Error Exponents for the Gaussian Channel with Noisy Active Feedback

Young-Han Kim
Department of ECE
University of California, San Diego
La Jolla, CA 92093, USA
Email: yhk@ucsd.edu

Amos Lapidoth
Signal and Information Processing Lab
ETH Zurich
CH-8092, Zurich, Switzerland
Email: lapidoth@isi.ee.ethz.ch

Tsachy Weissman
Information Systems Lab
Stanford Univeristy
Stanford, CA 94305, USA
Email: tsachy@stanford.edu

***Abstract*—We study the best exponential decay in the (deterministic) blocklength of the probability of error that can be achieved in the transmission of a single bit over the Gaussian channel with an active noisy Gaussian feedback link. We impose an *expected* block power constraint on the forward link and study both *almost-sure* and *expected* block power constraints on the feedback link. In both cases the best achievable error exponent is finite and grows approximately proportionally to the larger of the signal-to-noise ratios on the forward and feedback links. The error exponents under almost-sure block power constraints are typically strictly smaller than under expected constraints. The error exponents achievable with active feedback are shown to be superior to those that are achievable with passive feedback. Some of the results extend to communication at arbitrary rates below capacity and to general discrete memoryless channels.**

I. Introduction

This paper studies error exponents for the Gaussian channel with noisy feedback. Unlike our previous work, which focused on *passive* feedback [1], [2], [3], here we focus on *active* feedback. Thus, the time-k symbol U_k fed to the feedback channel need not be the time-k received symbol Y_k: it can be a function of Y_k and of the previous received symbols $Y_1, \ldots, Y_{k-1}$. As in our previous work, we consider only transmission schemes of a deterministic blocklength n. (Random transmission times for discrete memoryless channels with active feedback are discussed in [4].) And, although some of our results extend to more general models, we focus on the Gaussian model where both the forward channel and the feedback channel are additive white Gaussian noise channels. To simplify the analysis we focus on the case where the message to be transmitted is binary, i.e., takes on the values 0 and 1 equiprobably (but see (13) which is applicable to all rates of communication between zero and capacity). Our communication scheme is depicted in Figure 1.

Critical to our analysis is the precise nature of the power constraints that are imposed on the forward and feedback channels. On the forward channel we impose an *expected block power constraint*, where the time-average of the squared channel inputs is a random variable (whose realization may depend on the message and on the realization of the forward and feedback channels) whose expectation (over the message and over the noise sequences on the forward and feedback channels) is upper-bounded by some fixed (deterministic) positive constant P; see (8) ahead. For the feedback link, we consider two types of power constraints: an expected block power constraint ((9) ahead) and an *almost-sure block power constraint* ((10)). In the latter, the time-average of the squared inputs to the feedback channel must not exceed P_{FB} *irrespective of the message and of the channel realizations.* Clearly, an almost-sure power constraint is more restrictive than an expected power constraint.

We do not consider an almost-sure block power constraint on the forward channel because under this constraint even a noise-free feedback link does not improve the two-codewords error exponent [5], [6].

Our main result is that—although a noise-free feedback link allows the probability of error to decay faster than exponentially in n [7] [8] [9]—if the feedback link is noisy the probability of error cannot decay faster than exponentially. This is true even if we only impose an expected block power constraint on the feedback link. Moreover, we provide upper and lower bounds on the best achievable exponent both for expected and almost-sure block power constraints. At high signal-to-noise ratios (SNRs) on the feedback link, the error exponents in both cases grow as an affine function of the SNR. A more formal statement of the results will be given in Section II once we have formalized the problem's statement. For proofs please see [10].

II. The Problem Statement and Main Results

We consider the transmission of a single bit H, where H takes on the values 0 and 1 equiprobably. Let the sets $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{U}$, and $\mathcal{Z}$ all be the reals. A blocklength-n code for transmitting H over our channel consists of a forward-channel encoding rule, a feedback-channel encoding rule, and a decoder as described next. A forward-channel encoding rule is specified by n functions[1] $f_1, \ldots, f_n$, where

$$f_k\colon \{0,1\} \times \mathcal{Z}^{k-1} \to \mathcal{X}, \quad k = 1, \ldots n. \tag{1}$$

It is understood that the time-k channel input X_k is computed according to the rule

$$X_k = f_k\big(H, Z^{k-1}\big), \quad k = 1, \ldots, n, \tag{2}$$

[1] All functions from $\mathbb{R}$ to $\mathbb{R}$ in this paper are assumed to be Borel Measurable.

