z)$$

has a factor $Q(z)$.

Remark 5.2: From the structure of the optimal $B(z)$, the feedback capacity can be characterized as

$$C_{\text{FB}} = \max_{A,R}\int_{-\pi}^{\pi}\log|A(e^{i\theta})|\,\frac{d\theta}{2\pi}$$

where the maximum is taken over all k-degree polynomials $A(z) = \prod_{i=1}^k\left(1-\alpha_i^{-1}z\right)$ with k zeros inside the unit circle and all polynomials $R(z)$ with degree at most k such that $B(z)$ given by (41) satisfies the power constraint

$$\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} \le P.$$

Proof of Proposition 5.1: We first prove the optimality of the structure (41). Since S_Z is bounded away from zero, the supremum is attained by a strictly causal $B^\star$. From the weak orthogonality condition iv) in Proposition 4.2

$$\overline{B^\star}(1+B^\star)S_Z = S_Y - \frac{|P|^2}{|Q|^2}(1+B^\star) =: f$$

is causal. Now consider

$$S = |Q|^2 S_Y^\star = |P|^2(1+B^\star) + |Q|^2 f.$$

Since P and Q are polynomials of degree at most k, and $(1+B^\star)$ and f are causal, it is easy to see that $S(z)$ is of the form

$$S(z) = s_{-k}z^{-k} + s_{-k+1}z^{-k+1} + \cdots$$

that is

$$\int_{-\pi}^{\pi} S(e^{i\theta})e^{-ij\theta} \frac{d\theta}{2\pi} = 0$$

for $j < -k$. But from the symmetry $S(z) = S(z^{-1})$, this implies that $S(z)$ is of the form

$$S(z) = s_k z^{-k} + \cdots + s_k z^k$$

or equivalently, $S(z)$ has the canonical factorization $S(z) = \sigma_0^2 R(z)R(z^{-1})$ for some stable monic polynomial R of degree at most k. Therefore

$$S_Y^\star = \sigma_0^2 \frac{|R|^2}{|Q|^2} = |1+B^\star|^2 \frac{|P|^2}{|Q|^2},$$

by the inner-outer factorization theorem in Section II-B. Since P, Q, R are all stable polynomials, $1+B^\star$ must be of the form

$$1 + B^\star(z) = b(z)\frac{R(z)}{P(z)}$$

for some normalized Blaschke product $b(z) = A(z)/A^\#(z)$ with (potentially an infinite number of) zeros $\{\alpha_n\}$ such that $|b(e^{i\theta})|^2 = \prod_n (1/|\alpha_n|^2) = \sigma_0^2$.

Furthermore, from the strong orthogonality condition iii) of Theorem 4.1

$$\frac{\lambda}{(1+B^\star(z))H_Z(z)} - B^\star(z^{-1})H_Z(z^{-1}) = \frac{\lambda - S_Y^\star(z) + (1+B^\star(z))S_Z(z)}{(1+B^\star(z))H_Z(z)} \quad (42)$$

should be causal (i.e., analytic inside the unit disc $\mathbb{D}$). This implies that $\lambda - S_Y^\star(z) + (1+B^\star(z))S_Z(z)$ has a factor $A(z)$, i.e.

$$\lambda - S_Y^\star(\alpha_n) + (1 + B^\star(\alpha_n))\, S_Z(\alpha_n) = 0$$

for every zero α_n of $A(z)$, which in turn implies that $\lambda - S_Y^\star(z)$ has a factor $A(z)$ (since $(1+B^\star)S_Z$ has a factor $A(z)$). But by symmetry, $\lambda - S_Y^\star(z)$ should have a factor $A(z^{-1})$ as well. Since $\lambda - S_Y^\star(z)$ is a rational spectrum with degree at most k, $A(z)$, or equivalently, $b(z)$ cannot have more than k zeros, which completes the proof of the optimality of the structure (41).

To prove sufficiency and necessity of conditions i)–iii), we first note that the output spectrum condition ii) and the strong orthogonality condition iii) of Theorem 4.1 can be simplified as the condition that there exists $0 < \lambda \le \eta := \min_\theta S_Y(e^{i\theta})$ such that we have the function shown at the bottom of the page is causal. Or equivalently, $\lambda - S_Y(z)$ has a factor $A(z)$ for some $0 < \lambda \le \eta = \min_\theta S_Y(e^{i\theta})$, i.e.

$$\lambda - S_Y(\alpha_n) = 0$$

for all zeros α_n of $A(z)$ (condition ii) of the current proposition) and

$$-\sigma_0^2 A^\#(z)R(z^{-1}) + A(z)P(z^{-1}) = -\sigma_0 z^n (A(z^{-1})R(z^{-1}) + A^\#(z^{-1})P(z^{-1}))$$

has a factor $Q(z^{-1})$ (condition iii) of the current proposition). Since the power condition i) is trivially equivalent to the corresponding condition i) in Theorem 4.1, we have the desired proof. □

As the simplest application of Proposition 5.1, we consider the first-order autoregressive moving-average noise spectrum, defined by

$$S_Z(e^{i\theta}) = \left| \frac{1+\alpha e^{i\theta}}{1+\beta e^{i\theta}} \right|^2 \quad (43)$$

for $\alpha \in [-1, 1]$ and $\beta \in (-1, 1)$. (The case $|\alpha| > 1$ can be taken care of by the canonical spectral factorization and proper scaling.) This spectral density corresponds to the stationary noise process given by

$$Z_i + \beta Z_{i-1} = U_i + \alpha U_{i-1}, \quad i \in \mathbb{Z}$$

where $\{U_i\}_{i=-\infty}^{\infty}$ is a white Gaussian process with zero mean and unit variance.

Theorem 5.3: Suppose the noise process $\{Z_i\}_{i=1}^{\infty}$ has the power spectral density $S_Z(e^{i\theta})$ defined in (43). Then, the feedback capacity of the additive Gaussian noise channel $Y_i = X_i + Z_i$, $i = 1, 2, \ldots$, under the power constraint P, is

$$C_{\mathrm{FB}} = -\log x_0$$

$$\frac{A^\#(z)\left(\lambda Q(z)Q(z^{-1}) - \sigma_0^2 R(z)R(z^{-1})\right) + A(z)R(z)P(z^{-1})}{A(z)Q(z^{-1})}$$

where x_0 is the unique positive root of the fourth-order polynomial

$$P\,x^2 = \frac{(1-x^2)(1+\sigma\alpha x)^2}{(1+\sigma\beta x)^2} \tag{44}$$

and

$$\sigma = \operatorname{sgn}(\beta-\alpha) = \begin{cases} 1, & \beta > \alpha \\ 0, & \beta = \alpha \\ -1, & \beta < \alpha. \end{cases}$$

Proof Sketch: Without loss of generality, we assume $|\alpha| < 1$; for the case $|\alpha| = 1$, we can perturb the noise spectrum with small power to transform it into another ARMA (1) spectrum with $|\alpha| < 1$. Under the assumption $|\alpha| < 1$, $S_Z(e^{i\theta})$ is bounded away from zero, so we can apply Proposition 5.1.

Here is the bare-bones summary of the proof. We will take the feedback filter of the form

$$B(z) = \frac{1+\beta z}{1+\alpha z} \cdot \frac{yz}{1-\sigma xz} \tag{45}$$

where $x \in (0,1)$ is an arbitrary parameter corresponding to each power constraint $P \in (0,\infty)$ under the choice of

$$y = \frac{x^2-1}{\sigma x} \cdot \frac{1+\alpha\sigma x}{1+\beta\sigma x} = -P\sigma x\left(\frac{1+\beta\sigma x}{1+\alpha\sigma x}\right). \tag{46}$$

Then, we can show that $B(z)$ satisfies conditions i)–iii) in Proposition 5.1 under the power constraint

$$\begin{aligned} P &= \int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta})\, \frac{d\theta}{2\pi} \\ &= \int_{-\pi}^{\pi} \frac{y^2}{|1-xe^{i\theta}|^2}\, \frac{d\theta}{2\pi} \\ &= \frac{y^2}{1-x^2} \\ &= \frac{1-x^2}{x^2}\left(\frac{1+\alpha\sigma x}{1+\beta\sigma x}\right)^2. \end{aligned}$$

The corresponding output spectrum is given by

$$\begin{aligned} S_Y(e^{i\theta}) &= |1+B(e^{i\theta})|^2 S_Z(e^{i\theta}) \\ &= \left|\frac{(1-re^{i\theta})(1-x^{-1}e^{i\theta})}{(1+\alpha e^{i\theta})(1-xe^{i\theta})}\,\frac{(1+\alpha e^{i\theta})}{(1+\beta e^{i\theta})}\right|^2 \\ &= \frac{1}{x^2}\left|\frac{1-re^{i\theta}}{1+\beta e^{i\theta}}\right|^2 \end{aligned} \tag{47}$$

which results in the information rate

$$h(\mathcal{Y}) - h(\mathcal{Z}) = \int_{-\pi}^{\pi} \frac{1}{2}\log|1+B(e^{i\theta})|^2\, \frac{d\theta}{2\pi} = -\frac{1}{2}\log x^2.$$

The rest of the proof is an actual implementation of this idea, which is given in Appendix B. □

We can interpret in several ways the optimal feedback filter $B^\star(z)$ we found in (45). First, we show that the celebrated Schalkwijk–Kailath coding is asymptotically equivalent to our feedback filter $B^\star$, establishing the optimality of the Schalkwijk–Kailath coding for the ARMA (1) noise spectrum.

