Uniform Power Allocation with Thresholding over Rayleigh Slow Fading Channels with QAM Inputs

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***Abstract*— In this paper, we consider the power allocation problem that minimizes the outage probability for Rayleigh slow fading channels with equiprobable QAM inputs. We focus on the *uniform power allocation with thresholding* (UPAT) policy that assigns nonzero constant power only to a subset of the subchannels. This simple suboptimal policy can significantly alleviate the feedback overhead and the complexity compared to the optimal *mercury/water-filling* (MWF) solution. Through asymptotic analysis and numerical simulations, we first show that the *optimal UPAT*, namely, the UPAT with the optimal threshold, performs close to MWF if the constellation size M is large enough that $\log_2 M \gg R$, where R is the fixed target transmission rate. This condition $\log_2 M \gg R$ turns out to define a natural system operating point. As we show through numerical results, if $\log_2 M \approx R$, both MWF and the optimal UPAT perform poorly due to having too small M and their performance can be significantly improved by using a larger M. From these results, we conclude that for a given target transmission rate, the optimal UPAT performs close to MWF as long as the constellation size is chosen appropriately not to limit the performance.**

***Index Terms*—Constellation size, fading, mercury/water-filling, outage probability, power allocation, QAM.**

I. Introduction

The performance of a communication system over a fading channel can be substantially improved by adapting transmit power according to the channel gains [1]–[8]. If the channel is partitioned into independent AWGN subchannels, the waterfilling power allocation [1] along with a Gaussian input distribution maximizes the mutual information (MI) of the channel. In contrast, for non-Gaussian inputs such as pulse amplitude modulation (PAM) and quadrature amplitude modulation (QAM), the *mercury/water-filling* (MWF) power allocation maximizes the MI [2].

However, the amount of overhead to enable transmit power adaptation is often significant in practice [3], [4]. In addition, the MWF power allocation involves the inverse minimum mean-square error (MMSE) functions [2], an exact implementation of which can be excessively complex in practice [7]. To overcome these difficulties, a simple suboptimal power allocation policy, *uniform power allocation with thresholding* (UPAT) that assigns nonzero constant power only to a subset of the subchannels, has received much attention [3]–[6], due to the remarkably relaxed overhead requirements [3], [4] as well as the simple transceiver structure [6].

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Much effort has been devoted to developing simple methods to determine the threshold value (or equivalently, the subset of the subchannels to which nonzero constant power is assigned) for UPAT and to analyzing the resulting performance [3]–[6]. However, most of the existing results have focused on Gaussian inputs over ergodic (or fast fading) channels and thus the insight into practical system design has been rather limited. Moreover, the UPAT scheme with the optimal threshold, henceforth referred to as the *optimal UPAT*, has not been well studied.

In this paper, we study the outage performance of the optimal UPAT for Rayleigh slow fading channels with equiprobable QAM inputs. Note that it is mathematically challenging to derive the exact performance results, since the performance metric involves the MI of an AWGN channel with QAM input which does not have a closed-form expression. In our work, we rely on asymptotic analysis, engineering intuition, and comprehensive numerical work. Through asymptotic analysis and numerical simulations, we first show that the optimal UPAT performs close to MWF if the constellation size M is large enough that $\log_2 M \gg R$, where R is the fixed target transmission rate. This condition $\log_2 M \gg R$ turns out to define a natural system operating point. As we show through numerical results, if $\log_2 M \approx R$, both MWF and the optimal UPAT perform poorly due to having too small M and their performance can be significantly improved by using a larger M. From these results, we conclude that for a given target transmission rate, the optimal UPAT performs close to MWF as long as the constellation size is chosen appropriately not to limit the performance.

The rest of the paper is organized as follows. Section II presents the system model and the performance metric. Section III formulates the outage probability with MWF and the optimal UPAT. Section IV discusses the outage performance.

Throughout, for a vector $\mathbf{a} = (a_1, ..., a_n)$, $\langle \mathbf{a} \rangle \triangleq \frac{1}{n} \sum_{i=1}^{n} a_i$. Component-wise inequalities are denoted by $\preceq$ and $\succeq$.