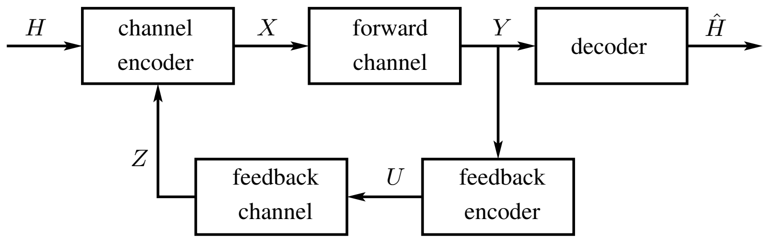

Fig. 1. The Gaussian channel with a coded noisy feedback link.

where we use A^ℓ to denote $A_1, \ldots, A_\ell$ and where, for convenience, we set

$$Z_0 = 0. \tag{3}$$

The feedback-channel encoding rule is a collection of n functions $g_1, \ldots, g_n$, where

$$g_k \colon \mathcal{Y}^k \to \mathcal{U}, \quad k = 1, \ldots, n. \tag{4}$$

It is understood that the symbol U_k that is fed to the feedback channel at time k is given by

$$U_k = g_k\big(Y^k\big), \quad k = 1, \ldots, n. \tag{5}$$

(The special case where $g_k(Y^k)$ is Y_k corresponds to passive—also knows as "uncoded" or "symbol-by-symbol"—feedback.)

A decoder ϕ is a decision rule for guessing H based on Y^n. Thus,

$$\phi \colon \mathcal{Y}^n \to \{0, 1\}. \tag{6}$$

We denote the decision regions by $\mathcal{D}_0$ and $\mathcal{D}_1$ so

$$\mathcal{D}_\nu = \big\{\mathbf{y} \in \mathcal{Y}^n : \phi(\mathbf{y}) = \nu\big\}, \quad \nu = 0, 1, \tag{7}$$

where we use $\mathbf{a}$ to denote the n-tuple $(a_1, \ldots, a_n)$.

The communication system that we consider operates as follows. The message, along with the forward and backward channel noise components, $H, N_1, \ldots, N_n, V_1, \ldots, V_n$, are independent random variables, where $N_k \sim \mathcal{N}(0, \sigma^2)$ and $V_k \sim \mathcal{N}(0, \sigma_{\text{FB}}^2)$ for every $k \in \{1, \ldots, n\}$. We assume throughout that σ and σ_{FB} are strictly positive. At time k, the input X_k to the forward channel is generated according to (2). This input is corrupted by the forward channel noise, yielding the forward-channel output $Y_k = X_k + N_k$. The feedback-channel encoder now computes the symbol U_k from Y^k according to (5). The symbol U_k, which forms the time-k input to the feedback channel is corrupted by the feedback-channel noise to yield $Z_k = U_k + V_k$ at the output of that channel. The conditional density $w(y_k|x_k)$ of Y_k given X_k is thus

$$w(y_k|x_k) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(y_k - x_k)^2}{2\sigma^2}}, \quad x_k, y_k \in \mathbb{R},$$

and the conditional density $w_{\text{FB}}(z_k|u_k)$ of Z_k given U_k is

$$w_{\text{FB}}(z_k|u_k) = \frac{1}{\sqrt{2\pi\sigma_{\text{FB}}^2}}\, e^{-\frac{(z_k - u_k)^2}{2\sigma_{\text{FB}}^2}}, \quad z_k, u_k \in \mathbb{R}.$$

We only consider forward-channel encoding rules that satisfy the expected block power constraint

$$\mathsf{E}\Bigg[\sum_{k=1}^{n} f_k^2\big(H, Z^{k-1}\big)\Bigg] \le n\mathsf{P}, \tag{8}$$

where $\mathsf{P} > 0$ is some given constant designating the allowed average power (per transmission) on the forward-channel.

For the feedback-channel encoding rules we consider two types of power constraints. An almost-sure block power constraint

$$\sum_{k=1}^{n} g_k^2\big(Y^k\big) \le n\mathsf{P}_{\text{FB}}, \quad \mathbf{Y} \in \mathcal{Y}^n, \tag{9}$$

and an expected block power constraint

$$\mathsf{E}\Bigg[\sum_{k=1}^{n} g_k^2\big(Y^k\big)\Bigg] \le n\mathsf{P}_{\text{FB}}. \tag{10}$$

In both cases we assume that P_{FB} is strictly positive. (The case where $\mathsf{P}_{\text{FB}} = 0$ corresponds to the no-feedback case.) We can now present our main results.