Consider the following coding method over the Gaussian channel $Y_i = X_i + Z_i$ with the noise spectral density $S_Z(e^{i\theta})$ given by (43). Let $V \sim N(0,1)$. The transmitter initially sends

$$X_1 = V \tag{48}$$

and subsequently sends

$$X_n = (\sigma x)^{-(n-1)}(V - \hat{V}_{n-1}), \quad n = 2,3,\ldots \tag{49}$$

where $\sigma = \operatorname{sgn}(\beta-\alpha)$, x is the unique positive root of the fourth-order polynomial (44), and

$$\hat{V}_n = \hat{V}_n(Y^n) = E(V|Y_1,\ldots,Y_n)$$

is the minimum mean-squared error estimate of V given the channel output signals $Y^n = (Y_1,\ldots,Y_n)$.

For all $m < n$, we have

$$X_n = (\sigma x)^{m-n}(X_m - E(X_m|Y^{n-1})) \tag{50}$$

$$\begin{aligned} &= (\sigma x)^{m-n}\left(Y_m - Z_m - E\left(Y_m - Z_m|Y^{n-1}\right)\right) \\ &= -(\sigma x)^{m-n}\left(Z_m - E\left(Z_m|Y^{n-1}\right)\right). \end{aligned} \tag{51}$$

Furthermore, since $Z_n = -\beta Z_{n-1} + U_n + \alpha U_{n-1}$ with white $\{U_i\}$, we can show that

$$Z_n \approx -(\beta-\alpha)\sum_{k=1}^{n-1}(-\alpha)^{k-1}Z_{n-k} + U_n$$

for large n, which, combined with (51), implies that

$$Z_n - E\left(Z_n|Y^{n-1}\right) \approx \left(\frac{\beta-\alpha}{\alpha+(\sigma x)^{-1}}\right)X_n + U_n \tag{52}$$

for large n. When $\alpha \neq \beta$, that is, when the noise spectrum is nonwhite, (52) is equivalent to

$$X_n \approx \frac{\alpha+(\sigma x)^{-1}}{\beta-\alpha}\left(E\left(Z_n|Z^{n-1}\right) - E\left(Z_n|Y^{n-1}\right)\right). \tag{53}$$

Now by taking $m = n-1$ for (50) and the orthogonality of X_{n-1} and Y^{n-2}

$$\begin{aligned} X_n &= (\sigma x)^{-1}\left(X_{n-1} - E\left(X_{n-1}|Y^{n-1}\right)\right) \\ &= (\sigma x)^{-1}\left(X_{n-1} - E\left(X_{n-1}|Y^{n-2}\right) - E\left(X_{n-1}|\tilde{Y}_{n-1}\right)\right) \\ &= (\sigma x)^{-1}\left(X_{n-1} - E\left(X_{n-1}|\tilde{Y}_{n-1}\right)\right) \end{aligned} \tag{54}$$

where $\tilde{Y}_{n-1} := Y_{n-1} - E\left(Y_{n-1}|Y^{n-2}\right)$ is the innovation of the output process at time $n-1$. Also from (52) and the orthogonality of X_{n-1} and Y^{n-2}, we have

$$\begin{aligned} \tilde{Y}_{n-1} &= X_{n-1} + Z_{n-1} - E\left(X_{n-1} + Z_{n-1}|Y^{n-2}\right) \\ &= X_{n-1} + Z_{n-1} - E\left(Z_{n-1}|Y^{n-2}\right) \\ &\approx cX_{n-1} + U_{n-1} \end{aligned}$$

where

$$c = 1 + \frac{\beta-\alpha}{\alpha+(\sigma x)^{-1}} = \frac{1+\beta\sigma x}{1+\alpha\sigma x}.$$

Finally, returning to (54), we can easily see that

$$X_n \approx \frac{(\sigma x)^{-1}}{c^2P+1}(X_{n-1} - cPU_{n-1}) = \sigma x X_{n-1} - yU_{n-1}$$

where x and y are the constants given by (44) and (46). Therefore, the feedback coding (48) and (49) is asymptotically equivalent to filtering the noise through the feedback filter

$$B(z) = \frac{1+\beta z}{1+\alpha z} \cdot \frac{yz}{1-\sigma x z}$$

which is exactly equal to the optimal feedback filter (45) we found in the proof of Theorem 5.3.

For a more rigorous analysis, we can also show that

$$\liminf_{n\to\infty} \frac{1}{2n} I(V;\hat{V}_n) \geq \frac{1}{2}\log\left(\frac{1}{x^2}\right)$$

while

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^2 \leq P$$

under the coding scheme (49). Recall that

$$Y_n = (\sigma x)^{-(n-1)}\left(V - \hat{V}_{n-1}(Y^{n-1})\right) + Z_n$$

and define

$$Y_1' = Y_1$$
$$Y_n' = \left(Y_n + (\sigma x)^{-(n-1)}\hat{V}_{n-1}\right) + \beta\left(Y_{n-1} + (\sigma x)^{-(n-2)}\hat{V}_{n-2}\right)$$

for $n \geq 2$, and

$$Y_n'' = \sum_{k=1}^n (-\alpha)^{n-k} Y_k', \quad n \geq 1.$$

Clearly, Y_n'' can be represented as a linear combination of $Y_1, \ldots, Y_n$ and therefore, for any $c_2, c_3, \ldots$,

$$E(V-\hat{V}_n)^2 \leq E\left(V - \left(\sum_{k=2}^n c_k Y_k''\right)\right)^2. \qquad (55)$$

Now we express

$$Y_n' = (\sigma x)^{-(n-1)}(1+\sigma\beta x)V + Z_n + \beta Z_{n-1} = (\sigma x)^{-(n-1)}(1+\sigma\beta x)V + U_n + \alpha U_{n-1}$$

and

$$Y_n'' = d_n V + U_n + (-\alpha)^{n-1}U'$$

where $U' = \alpha U_0 - \beta Z_0$ and

$$d_n = (1+\sigma\beta x)\left(\sum_{k=1}^n (-\alpha)^{n-k}(\sigma x)^{-(k-1)}\right) = \left(\frac{1+\sigma\beta x}{1+\sigma\alpha x}\right)(1-(-\sigma\alpha x)^n)(\sigma x)^{-(n-1)}.$$

By taking $c_k = d_k$ in (55), we can easily verify that

$$\frac{E\left(V - \left(\sum_{k=2}^n d_k Y_k''\right)\right)^2}{E\left(V - \left(\sum_{k=2}^{n-1} d_k Y_k''\right)\right)^2} \to \frac{1}{x^2}$$

whence

$$\limsup_{n\to\infty} \frac{1}{n}\log E(V-\hat{V}_n)^2 \leq \log\left(\frac{1}{x^2}\right)$$

and

$$\liminf_{n\to\infty} \frac{1}{2n} I(V;\hat{V}_n) \geq \frac{1}{2}\log\left(\frac{1}{x^2}\right).$$

On the other hand

$$EX_n^2 = x^{-2(n-1)}E(V-\hat{V}_{n-1})^2 \leq x^{-2(n-1)}E\left(V - \left(\sum_{k=2}^{n-1} d_k Y_k''\right)\right)^2$$

which converges to

$$\lim_{n\to\infty} \frac{x^{-2(n-1)}}{\sum_{k=2}^{n-1} d_k^2} = \frac{(1+\sigma\alpha x)^2}{(1+\sigma\beta x)^2}\cdot(x^{-2}-1) = P.$$

The coding scheme described above uses the minimum mean-square error decoding of the message V, or equivalently, the joint typicality decoding of the Gaussian random codeword V, based on the general asymptotic equipartition property of Gaussian processes shown by Cover and Pombra [8, Theorem 2]. It is fairly straightforward to transform the Gaussian random coding to the original (constructive) Schalkwijk–Kailath coding. Here we sketch the standard procedure. A detailed analysis is given in Butman [5], [6].