II. System Model and Performance Metric

A. System Model

Consider transmission over a nonergodic block-fading channel [7], also called slow fading channel, consisting of B blocks of L channel uses, where block $i = 1, 2, ..., B$ undergoes a random channel gain H_i that is constant during the block and is independently and identically distributed (i.i.d.) across the blocks. Assume that the channel inputs to the blocks are

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independently and uniformly distributed over the standard M-QAM [9] constellation set $\mathcal{S}_M$, where $M \in \{2^{2j} : j = 1, 2, ...\}$ and $\frac{1}{M}\sum_{s\in\mathcal{S}_M} |s|^2 = 1$. Suppose that $\{|H_i|\}_{i=1}^B$ is known to the transmitter so that transmit power for each block can be adapted to the channel strength, subject to an average power constraint P. To describe a power allocation scheme, it is convenient to define $\gamma_i \triangleq P|H_i|^2$ which indicates the SNR in block i with the uniform power allocation (UPA). Let $\boldsymbol{\gamma} = \{\gamma_i\}_{i=1}^B$. Then, a power allocation scheme is described as $\boldsymbol{p}(\boldsymbol{\gamma}; M) = \{p_i(\boldsymbol{\gamma}; M)\}_{i=1}^B$, where $p_i(\boldsymbol{\gamma}; M) \geq 0$ indicates the normalized transmit power of block i, i.e., $\langle \boldsymbol{p}(\boldsymbol{\gamma}; M)\rangle \leq 1$. Note that $\boldsymbol{p}(\boldsymbol{\gamma}; M)$ depends on $\{|H_i|^2\}_{i=1}^B$, P, and B through $\boldsymbol{\gamma}$. The channel output vector $\mathbf{Y}_i \in \mathbb{C}^L$ in block i, is given by

$$\mathbf{Y}_i = H_i\sqrt{p_i(\boldsymbol{\gamma}; M)P}\ \mathbf{S}_i + \mathbf{Z}_i, \quad i = 1, 2, ..., B, \tag{1}$$

where $\mathbf{S}_i \in \mathcal{S}_M^L$ is the M-QAM channel input vector and $\mathbf{Z}_i \sim \mathcal{N}_{\mathbb{C}}(0, \mathrm{I})$ is the Gaussian channel noise vector. Note that $p_i(\boldsymbol{\gamma}; M)\gamma_i$ corresponds to the instantaneous SNR in block i. Throughout, we assume the Rayleigh fading $H_i \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ and therefore the probability density function (pdf) of γ_i is given by $f_{\gamma_i}(\xi) = \frac{1}{P}e^{-\frac{1}{P}\xi}, \xi \geq 0$.

B. Performance Metric: Outage Probability

Recall that the MI of the AWGN channel $Y = \sqrt{\rho}S + Z$ under the uniform distribution of the input S over $\mathcal{S}_M$ is [10]

$$I_M^{\text{AW}}(\rho) = \log_2 M - \frac{1}{M}\sum_{s\in\mathcal{S}_M} \mathsf{E}_Z\left[\log_2\left(\sum_{s'\in\mathcal{S}_M} e^{-|\sqrt{\rho}(s-s')+Z|^2+|Z|^2}\right)\right]. \tag{2}$$

Then, the instantaneous MI is defined [8] by

$$I_M(\boldsymbol{\gamma}, \boldsymbol{p}(\boldsymbol{\gamma}; M)) \triangleq \frac{1}{B}\sum_{i=1}^B I_M^{\text{AW}}(p_i(\boldsymbol{\gamma}; M)\gamma_i). \tag{3}$$

For a fixed target transmission rate R, the *outage probability* is defined [8] by

$$P_M^{\text{out}}(\boldsymbol{p}(\boldsymbol{\gamma}; M), P, R) \triangleq \mathsf{P}\{I_M(\boldsymbol{\gamma}, \boldsymbol{p}(\boldsymbol{\gamma}; M)) < R\} \tag{4}$$

where the probability is with respective to the random $\boldsymbol{\gamma}$.