Almost-Sure Block Power Constraints: Let us denote by $p_{\text{e}}^{\text{a.s.}}\big(\mathsf{P}/\sigma^2, \mathsf{P}_{\text{FB}}/\sigma_{\text{FB}}^2, n\big)$ the least probability of error that can be achieved by a blocklength-n coding scheme subject to the almost-sure constraint (9). We show that

$$\varlimsup_{n\to\infty} -\frac{1}{n} \log p_{\text{e}}^{\text{a.s.}}\big(\mathsf{P}/\sigma^2, \mathsf{P}_{\text{FB}}/\sigma_{\text{FB}}^2, n\big) \le \frac{\mathsf{P}}{2\sigma^2} + \frac{2\sqrt{(\mathsf{P}_{\text{FB}} + \sigma_{\text{FB}}^2)\mathsf{P}_{\text{FB}}}}{\sigma_{\text{FB}}^2}, \tag{11}$$

and we present a sequence of codes that proves that

$$\varliminf_{n\to\infty} -\frac{1}{n} \log p_{\text{e}}^{\text{a.s.}}\big(\mathsf{P}/\sigma^2, \mathsf{P}_{\text{FB}}/\sigma_{\text{FB}}^2, n\big) \ge \frac{\mathsf{P}}{2\sigma^2} + \frac{2\mathsf{P}_{\text{FB}}}{\sigma_{\text{FB}}^2}. \tag{12}$$

Moreover, (11) generalizes to the case where there are more than two codewords. If we denote by R the rate of communication, i.e., the ratio of the logarithm of the number of messages to the block length, and if we denote by $E_{\text{FB}}^{\text{a.s.}}(R)$ the best achievable error exponent then

$$E_{\text{FB}}^{\text{a.s.}}(R) \le E_{\text{NoFB}}(R) + \frac{2\sqrt{(\mathsf{P}_{\text{FB}} + \sigma_{\text{FB}}^2)\mathsf{P}_{\text{FB}}}}{\sigma_{\text{FB}}^2}, \tag{13}$$

where $E_{\text{NoFB}}(R)$ is the reliability function of the forward channel in the absence of feedback.

Expected Block Power Constraints: Let us denote by $p_e^{\exp}\big(\mathsf{P}/\sigma^2, \mathsf{P}_{\mathrm{FB}}/\sigma_{\mathrm{FB}}^2, n\big)$ the least probability of error that can be achieved by a blocklength-n coding scheme subject to the expected block power constraint (10). We show that[2]

$$\overline{\lim_{n\to\infty}} -\frac{1}{n}\log p_e^{\exp}\big(\mathsf{P}/\sigma^2, \mathsf{P}_{\mathrm{FB}}/\sigma_{\mathrm{FB}}^2, n\big) \leq \frac{\left(\sqrt{\mathsf{P}+\sigma^2}+\sqrt{\mathsf{P}}\right)^2}{\sigma^2} + \frac{\left(\sqrt{\mathsf{P}_{\mathrm{FB}}+\sigma_{\mathrm{FB}}^2}+\sqrt{\mathsf{P}_{\mathrm{FB}}}\right)^2}{\sigma_{\mathrm{FB}}^2}, \quad (14)$$

and we present a sequence of codes that achieves

$$\underline{\lim_{n\to\infty}} -\frac{1}{n}\log p_e^{\exp}\big(\mathsf{P}/\sigma^2, \mathsf{P}_{\mathrm{FB}}/\sigma_{\mathrm{FB}}^2, n\big) \geq \frac{2\mathsf{P}}{\sigma^2} + \frac{2\mathsf{P}_{\mathrm{FB}}}{\sigma_{\mathrm{FB}}^2}. \quad (15)$$

III. Discussion

We have seen that even if both the forward link and the feedback link are subjected to expected block power constraints, the best achievable error exponent is finite. Roughly speaking—irrespective of the nature of the feedback power constraint—the best error exponent is roughly proportional to the larger of the signal-to-noise ratio on the forward link P/σ^2 and the signal-to-noise ratio on the feedback channel $\mathsf{P}_{\mathrm{FB}}/\sigma_{\mathrm{FB}}^2$. In this very rough sense, active feedback is not much different from passive symbol-by-symbol feedback [2].

However, a more careful analysis based on our previous results [1], [2] shows that the best error exponent for two messages with passive (symbol-by-symbol) feedback is upper-bounded by

$$\frac{1}{2}\left(\frac{\mathsf{P}}{\sigma^2} + \frac{\mathsf{P}}{\mathsf{P}+\sigma^2}\cdot\frac{\mathsf{P}_{\mathrm{FB}}}{\sigma_{\mathrm{FB}}^2}\right), \quad (16)$$

which can be further upper-bounded by

$$\frac{1}{2}\left(\frac{\mathsf{P}}{\sigma^2} + \frac{\mathsf{P}_{\mathrm{FB}}}{\sigma_{\mathrm{FB}}^2}\right). \quad (17)$$

On the other hand, an achievable error exponent (15) for an active feedback with the same feedback signal-to-noise ratio is

$$2\left(\frac{\mathsf{P}}{\sigma^2} + \frac{\mathsf{P}_{\mathrm{FB}}}{\sigma_{\mathrm{FB}}^2}\right).$$

Hence, the freedom to code over the feedback link can at least quadruple the error exponent of binary communication. It would be interesting to see how much active feedback gains over passive feedback for a positive rate R.