Instead of the Gaussian codebook V, divide the interval $[-1,1]$ into 2^{nR} equal-length "message intervals" and represent each message $w \in [1:2^{nR}]$ by the midpoint $\theta(w)$ of its interval with distance $\Delta = 2^{-nR+1}$ between neighboring messages. The transmitter initially sends the signal $\theta = \theta(w)$ and subsequently sends $\theta - \hat{\theta}_n$ (up to the same scaling as before) at time n, where $\hat{\theta}_n$ is the minimum variance unbiased linear estimate of θ given Y^{n-1}. Now we can verify that the optimal *maximum-likelihood* decoding is equivalent to find $\theta^\star \in \Theta$ that is closest to $\hat{\theta}_n$, which results in the error probability

$$P_e^{(n)} \leq \operatorname{erfc}\left(c_0 x_0^{-n}/2^{nR}\right)$$

where x_0 is the unique positive root of (44), c_0 is a constant independent of n, and

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)dt$$

is the complementary error function. Now we can easily see that $P_e^{(n)}$ decays doubly exponentially fast as long as $R < -\log x_0 = C_{\text{FB}}$. Finally note that the doubly exponential decay of error probability can be raised to an arbitrary higher order by modifying the adaptive power allocation scheme by Pinsker [56], Kramer [38], and Zigangirov [80]. Also note that (50), (51), and (53) give interesting alternative interpretations of the Schalkwijk–Kailath coding; the optimal transmitter refines the receiver's knowledge of any past input (50), or equivalently, any past noise (51). Asymptotically, the optimal transmitter sends the difference between what he knows about the upcoming noise and what the receiver knows about it (53).

We now recast the optimal feedback coding from another angle, which will be fully developed in Section VI. Consider the following state–space model of the ARMA (1) noise process:

$$\begin{aligned} S_{n+1} &= -\beta S_n + U_n \\ Z_n &= (\alpha - \beta) S_n + U_n \end{aligned}$$

where $\{U_i\}_{i=0}^{\infty}$ are independent and identically distributed zero-mean unit-variance Gaussian random variables, and the *state* S_n is independent of U_n for each n. It is easy to check that this state–space model represents the noise spectrum

$$S_Z(e^{i\theta}) = |H_Z(e^{i\theta})|^2 = \left| \frac{1+\alpha e^{i\theta}}{1+\beta e^{i\theta}} \right|^2. \tag{43}$$

For simplicity, we consider a slightly nonstationary noise model by assuming $S_0 = U_0 = 0$. One can prove that this does not change the feedback capacity [33, Appendix], which implies that the Gaussian feedback channel $Y_i = X_i + Z_i$ with the noise spectrum (43) is asymptotically equivalent to the intersymbol interference channel

$$Y'_k = \sum_{j=1}^{k} g_{k-j} X_j + U_k$$

where $\{g_k\}_{k=0}^{\infty}$ is the Fourier coefficient of the whitening filter

$$G(e^{i\theta}) = \frac{1}{H_Z(e^{i\theta})} = \frac{1+\beta e^{i\theta}}{1+\alpha e^{i\theta}}$$

and $\{U_k\}_{k=1}^{\infty}$ is the white innovations process.

Consider the following coding scheme, which is "stationary" from time 2. At time 1, the transmitter sends $X_1 = V \sim N\left(0, \sigma_V^2\right)$ to learn $U_1 = Y_1 - X_1$ and subsequently sends

$$X_n = \chi \left(S_n - E\left(S_n | Y_1^{n-1}\right)\right), \quad n = 2, 3, \ldots \tag{56}$$

where

$$\chi = \frac{1 - \sigma\alpha x}{\sigma x}$$

and x is the unique positive root of (44).

We can easily prove the optimality of this coding scheme from our previous analysis of the coding scheme (49). Indeed, it is straightforward to transform the refinement of the message V in (49) to the refinement of the noise state S_n in (56) and vice versa. However, the direct analysis has two important benefits. First, as we will see in Section VI, the optimal feedback coding scheme for a general finite-order ARMA channel can be represented most naturally as the refinement of current noise state. Second, we can interpret the role of the message bearing signal V as a perturbation to boost the output entropy rate; refer to Section II-C for background materials.

For the analysis of the feedback coding (56), we introduce the notation

$$\begin{aligned} \hat{S}_n &= E\left(S_n | Y_1^{n-1}\right) \\ \tilde{S}_n &= S_n - E\left(S_n | Y_1^{n-1}\right) = S_n - \hat{S}_n \end{aligned}$$

and similarly define $\hat{Y}_n = E\left(Y_n | Y_1^{n-1}\right)$ and $\tilde{Y}_n = Y_n - E\left(Y_n | Y_1^{n-1}\right)$. Under this notation, we can express the channel output as

$$\begin{aligned} Y_n &= X_n + Z_n \\ &= \chi \tilde{S}_n + (\alpha - \beta) S_n + U_n \\ &= (\alpha - \beta + \chi) \tilde{S}_n + (\alpha - \beta) \hat{S}_n + U_n. \end{aligned}$$

Let $\sigma_n^2 = E\tilde{Y}_n^2$ and $s_n^2 = E\tilde{S}_n^2$. Then, we have

$$\begin{aligned} \hat{S}_{n+1} &= E\left(S_{n+1} | Y_1^n\right) \\ &= E\left(S_{n+1} | Y_1^{n-1}, \tilde{Y}_n\right) \\ &= E\left(S_{n+1} | Y_1^{n-1}\right) + E\left(S_{n+1} | \tilde{Y}_n\right) \\ &= E\left(-\beta S_n + U_n | Y_1^{n-1}\right) + E\left(-\beta S_n + U_n | \tilde{Y}_n\right) \\ &= -\beta \hat{S}_n + \gamma_n \tilde{Y}_n \end{aligned}$$

where

$$\gamma_n = \frac{1}{\sigma_n^2}\left(-\beta(\alpha - \beta + \chi) s_n^2 + 1\right).$$

From this we get the state–space model for $\tilde{Y}_n$ as

$$\begin{aligned} \tilde{S}_{n+1} &= (-\beta - \gamma_n(\alpha - \beta + \chi)) \tilde{S}_n + (1 - \gamma_n) U_n \\ \tilde{Y}_n &= (\alpha - \beta + \chi) \tilde{S}_n + U_n, \end{aligned}$$

which implies the following recursive relationship for σ_n^2 and s_n^2 for $n \geq 2$:

$$\sigma_n^2 = 1 + (\alpha - \beta + \chi)^2 s_n^2$$

and

$$\begin{aligned} s_{n+1}^2 &= (\beta + \gamma_n(\alpha - \beta + \chi))^2 s_n^2 + (1 - \gamma_n)^2 \\ &= \beta^2 s_n^2 + 1 - \frac{\left(-\beta(\alpha - \beta + \chi) s_n^2 + 1\right)^2}{1 + (\alpha - \beta + \chi)^2 s_n^2}. \end{aligned} \tag{57}$$

We recall from Section II-C that the above recursion for s_n^2 is nothing but a one-dimensional discrete Riccati recursion.

Suppose we have $V = 0$. Then $s_n^2 \equiv 0$ for all n and $\sigma_n^2 = 1$ for all n. In other words, the information rate $h(\mathcal{Y}) - h(\mathcal{Z}) = 0$; obviously, if we send nothing, the information rate should be zero.

Now take any $\epsilon > 0$. If $V \sim N(0, \epsilon)$, Lemma 2.5 iv) shows that $s_n^2 \to s^2$ where s^2 is the positive solution to the one-dimensional Riccati equation

$$s^2 = \beta^2 s^2 + 1 - \frac{(-\beta(\alpha - \beta + \chi) s^2 + 1)^2}{1 + (\alpha - \beta + \chi)^2 s^2}$$

so that $\sigma_n^2 \to 1+(\alpha-\beta+\chi)^2 s^2$. With a little algebra, we can solve the Riccati equation to get

$$s^2 = \frac{(\chi+\alpha)^2 - 1}{(\chi+\alpha-\beta)^2}$$

which, combined with our choice of $\chi = (1+\sigma\beta x)/(\sigma x)$, implies that $1+(\alpha-\beta+\chi)^2 s^2 = 1/x^2$. On the other hand

$$EX_n^2 = \chi^2 s_n^2 \to \chi^2 s^2 = P.$$

Hence, the coding scheme given by (56) achieves the information rate $-\log x$ under the power constraint P, and hence is optimal.