III. Power Allocation Schemes

A. Optimal Power Allocation: Mercury/water-filling

The outage probability minimization problem is

$$\begin{aligned} \text{minimize} \quad & P_M^{\text{out}}(\boldsymbol{p}(\boldsymbol{\gamma}; M), P, R) \\ \text{subject to} \quad & \langle \boldsymbol{p}(\boldsymbol{\gamma}; M)\rangle \leq 1 \\ & \boldsymbol{p}(\boldsymbol{\gamma}; M) \succeq 0. \end{aligned} \tag{5}$$

We refer to the solution of the above problem as the *optimal power allocation*, denoted by $\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma}; M)$. Since the outage probability is minimized when the instantaneous MI (3) is maximized for each realization of $\boldsymbol{\gamma}$, we have [8]

$$\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma}; M) = \arg\max_{\substack{\langle \boldsymbol{p}(\boldsymbol{\gamma}; M)\rangle \leq 1 \\ \boldsymbol{p}(\boldsymbol{\gamma}; M) \succeq 0}} I_M(\boldsymbol{\gamma}, \boldsymbol{p}(\boldsymbol{\gamma}; M)). \tag{6}$$

Let $\mathsf{MMSE}_M(\rho)$ denote the MMSE incurred in the estimation of an equiprobable M-QAM symbol over the AWGN channel with SNR ρ. By the Lagrangian duality, the Karush–Kuhn–Tucker (KKT) conditions [11], and the relationship $\frac{d}{d\rho}I_M^{\text{AW}}(\rho) = \frac{1}{\ln 2}\mathsf{MMSE}_\mathsf{M}(\rho)$ between MI and MMSE [12], the solution of (6) is given [2] by

$$p_i^{\text{opt}}(\boldsymbol{\gamma}; M) = \begin{cases} \frac{1}{\gamma_i}\mathsf{MMSE}_M^{-1}\left(\frac{\lambda}{\gamma_i}\right), & \gamma_i \geq \lambda \\ 0, & \gamma_i < \lambda \end{cases} \tag{7}$$

where the SNR threshold λ is chosen so that the average power constraint is satisfied with equality. The optimal solution (7) is often referred to as *mercury/water-filling* (MWF) [2]. Substituting (7) into (4) yields the outage probability with the optimal power allocation

$$\mathsf{P}\left\{\frac{1}{B}\sum_{i:\gamma_i\geq\lambda} I_M^{\text{AW}}\left(\mathsf{MMSE}_M^{-1}\left(\frac{\lambda}{\gamma_i}\right)\right) < R\right\}. \tag{8}$$

B. Uniform Power Allocation with Thresholding

In a UPAT scheme, nonzero constant power is assigned to a set of selected blocks while zeropower is assigned to the other blocks. Let $\gamma_1^o \leq \gamma_2^o \leq \cdots \leq \gamma_B^o$ be the ordered $\boldsymbol{\gamma}$ sequence. Then, a UPAT scheme is given by

$$p_i^{\text{UPAT}}(\boldsymbol{\gamma}; M) = \begin{cases} \frac{B}{B-N_{\text{UPAT}}}, & \gamma_i \geq \gamma_{N_{\text{UPAT}}+1}^o \\ 0, & \gamma_i < \gamma_{N_{\text{UPAT}}+1}^o \end{cases} \tag{9}$$

where $0 \leq N_{\text{UPAT}} < B$ is the number of zero-power blocks. Therefore, a UPAT scheme is completely defined by how to determine the value of N_{UPAT}. The optimal value of N_{UPAT} that maximizes the instantaneous MI is

$$N_{\text{UPAT}}^* = \arg\max_{0\leq n<B}\sum_{i=n+1}^B I_M^{\text{AW}}\left(\frac{B}{B-n}\gamma_i^o\right). \tag{10}$$

The UPAT with N_{UPAT}^* is referred to as the *optimal UPAT*, denoted by $\boldsymbol{p}^{\text{UPAT}*}(\boldsymbol{\gamma}; M)$. Substituting (9) and (10) into (4) yields the outage probability with the optimal UPAT

$$\mathsf{P}\left\{\frac{1}{B}\sum_{i=N_{\text{UPAT}}^*+1}^B I_M^{\text{AW}}\left(\frac{B}{B-N_{\text{UPAT}}^*}\gamma_i^o\right) < R\right\}. \tag{11}$$

C. Examples

We now discuss[1] a set of examples that show power allocations of MWF and the optimal UPAT, corresponding instantaneous MIs, and the power loss ΔP that indicates the additional P required for the optimal UPAT to achieve the same instantaneous MI as MWF. The examples are with respect to two specific realizations of $\boldsymbol{\gamma}$ but provide some insight into the outage probability results that come from a random $\boldsymbol{\gamma}$, discussed in the next section.