While our focus has been on Gaussian channels with Gaussian feedback channels, some of our techniques are more general. For example, consider a setting where the forward and feedback channels are binary symmetric channels (BSCs) with crossover probabilities $\epsilon, \epsilon_{\mathrm{FB}} \leq 1/2$. In this case we obtain using similar techniques that the reliability function with noisy active feedback cannot exceed

$$\ln\frac{1-\epsilon_{\mathrm{FB}}}{\epsilon_{\mathrm{FB}}} + E_{\mathrm{NoFB}}(R;\epsilon), \quad (18)$$

where $E_{\mathrm{NoFB}}(R;\epsilon)$ is the reliability function of the BSC of crossover probability ϵ.

In some cases, particularly when the feedback channel is very noisy, this bound can be tighter than the trivial bound that bounds the reliability function by that with perfect feedback and bounds the latter by the best two-codeword error exponent

$$\frac{1}{2}\ln\frac{1}{4\epsilon(1-\epsilon)}. \quad (19)$$

(The fact that feedback does not improve the best two-codeword error exponent on a discrete memoryless channel appears in the Ph.D. thesis of Berlekamp [11] who attributes this result to Gallager and Shannon.)

The bound in (18) complements the recent work of Burnashev and Yamamoto [12] on the reliability function of the binary symmetric channel with a passive binary symmetric feedback link. (Upper bounds on the latter reliability function can be derived using techniques similar to those we used in [2].)

[2]In [10] we also present a tighter bound than (14).

References

[1] Y.-H. Kim, A. Lapidoth, and T. Weissman, "On the reliability of Gaussian channels with noisy feedback," in *Proceedings Forty-First Allerton Conference on Communication, Control and Computing*, pp. 364–371, September 2006.

[2] Y.-H. Kim, A. Lapidoth, and T. Weissman, "The Gaussian channel with noisy feedback," in *Proc. of the International Symposium on Information Theory (ISIT'07)*, (Nice, France), pp. 1416–1420, June 2007.

[3] Y.-H. Kim, A. Lapidoth, and T. Weissman, "Bounds on the error exponents of the AWGN channel with AWGN-corrupted feedback," in *Proceedings of 24th IEEE Convention of Electrical and Electronics Engineers in Israel*, pp. 184–188, Nov. 2006.

[4] S. Draper and A. Sahai, "Variable-length coding with noisy feedback," *European Transactions on Telecommunications*, vol. 19, pp. 355–370, May 2008.

[5] M. S. Pinsker, "The probability of error in block transmission in a memoryless Gaussian channel with feedback," *Problems of Information Transmission*, vol. 4, no. 4, pp. 3–19, 1968.

[6] L. A. Shepp, J. K. Wolf, A. D. Wyner, and J. Ziv, "Binary communication over the Gaussian channel using feedback with a peak energy constraint," *IEEE Trans. on Inform. Theory*, vol. 15, pp. 476–478, July 1969.

[7] J. P. M. Schalkwijk and T. Kailath, "A coding scheme for additive noise channels with feeback—part I: no bandwidth constraint," *IEEE Trans. on Inform. Theory*, vol. 12, pp. 172–182, April 1966.

[8] A. J. Kramer, "Improving communication reliability by use of an intermittent feedback channel," *IEEE Trans. on Inform. Theory*, vol. 15, pp. 52–60, Jan. 1969.

[9] R. G. Gallager and B. Nakiboğlu, "Variations on a theme by Schalkwijk and Kailath," *arXiv:0812.2709v2*, 16 August 2009.

[10] Y.-H. Kim, A. Lapidoth, and T. Weissman, "Error exponents for the Gaussian channel with noisy active feedback," *IEEE Trans. on Inform. Theory (submitted). See also http://arxiv.org/pdf/0909.4203*, 2009.

[11] E. R. Berlekamp, *Block coding with noiseless feedback*. PhD thesis, Massachusetts Institute of Technology, 1964.

[12] M. V. Burnashev and H. Yamamoto, "Noisy feedback improves the BSC reliability function," in *Proceedings of the International Symposium on Information Theory ISIT'09*, (Seoul, Korea), pp. 1501–1505, June-July 2009.