The above analysis gives two complementary interpretations for the role of the signal V. Most naturally, we view the feedback capacity problem as that of maximizing the information rate and V obviously has the role of carrying the information we wish to transmit. On the other hand, if we view the feedback capacity problem as that of maximizing the output entropy rate, then V has the role of perturbing the (nonstationary) filtered output process so that the resulting perturbed process has the same entropy rate as its stationary version. This second interpretation leads to the following observation in the spectral domain.

In the notation of the Cover–Pombra n-block capacity $C_{\mathrm{FB},n}$, let $B_n^\star$ denote the "almost Toeplitz" feedback matrix corresponding to the optimal coding scheme and $K_{V,n}^\star$ denote the message covariance matrix of rank one. If $\{\lambda_1,\ldots,\lambda_n\}$ denote the eigenvalues of $(I+B_n^\star)\,K_{Z,n}\,(I+B_n^\star)'$, then the asymptotic distribution of $\{\lambda_i\}_{i=1}^n$ follows the optimal output spectrum $S_Y^\star$ in (47).

Now we argue that there must be one eigenvalue, say λ_1, that goes down to zero exponentially fast (as $n\to\infty$) and the rate of decay is in fact the feedback capacity. Why? The rank of $K_{V,n}^\star$ is 1. Hence, roughly speaking, $K_{V,n}^\star$ is water-filling the eigenmode corresponding to λ_1 with small power ϵ. This results in

$$\det(K_{Y,n}^\star) \doteq (\lambda_1+\epsilon)\prod_{i=2}^n \lambda_i \doteq x^{-2n}\det(K_{Z,n}).$$

But we have

$$1 \doteq \det(K_{Z,n}) = \det\left((I+B_n^\star)\,K_{Z,n}\,(I+B_n^\star)'\right) = \prod_{i=1}^n \lambda_i$$

thus $\lambda_1 \doteq x^{2n}$. Therefore, we can view the role of the rank-one $K_{V,n}^\star$ as the tiny drop of water that fills the exponentially deep hole in the modified terrain $(I+B_n^\star)\,K_{Z,n}\,(I+B_n^\star)'$ shaped by the optimal feedback filter $B_n^\star$.

VI. State-Space Approach to Feedback Coding

While Theorem 4.1 and Proposition 5.1 provide a characterization of the optimal feedback filter, except for the first-order ARMA spectrum, it is still a nontrivial task to find analytically (or even numerically) the optimal filter and corresponding feedback capacity. As was hinted at the end of the previous section, the state-space representation provides a much richer development, leading to a more explicit answer.

We start by introducing the state-space model for the ARMA(k) noise spectrum (40). Given stable monic polynomials $P(z)$ and $Q(z)$ with coefficients $\{p_n\}_{n=1}^k$ and $\{q_n\}_{n=1}^k$, respectively, as in (40), we construct real matrices F, G, and H of sizes $k\times k$, $k\times 1$, and $1\times k$ as

$$F = \begin{bmatrix} -q_1 & -q_2 & \cdots & & -q_k \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$
$$G = [1 \quad 0 \quad \ldots \quad 0]'$$
$$H = [(p_1-q_1) \quad \ldots \quad (p_k-q_k)].$$

Let $\{U_n\}_{n=-\infty}^{\infty}$ be i.i.d. Gaussian random variables with zero mean and unit variance. We introduce a state-space model of a linear system driven by $\{U_n\}$ as the input:

$$\begin{aligned} S_{n+1} &= FS_n + GU_n \\ Z_n &= HS_n + U_n \end{aligned} \tag{58}$$

where the state S_n and the input U_n are independent of each other. We can easily check that the output $\{Z_n\}_{n=-\infty}^{\infty}$ is a stationary Gaussian process with power spectral density $S_Z(e^{i\theta}) = |H_Z(e^{i\theta})|^2$, where

$$\begin{aligned} H_Z(z) &= \frac{P(z)}{Q(z)} \\ &= \frac{\det(I - z(F-GH))}{\det(I-zF)} \\ &= zH(I-zF)^{-1}G + 1. \end{aligned}$$

Under the above state-space representation, the channel output can be expressed as

$$Y_n = X_n + Z_n = X_n + HS_n + U_n. \tag{59}$$

We state our main result in this section.

Theorem 6.1: Suppose the stationary Gaussian noise process $\{Z_i\}_{i=1}^{\infty}$ has the state-space representation (58). Then, the feedback capacity of the additive Gaussian noise channel $Y_i = X_i + Z_i$, $i=1,2,\ldots$, under the power constraint P, is

$$C_{\mathrm{FB}} = \max_X \frac{1}{2}\log\left(1 + (X+H)\Sigma_+(X)(X+H)'\right). \tag{60}$$

Here the maximum is taken over all $X \in \mathbb{R}^{1\times k}$ such that $F - G(X+H)$ has no unit-circle eigenvalue and $X\Sigma_+(X)X' \le P$, where $\Sigma_+(X)$ is the maximal solution to the discrete algebraic Riccati equation

$$\Sigma = F\Sigma F' + GG' - \Gamma(1+(X+H)\Sigma(X+H)')\Gamma' \tag{61}$$

and

$$\Gamma = \Gamma(\Sigma, X) = \frac{F\Sigma(X+H)' + G}{1+(X+H)\Sigma(X+H)'}.$$

This theorem characterizes the feedback capacity as conjectured by Yang, Kavčić, and Tatikonda [7, Theorem 6 and Conjecture 1] in a simpler form (without any extra innovations process for the input). In particular, as we will see shortly, this result confirms the optimality of the k-dimensional variant of the Schalkwijk–Kailath coding. When specialized to the first-order ARMA spectrum, one can also deduce Theorem 5.3 from Theorem 6.1 through elementary algebra.

We prove Theorem 6.1 in two steps. The first step is the following structural result, which is reminiscent of a similar result in [78] for the finite-dimensional case $C_{\mathrm{FB},n}$ and is proved rather directly from necessary conditions in Theorem 4.1 instead of less manageable Proposition 5.1.

Lemma 6.1: Suppose the Gaussian noise process has the state-space representation (58). Then the feedback capacity is achieved by the input process $\{X_n\}_{n=-\infty}^{\infty}$ of the form

$$X_n = X\left(S_n - E\left(S_n|Y_{-\infty}^{n-1}\right)\right)$$

for some $X \in \mathbb{R}^{1\times k}$ such that $F-G(X+H)$ has no unit-circle eigenvalue.

Proof: Suppose that $B(e^{i\theta}) = \sum_{j=1}^{\infty} b_j e^{ij\theta}$ attains the maximum in (26), or equivalently, the stationary process $\{X_i\}_{i=-\infty}^{\infty}$ defined by $X_n = \sum_{j=1}^{\infty} b_j Z_{n-j}$ achieves the feedback capacity. If we regard X_n as a vector in the Hilbert space generated by linear spans of $\{Z_i\}_{i=\infty}^{\infty}$, X_n lies in the closed linear span of all past Z_i, that is, $X_n \in \mathrm{clin}\{Z_{-\infty}^{n-1}\}$. Equivalently

$$X_n \in \mathcal{H}_n := \mathrm{clin}\left\{S^n, Y_{-\infty}^{n-1}\right\}.$$

We decompose X_n into two orthogonal parts as

$$X_n = \xi_n + \zeta_n$$

where ξ_n lies in the closed linear span $\mathcal{G}_n$ of S_n and $Y_{-\infty}^{n-1}$, and ζ_n lies in the orthogonal complement of $\mathcal{G}_n$ in $\mathcal{H}_n$, namely

$$\xi_n \in \mathcal{G}_n := \mathrm{clin}\left\{S_n, Y_{-\infty}^{n-1}\right\}$$
$$\zeta_n \in (\mathcal{H}_n \ominus \mathcal{G}_n).$$

Since $\{X_n\}$ achieves the feedback capacity, from the weak orthogonality condition iv) in Proposition 4.2

$$\xi_n = X\left(S_n - E\left(S_n|Y_{-\infty}^{n-1}\right)\right)$$

for some $X \in \mathbb{R}^{1\times k}$. In other words, for each orthogonal feedback filter $B(z)$, we have a representation

$$X_n = X\left(S_n - E\left(S_n|Y_{-\infty}^{n-1}\right)\right) + \zeta_n \tag{62}$$

for some $X \in \mathbb{R}^{1\times k}$.

To ease the notation a little, we shall subsequently write

$$\hat{A}_n := E(A_n|Y_{-\infty}^{n-1})$$
$$\tilde{A}_n := A_n - \hat{A}_n$$

for a generic random variable (or a random vector) A_n. Under this notation, we have

$$Y_n = X\tilde{S}_n + HS_n + \zeta_n + U_n$$

so that

$$\tilde{Y}_n = (X+H)\tilde{S}_n + \zeta_n + U_n. \tag{63}$$

Let $P_\zeta = E\zeta_n^2$, $\sigma^2 = E\tilde{Y}_n^2$, and $\Sigma = \mathrm{Cov}(\tilde{S}_n)$. Then, from the mutual orthogonality of ζ_n, U_n, and $\tilde{S}_n$

$$\sigma^2 = (X+H)\Sigma(X+H)' + P_\zeta + 1.$$

On the other hand, it is easy to check that

$$\begin{aligned}\hat{S}_{n+1} &= E(FS_n + GU_n|Y_{-\infty}^{n})\\ &= E(FS_n + GU_n|Y_{-\infty}^{n-1}) + E(FS_n + GU_n|\tilde{Y})\\ &= F\hat{S}_n + \Gamma\tilde{Y}_n\end{aligned} \tag{64}$$

where

$$\Gamma := \frac{1}{\sigma^2}\left(F\Sigma(X+H)' + G\right).$$

Thus, we have the state-space representation of $\tilde{Y}_n$ as

$$\begin{aligned}\tilde{S}_{n+1} &= \left(F - \Gamma(X+H)\right)\tilde{S}_n - \Gamma\zeta_n + (G-\Gamma)U_n\\ \tilde{Y}_n &= (X+H)\tilde{S}_n + \zeta_n + U_n\end{aligned} \tag{65}$$

which implies that Σ satisfies the following discrete algebraic Riccati equation (DARE):

$$\begin{aligned}\Sigma &= \tilde{F}\Sigma\tilde{F}' + (G-\Gamma)(G-\Gamma)' + P_\zeta\Gamma\Gamma'\\ &= F\Sigma F' + GG' - \sigma^2\Gamma\Gamma'\end{aligned} \tag{66}$$

where $\tilde{F} = F - \Gamma(X+H)$.

Now we prove that P_ζ is necessarily zero. We first observe that the derivation of the state-space (65) depends on the fact that $\zeta_n \in \mathcal{H}_n = \mathrm{clin}\left\{S^n, Y_{-\infty}^{n-1}\right\}$ only via the orthogonality of ζ_n and $\left(S_n, Y_{-\infty}^{n-1}\right)$. Therefore, if the input process

$$X_n = X\left(S_n - E\left(S_n|Y_{-\infty}^{n-1}\right)\right) + \zeta_n$$

achieves the feedback capacity and induces the output distribution uniquely defined by (59)–(66), any other input process of the form

$$X_n = X\left(S_n - E\left(S_n|Y_{-\infty}^{n-1}\right)\right) + W_n$$

results in the same output distribution and hence achieves the feedback capacity, provided that $EW_n^2 = P_\zeta$ and W_n is orthogonal to $\left(S_n, Y_{-\infty}^{n-1}, U_n\right)$.[1] In particular, we can take $W_n = V_n$,

[1]Although $E(S_n|Y_{-\infty}^{n-1})$ is symbolically the same for any choice of W_n, each could result in different output processes defined recursively by $Y_n = X\left(S_n - E\left(S_n|Y_{-\infty}^{n-1}\right)\right) + W_n + U_n$. However, our analysis of the Riccati equation shows that the output process is uniquely defined for any choice of W_n.

where $\{V_n\}_{n=-\infty}^{\infty}$ is a white Gaussian process with power spectral density $S_V(e^{i\theta}) \equiv P_\zeta$, independent of $\{Z_n\}_{n=-\infty}^{\infty}$.

But as Remark 4.5 shows, a nonzero white $S_V^\star$ achieves the feedback capacity only if the noise spectrum itself is white. Since S_Z is nonwhite, P_ζ must be zero. Therefore, the optimal input process must be of the form

$$X_n = X\left(S_n - E\left(S_n | Y_{-\infty}^{n-1}\right)\right)$$

for some X such that $F - G(X + H)$ has no unit-circle eigenvalue. □

Equipped with Lemma 6.1, the proof of Theorem 6.1 is straightforward.

Proof of Theorem 6.1: We know that the capacity achieving input process is of the form

$$X_n = X\left(S_n - E\left(S_n | Y_{-\infty}^{n-1}\right)\right).$$

From (65), the state-space equation for $\tilde{Y}_n$ becomes

$$\begin{aligned} \tilde{S}_{n+1} &= (F - \Gamma(X+H))\,\tilde{S}_n + (G - \Gamma)U_n \\ \tilde{Y}_n &= (X+H)\tilde{S}_n + U_n \end{aligned} \tag{67}$$

where

$$\Gamma = \Gamma(\Sigma_+, X) = \frac{F\Sigma_+(X+H)' + G}{1 + (X+H)\Sigma_+(X+H)'}$$

and $\Sigma_+ = \Sigma_+(X)$ is the unique positive semidefinite stabilizing solution to the DARE

$$\Sigma = F\Sigma F' + GG' - \Gamma(1 + (X+H)\Sigma(X+H)')\Gamma'.$$

Since $\tilde{Y}_n$ is a white process with variance

$$\begin{aligned} \sigma^2 &= 1 + (X+H)\Sigma_+(X)(X+H)' \\ &= \frac{\det(F - G(X+H))}{\det(F - \Gamma(X+H))} \end{aligned}$$

and $h(\tilde{\mathcal{Y}}) = h(\mathcal{Y})$, the corresponding information rate is

$$\frac{1}{2}\log\left(1 + (X+H)\Sigma_+(X)(X+H)'\right)$$

along with the power consumption $X\Sigma_+(X)X'$. Hence, the feedback capacity $C_{\mathrm{FB}}(P)$ is the maximal information rate over all X satisfying the power constraint $X\Sigma_+(X)X' \le P$. □

The proofs of Lemma 6.1 and Theorem 6.1 reveal the structure of the optimal feedback filter and the corresponding output spectrum once again (cf. Proposition 5.1). Indeed, we have

$$Y_n = \hat{Y}_n + \tilde{Y}_n = H\hat{S}_n + \tilde{Y}_n.$$

This, combined with (64), implies

$$S_Y(e^{i\theta}) = \sigma^2 \left| \frac{\det(I - e^{i\theta}(F - \Gamma H))}{\det(I - e^{i\theta}F)} \right|^2 \tag{68}$$

which is bounded away from zero [32, Lemma 8.3.1] (cf. Theorem 4.1 ii)). Furthermore, since the optimal input can be expressed as $X_n = X\tilde{S}_n$, we can easily check from (67) that the corresponding feedback filter satisfies

$$\begin{aligned} &B(z) + 1 \\ &\quad = zX(I - z(F - \Gamma(X+H)))^{-1}(G - \Gamma)H_Z^{-1}(z) + 1 \\ &\quad = \frac{\det(I - z(F - G(X+H)))}{\det(I - z(F - \Gamma(X+H)))}\,\frac{\det(I - z(F - \Gamma H))}{\det(I - z(F - GH))}. \end{aligned}$$

From Lemma 2.4 iv), it is easy to see that

$$\frac{\det(I - z(F - G(X+H)))}{\det(I - z(F - \Gamma(X+H)))} =: \frac{A(z)}{A^{\#}(z)} \tag{69}$$

is a normalized Blaschke product, the zeros of which determine the entropy rate of the output process.

Finally we relate Theorem 6.1 to a multidimensional variant of the Schalkwijk–Kailath coding. Since we already went through detailed discussions of the Schalkwijk–Kailath coding for the first-order ARMA spectrum in the previous section, we give here a rather sketchy argument. For simplicity, assume the state-space representation (58) of the noise process $\{Z_i\}_{i=1}^{\infty}$ with $S_0 = 0$ and $U_0 = 0$. For the initial k transmissions, the transmitter sends $X_n = V_n$, $n = 1, \ldots, k$, with $V^k \sim N_k(0, K_V)$ and subsequently

$$X_n = X\left(S_n - E\left(S_n | Y^{n-1}\right)\right), \quad n = k+1, k+2, \ldots \tag{70}$$

where $X \in \mathbb{R}^{1\times k}$ achieves the maximum in (60). In other words, after the initial k transmissions, the transmitter refines the receiver's error of the current noise state. Since the error is k-dimensional, one must project it down in the direction X.

Lemma 2.5 shows that, as far as K_V is positive definite, or equivalently, as far as $\mathrm{Cov}\left(S_{k+1} | Y_1^k\right)$ is positive definite, $\mathrm{Cov}\left(S_n | Y_1^{n-1}\right)$ converges to the unique stabilizing solution $\Sigma_+(X)$ of the DARE (61) and thus $h\left(Y_n | Y_1^{n-1}\right)$ converges to C_{FB}. It is also straightforward to rewrite the coding (70) as the iterative refinement of the receiver's knowledge of the message-bearing signal $(V_1, \ldots, V_k)$. From this we can generalize the original one-dimensional Schalkwijk–Kailath coding into the k-dimensional one with $(\theta_1, \ldots, \theta_k)$ in some equally spaced constellation. (Instead of using the minimum mean square error estimate of V^k, we use the minimum variance unbiased estimate of θ^k; both estimates are linearly related [32, Section 3.4], [43, Section 4.5].) As before, we can also interpret the role of V^k as tiny drops of water that fill the noise terrain modified by the optimal feedback filter $B^\star(z)$.

VII. Concluding Remarks

Starting from the Cover–Pombra formulation, we have developed a sequence of progressively more concrete characterizations of the feedback capacity. For example, for the first-order ARMA noise spectrum, the feedback capacity can be characterized as expressions at the bottom of the page. The last two characterizations are achieved by a natural and concrete coding technique:

$$X_n = X\left(S_n - E\left(S_n | Y^{n-1}\right)\right) \\ \propto\ \theta - \hat{\theta}(Y^{n-1})$$

which resolves the long-standing open question of the optimality of the Schalkwijk–Kailath feedback coding and provides a single-letter characterization of the feedback capacity.

The solution to the Gaussian feedback capacity problem reveals a rich connection between control, estimation, and communication; roughly speaking, when formulated in the infinite-dimensional variational problem, communication over the Gaussian feedback channel can be viewed as a stochastic control problem of the receiver's estimation error, which can be, in turn, viewed as the entropy maximization problem of the output spectrum. We conclude this paper by posing a few related questions that will invite further investigations to illuminate a complete picture of this fascinating interplay between control, estimation, and communication.

First, from Theorem 4.6 and the Szegő–Kolmogorov–Krein theorem, we get the following maximin characterization of the feedback capacity:

$$\sup_B \inf_A \frac{1}{2} \log \left(\int_{-\pi}^{\pi} |1 - A(e^{i\theta})|^2 |1 - B(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \right) \tag{71}$$

where the infimum is taken over all strictly causal polynomials $A(z) = \sum_k a_k z^k$ and the supremum is taken over all strictly causal polynomials $B(z) = \sum_k b_k z^k$ satisfying

$$\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta}) \, \frac{d\theta}{2\pi} \le P.$$

Thus, the feedback capacity problem can be viewed as a game between the controller (feedback filter) B and the estimator A. Does this game has a saddle point? If so, can we get an explicit characterization of the saddle point and the associated value of the game? The objective of the optimization problem (71) is not quasi-convex-concave in (A, B) and standard Fan-Sion minimax theorems [18], [65] do not apply. Nonetheless, the problem is quadratic, so a careful application of the S-procedure (see Yakubovich [74]) might lead to an interesting answer.

Second, the feedback capacity characterized in Theorem 6.1 is equivalent to the following optimization problem in (X, Σ)

$$\begin{aligned} \text{maximize} \quad & \log(1 + \tilde{H}\Sigma\tilde{H}') \\ \text{subject to} \quad & \Sigma \succeq 0 \\ & X\Sigma X' \le P \\ & \tilde{H} = X + H \\ & \Sigma = F\Sigma F' + GG' - \Gamma(1 + \tilde{H}\Sigma\tilde{H}')\Gamma' \\ & \Gamma = (F\Sigma\tilde{H}' + G)/(1 + \tilde{H}\Sigma\tilde{H}'). \end{aligned} \tag{72}$$

Now this problem can be recast as an instance of bilinear matrix inequalities [59]. This optimization problem is a slight generalization of linear matrix inequalities [4] and indeed for a fixed signal direction X, finding the optimal Σ can be done easily by, for example, the invariant subspace method [41]. But joint optimization of (X, Σ) makes the problem nonconvex, so finding the global maximum under the given power constraint P is computationally difficult.

In this regard, Proposition 5.1 has a rather interesting implication, when combined with Theorem 6.1. The condition ii) of Proposition 5.1 states that all zeros α_n of (69) (= eigenvalues of $F - G(X + H)$) should satisfy

$$0 < S_Y(\alpha_n) = \lambda \le \min S_Y(e^{i\theta})$$

where $S_Y(z)$ is given in (68). Can we utilize this condition to find the optimal signal direction X and the corresponding feedback filter analytically? We remark that this condition is reminiscent of the classical interpolation problem studied by Pick and Nevanlinna (see, for example, Ball, Gohberg, and Rodman [2]).

Finally, there is one more potential connection to optimal control theory. The variational characterization of the feedback capacity problem seems to have some relevance to the risk-sensitive or minimum-entropy control/estimation problem (see Whittle [72] and Mustafa and Glover [46]) as was pointed out by Babak Hassibi, Stephen Boyd, and Sanjoy Mitter in private communication. Indeed, the dual (38) to the feedback capacity problem has the leading entropy term $\int_{-\pi}^{\pi} \log(\nu - \psi_1(e^{i\theta})) \frac{d\theta}{2\pi}$ that looks similar to the one in the minimum-entropy control problem. Furthermore, the technique developed in the proof of Theorem 3.2 that transforms a sequence of finite-dimensional problems into a single infinite-dimensional problem can be

$$\begin{aligned} C_{\text{FB}} &= \lim_{n\to\infty} \max_{K_V, B} \frac{1}{2n} \log \frac{\det(K_V + (I+B)K_{Z,n}(I+B)')}{\det(K_{Z,n})} \\ &\qquad \text{(Lemma 3.1: Cover–Pombra Theorem [8, Theorem 1])} \\ &= \sup_{S_V, B} \int_{-\pi}^{\pi} \frac{1}{2} \log \frac{S_V + |1+B|^2 S_Z}{S_Z} \, \frac{d\theta}{2\pi} && \text{(Theorem 3.2)} \\ &= \max_B \int_{-\pi}^{\pi} \frac{1}{2} \log |1+B|^2 \, \frac{d\theta}{2\pi} && \text{(Theorem 4.1)} \\ &= \max_X \frac{1}{2} \log(1 + (X+H)\Sigma_+(X)(X+H)') && \text{(Theorem 6.1)} \\ &= -\log x_0^2 && \text{(Theorem 5.3).} \end{aligned}$$

useful to other control problems of similar structures. Can these connections be made more clear and precise?

APPENDIX A
EXISTENCE OF AN OPTIMAL $(S_V^\star, B^\star)$

Suppose that the noise spectrum S_Z is lower bounded by some $\delta > 0$. We write

$$f(S_V, B) = \int_{-\pi}^{\pi} \frac{1}{2} \log \frac{S_V + |1+B|^2 S_Z}{S_Z} \frac{d\theta}{2\pi}.$$

Then $C_{\mathrm{FB}} = \sup_{S_V, B} f(S_V, B)$ where the supremum is taken over all $S_V \geq 0$ and strictly causal B with

$$\int_{-\pi}^{\pi} S_V(e^{i\theta}) + |B(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} \leq P. \quad (73)$$

By change of variable $S_Y = S_V + |1+B|^2 S_Z$, we write

$$g(S_Y, B) = f(S_V, B) = \int_{-\pi}^{\pi} \frac{1}{2} \log(S_Y) \frac{d\theta}{2\pi}.$$

Let $\mathcal{H}_2(\mu_Z)$ denote the space of analytic functions square-integrable with respect to the noise spectral distribution $d\mu_Z = S_Z(e^{i\theta})d\theta$. Then by the power constraint (73), $S_Y \in \mathcal{L}_1$ and $B \in \mathcal{H}_2(\mu_Z)$.

Our goal is to show that the maximum of $g(S_Y, B)$ is achieved by an $(S_Y^\star, B^\star)$ in

$$\begin{aligned}\mathcal{K} = \Big\{ (S_Y, B) \in \mathcal{L}_1 \times \mathcal{H}_2(\mu_Z) : \\ S_Y - |1+B|^2 S_Z \geq 0 \\ B(0) = 0 \\ \int_{-\pi}^{\pi} S_Y - (2B+1) S_Z \frac{d\theta}{2\pi} \leq P \Big\}.\end{aligned}$$

Here the last constraint comes from the facts that

$$S_V + |B|^2 S_Z = S_Y - B S_Z - \overline{B} S_Z - S_Z$$

and that B and S_Z have real Fourier coefficients. Note that from the boundedness condition $S_Z(e^{i\theta}) \geq \delta > 0$, we have $B \in \mathcal{H}_2$ whenever $B \in \mathcal{H}_2(\mu_Z)$, since

$$\int_{-\pi}^{\pi} \delta |B|^2 d\theta \leq \int_{-\pi}^{\pi} |B|^2 d\mu_Z. \quad (74)$$

The rest of the proof relies on functional analysis on topological vector spaces. See, for example, Megginson [45] and Dunford and Schwartz [11] for terminology and proofs of classical theorems we refer to in the following discussion.

First, we relax the constraint set $\mathcal{K}$ by embedding the space of S_Y in $\mathcal{L}_1$ into the space $\mathcal{M}_+$ of positive measures μ_Y on $[-\pi, \pi)$. Noting from Lemma 2.2 that

$$g(S_Y, B) = \inf_{\{a_k\}} \frac{1}{2} \log \left(\int_{-\pi}^{\pi} |1 - \sum_k a_k e^{ik\theta}|^2 S_Y(e^{i\theta}) \frac{d\theta}{2\pi} \right)$$

with the infimum over all polynomials with coefficients $\{a_k\}$, we define

$$\tilde{g}(\mu_Y, B) = \inf_{\{a_k\}} \frac{1}{2} \log \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - \sum_k a_k e^{ik\theta}|^2 d\mu_Y \right).$$

If μ_Y is decomposed into absolutely continuous and singular parts as $d\mu_Y = S_Y(e^{i\theta})d\theta + d\mu_{Y,s}$, Lemma 2.2 shows that

$$\tilde{g}(\mu_Y, B) = g(S_Y, B)$$

independent of the singular part $\mu_{Y,s}$.

Now we prove that the maximum of $\tilde{g}(\mu_Y, B)$ is attained in

$$\begin{aligned}\tilde{\mathcal{K}} = \Big\{ (\mu_Y, B) \in \mathcal{M}_+ \times \mathcal{H}_2(\mu_Z) : \\ d\mu_Y - |1+B|^2 d\mu_Z \in \mathcal{M}_+ \\ B(0) = 0 \\ \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} d\mu_Y - \int_{-\pi}^{\pi} (2B+1)\, d\mu_Z \right) \leq P \Big\}.\end{aligned}$$

Recall that $\mathcal{M}_+$ is a subset of the space $\mathcal{M}$ of signed measures and $\mathcal{M}$ is isomorphic to the space of linear functionals on continuous functions on $[-\pi, \pi)$, that is, $\mathcal{M} \simeq C[\pi, \pi)^*$. Also $\mathcal{H}_2(\mu_Z)$ is a Hilbert space and the dual of itself. We will show that the constraint set $\tilde{\mathcal{K}}$ is compact in the product topology of weak* topology on $\mathcal{M}_+$ and weak (= weak* because $\mathcal{H}_2(\mu_Z)$ is a Hilbert space) topology on $\mathcal{H}_2(\mu_Z)$. And then we show that $\tilde{g}$ is upper semicontinuous under the same topology. This clearly implies that the maximum of $\tilde{g}$ is attained in $\tilde{\mathcal{K}}$. (That the maximum of an upper semicontinuous function is attained on a compact domain is well known. For the proof, see, for example, Luenberger [43, Sections 2.13, 5.10].) Finally, because $\tilde{g}(\mu_Y)$ depends only on the absolutely continuous part of μ_Y, if the maximum of $\tilde{g}$ is attained by $(\mu_Y^\star, B^\star) \in \tilde{\mathcal{K}}$, there exists $(S_Y^\star, B^\star) \in \mathcal{K}$ that attains the same maximum of g; clearly, any singular part of the spectral distribution wastes the power. The details of the proof follow.

All topological properties such as compactness, closedness, and continuity will be used with respect to the product topology of weak* topologies on $\mathcal{M}_+$ and $\mathcal{H}_2(\mu_Z)$, unless noted otherwise.

For compactness, we observe that

$$\mathcal{K}_1 := \left\{ \mu_Y \in \mathcal{M}_+ : \frac{1}{2\pi} \int_{-\pi}^{\pi} d\mu_Y \leq P \right\}$$

and

$$\mathcal{K}_2 := \left\{ B \in \mathcal{H}_2(\mu_Z) : \frac{1}{2\pi} \int_{-\pi}^{\pi} |B|^2\, d\mu_Z \leq P \right\}$$

are norm balls in respective norm topologies; thus both are weak* compact by the Alaoglu–Banach theorem, and so is $\mathcal{K}_1 \times \mathcal{K}_2$. Since $\tilde{\mathcal{K}} \subseteq \mathcal{K}_1 \times \mathcal{K}_2$, closedness of $\tilde{\mathcal{K}}$ will guarantee its compactness. First note that $B(0) = 0$ if and only if $\int_{-\pi}^{\pi} B(e^{i\theta}) d\theta = 0$. Since the functional $T_1(B) := \int_{-\pi}^{\pi} B(e^{i\theta})\, d\theta$ is bounded (cf. (74) and the Cauchy–Schwartz inequality), linear, and thus weakly* continuous, $\{B(0) = 0\} = T_1^{-1}(\{0\})$ is closed. Similarly,

$T_2(\mu_Y, B) := \int_{-\pi}^{\pi} d\mu_Y - \int_{-\pi}^{\pi}(2B+1)d\mu_Z$ is continuous, so the set

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\mu_Y - \frac{1}{2\pi}\int_{-\pi}^{\pi}(2B+1)d\mu_Z \le P\right\}$$

is closed. Finally, $d\mu_Y - |1+B|^2 d\mu_Z$ is a positive measure if and only if

$$T_\phi(\mu_Y, B) := \int_{-\pi}^{\pi} \phi\, d\mu_Y - \int_{-\pi}^{\pi} \phi\, |1+B|^2 d\mu_Z \ge 0$$

for all $0 \le \phi \in C[-\pi,\pi)$. But for each $\phi \ge 0$, $\int_{-\pi}^{\pi} \phi|1+B|^2 d\mu_Z$ is (strongly) continuous and convex. Therefore, it is also weakly (=weakly*) lower semicontinuous; see Ekeland and Temam [15, Section 2.2]. This implies that T_ϕ is upper semicontinuous and $T_\phi^{-1}([0,\infty))$ is closed. Since the intersection of an arbitrary collection of closed sets is closed, $\{(\mu_Y, B) : d\mu_Y - |1+B|^2 d\mu_Z \in \mathcal{M}_+\} = \cap_\phi T_\phi^{-1}([0,\infty))$ is closed. For the same reason, $\tilde{\mathcal{K}}$ is closed, and as a closed subset of a compact set, it is compact as well.

For weak* upper semicontinuity of $\tilde{g}(\mu_Y) = \tilde{g}(\mu_Y, B)$, we first fix $\mu_Y \in \mathcal{M}_+$ and note from the definition of weak* convergence that

$$\alpha_n(p) := \int_{-\pi}^{\pi} |1-p|^2 d\mu_{Y,n} \to \int_{-\pi}^{\pi} |1-p|^2 d\mu_Y =: \alpha(p)$$

for any fixed strictly causal polynomial p and any sequence $\mu_{Y,n}$ weakly* convergent to μ_Y. Hence

$$\inf_p \lim_{n\to\infty} \alpha_n(p) = \inf_p \alpha(p) = \tilde{g}(\mu_Y).$$

But for all p

$$\overline{\lim_{n\to\infty}} \inf_p \alpha_n(p) \le \lim_{n\to\infty} \alpha_n(p)$$

and thus

$$\overline{\lim_{n\to\infty}} \inf_p \alpha_n(p) \le \inf_p \alpha(p).$$

Therefore

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, g(\mu_{Y,n}) &= \overline{\lim_{n\to\infty}} \inf_p \int_{-\pi}^{\pi} |1-p|^2 d\mu_{Y,n} \\ &\le \inf_p \int_{-\pi}^{\pi} |1-p|^2 d\mu_Y = \tilde{g}(\mu_Y)\end{aligned}$$

for any $\mu_{Y,n}$ weakly* convergent to μ_Y. In other words, $\tilde{g}(\mu_Y)$ is weakly* upper semicontinuous and its maximum should be attained in $\tilde{\mathcal{K}}$ (and in turn in $\mathcal{K}$).

Finally we remark that the condition that S_Z is bounded away from zero, which was used to prove the compactness of $\tilde{\mathcal{K}}$ (in particular, weak* continuity of $T_1(B)$), is necessary. As a simple example, if $S_Z(e^{i\theta}) = |1+e^{i\theta}|^2$, it is shown in Section V that the feedback capacity of this noise spectrum corresponds to the output spectrum S_Y of the form

$$S_Y(e^{i\theta}) = \frac{1}{x^2}|1+x^2 e^{i\theta}|^2.$$

But we can easily check that there is no optimal $(S_V^\star \equiv 0, B^\star)$ resulting in this output spectrum.

Appendix B
Proof of Theorem 5.3

Assume $-1 < \alpha < \beta < 1$. Given $x \in (0,1)$, take y as in (46). Then, we can factor $1 + B(z)$ as

$$\begin{aligned}1 + B(z) &= 1 + \frac{1+\beta z}{1+\alpha z}\cdot\frac{yz}{1-xz} \\ &= \frac{1-(\alpha - x + y)z + (\beta y - \alpha x)z^2}{(1+\alpha z)(1-xz)} \\ &= \frac{(1+(\alpha x - \beta y)xz)(1-x^{-1}z)}{(1+\alpha z)(1-xz)} \\ &= \frac{(1-rz)(1-x^{-1}z)}{(1+\alpha z)(1-xz)}\end{aligned}$$

where $r = -(\alpha x - \beta y)x$. The corresponding output spectrum is given by

$$\begin{aligned}S_Y(e^{i\theta}) &= |1+B(e^{i\theta})|^2 S_Z(z) \\ &= \left|\frac{(1-re^{i\theta})(1-x^{-1}e^{i\theta})}{(1+\alpha e^{i\theta})(1-xe^{i\theta})}\frac{(1+\alpha e^{i\theta})}{(1+\beta e^{i\theta})}\right|^2 \\ &= \frac{1}{x^2}\left|\frac{1-re^{i\theta}}{1+\beta e^{i\theta}}\right|^2 \end{aligned} \tag{75}$$

or equivalently

$$S_Y(z) = \frac{1}{x^2}\frac{(1-rz)(1-rz^{-1})}{(1+\beta z)(1+\beta z^{-1})}$$

for $z \in \mathbb{D}$.

We first check that $|r| < 1$. Indeed, from (46), we can express $r = r(x)$ as

$$r(x) = \frac{(\beta-\alpha)x^2 - \alpha\beta x - \beta}{1+\beta x} \tag{76}$$

$$= -\beta + (\beta - \alpha)\cdot\frac{\beta + x}{\beta + x^{-1}} \tag{77}$$

$$= -\beta\left(1 - \frac{\beta+x}{\beta+x^{-1}}\right) - \alpha\left(\frac{\beta+x}{\beta+x^{-1}}\right).$$

When $\beta \ge 0$, $0 < (\beta+x)/(\beta+x^{-1}) < 1$ (recall $0 < x < 1$) so that $-r$ is a convex combination of α and β; hence $-1 < r < 1$. When $\beta < 0$, we differentiate (76) to find that $r'(0) < 0$, $r'(1) > 0$, $\max_{x\in[0,1]} r(x) = \max\{r(0), r(1)\}$, and that there exists a unique $x^\star \in (0,1)$ attaining the minimum of $r(x)$ on $[0,1]$. Since $r(0) = -\beta$ and $r(1) = -\alpha$, it suffices to check that $r(x^\star) > -1$. We have

$$\begin{aligned}0 &= \left(\frac{\partial}{\partial x}\big((\beta-\alpha)x^2 - \alpha\beta x - \beta\big)\right)(1+\beta x) \\ &\quad - \big((\beta-\alpha)x^2 - \alpha\beta x - \beta\big)\frac{\partial}{\partial x}(1+\beta x) \\ &= \big(2(\beta-\alpha)x - \alpha\beta\big)(1+\beta x) - \beta\big((\beta-\alpha)x^2 - \alpha\beta x - \beta\big) \\ &= (\beta-\alpha)(\beta x^2 + 2x + \beta)\end{aligned}$$

at $x = x^\star$, whence

$$r(x^\star) = \frac{2(\beta-\alpha)x^\star - \alpha\beta}{\beta} = 2x^\star + \alpha(x^\star)^2 \ge 0.$$

Therefore, $|r| < 1$.

Now let

$$\lambda = S_Y(x) = \frac{1}{x^2}\frac{(1-rx)(1-rx^{-1})}{(1+\beta x)(1+\beta x^{-1})} = \frac{1}{x^2}\frac{(1-rx)(x-r)}{(1+\beta x)(x+\beta)}.$$

We will show that

$$0 < \lambda \le \min_{\theta\in[-\pi,\pi)} S_Y(e^{i\theta}).$$

For the positivity of λ, it suffices to show that $(x-r)/(x+\beta)$ is positive. From (77), we have

$$x-r = (x+\beta)\left(1-\frac{\beta-\alpha}{\beta+x^{-1}}\right) = (x+\beta)\left(\frac{\alpha+x^{-1}}{\beta+x^{-1}}\right) \quad (78)$$

so that $(x-r)/(x+\beta)$ is positive. (The case $x+\beta=0$ is trivial since $r(-\beta)=-\beta$.)

The upper bound requires a little more work. Let

$$f(u) = \frac{(1+r^2)-2ru}{(1+\beta^2)+2\beta u}$$

for $-\infty < u < \infty$. Then, we can express

$$S_Y(e^{i\theta}) = \frac{(1-re^{i\theta})(1-re^{-i\theta})}{(1+\beta e^{i\theta})(1+\beta e^{-i\theta})} = f(\cos\theta)$$

for $\theta\in[-\pi,\pi)$ and similarly express

$$S_Y(x) = f\left(\frac{x+x^{-1}}{2}\right).$$

Since the linear fractional function $f(u)$ does not have a singularity in $[-1,1]$, the minimum occurs at one of the end points and

$$\min_\theta S_Y(e^{i\theta}) = \min_\theta f(\cos\theta) = \min\{f(1), f(-1)\}.$$

We consider different cases as follows:

Case 1: $\beta \ge 0$. Then, $f(u)$ is decreasing on $\left(-\frac{1}{2}(\beta+\beta^{-1}),\infty\right)$ since

$$f'(u) = -\frac{2(\beta+r)(1+\beta r)}{((1+\beta^2)+2\beta u)^2}$$

and

$$\beta + r = (\beta-\alpha)\left(\frac{\beta+x}{\beta+x^{-1}}\right)$$

is positive. (Recall the standing assumption $\beta-\alpha>0$.) By Jensen's inequality

$$\frac{x+x^{-1}}{2} > 1$$

so that

$$f(-1)\ge f(1)\ge f\left(\frac{x+x^{-1}}{2}\right)=\lambda.$$

Case 2: $0<-\beta<x$. Same as the previous case since $\beta+r$ is positive.

Case 3: $0<-\beta=x$. As we saw before, $r=-\beta$ so that $f(u)$ is constant for all u.

Case 4: $0<x<-\beta$. Since $f'(u)>0$ with a singularity at $-(\beta+\beta^{-1})/2>1$ and

$$\frac{x+x^{-1}}{2} > -\frac{\beta+\beta^{-1}}{2}$$

we have

$$f\left(\frac{x+x^{-1}}{2}\right) \le \inf\{f(u): u< -(\beta+\beta^{-1})/2\} \le \min\{f(1), f(-1)\}.$$

Therefore, $B(e^{i\theta})$ and λ satisfy the output spectrum condition ii) of Proposition 5.1.

Finally we check the factorization condition iii) of Proposition 5.1, namely

$$(1+\alpha z)(1-xz)-(1-rz)(1-x^{-1}z)$$

has a factor $1+\beta z$, which follows immediately from (77). This establishes the optimality of $B(z)$ defined in (45) with $0<x<1$ and y satisfying (46).

From Jensen's formula (10), we see that the corresponding feedback capacity is given by

$$C_{FB}(x) = \int_{-\pi}^{\pi}\frac{1}{2}\log S_Y(e^{i\theta})\frac{d\theta}{2\pi} = -\frac{1}{2}\log x^2$$

under the power constraint

$$P(x) = \frac{y^2}{1-x^2} = \frac{(1-x^2)(1+\alpha x)^2}{x^2(1+\beta x)^2}.$$

The case $\beta<\alpha$ can be treated similarly with $x<0$, while the case $\beta=\alpha$ (i.e., $S_Z\equiv 1$) is trivial. This completes the proof of Theorem 5.3.

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