We consider a specific channel gain vector $\{h_i\}_{i=1}^B$ with $B = 100$, where its elements were i.i.d. drawn according to $\mathcal{N}_{\mathbb{C}}(0, 1)$, normalized and ordered[2] such that $\frac{1}{B}\sum_{i=1}^B |h_i|^2 = 1$ and $|h_1| \leq \cdots \leq |h_B|$ (see Fig. 1 and Fig. 2). Two different values of P, $P_1 = -10$ dB and $P_2 = 13$ dB are examined.

[1] Similar discussions are presented in [2], focusing on MWF and water-filling. In this paper, however, we are mainly interested in the comparison between MWF and the optimal UPAT.

[2] The ordering is merely intended to clearly show the dependency of power allocation on the channel gains

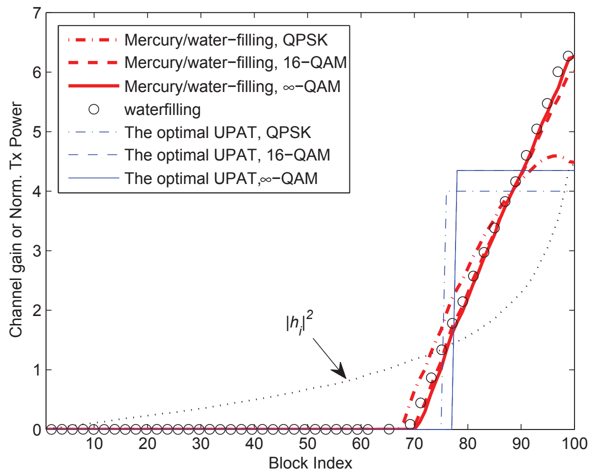

Fig. 1. Power allocation results of mercury/water-filling (optimal), waterfilling, and the optimal UPAT, when $\boldsymbol{\gamma} = \boldsymbol{\gamma}_1$ ($\frac{1}{B}\sum_{i=1}^{B}\gamma_i = -10$ dB).

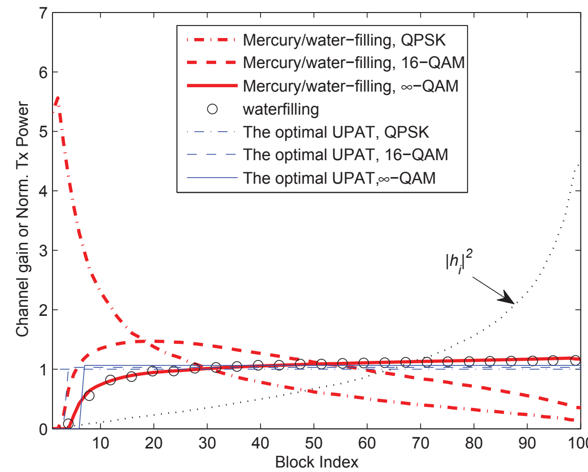

Fig. 2. Power allocation results of mercury/water-filling (optimal), waterfilling, and the optimal UPAT, when $\boldsymbol{\gamma} = \boldsymbol{\gamma}_2$ ($\frac{1}{B}\sum_{i=1}^{B}\gamma_i$ =13 dB).

TABLE I
INSTANTANEOUS MI AND ADDITIONAL P REQUIRED BY $\boldsymbol{p}^{\text{UPAT}*}$

$(\boldsymbol{\gamma}, M)$	$I_M(\boldsymbol{p}^{\text{opt}})$	$I_M(\boldsymbol{p}^{\text{UPAT}*})$	ΔI_M	ΔP
$(\boldsymbol{\gamma}_1, 4)$	0.2314	0.2288	1.2 %	0.07 dB
$(\boldsymbol{\gamma}_1, 16)$	0.2385	0.2341	1.8 %	0.12 dB
$(\boldsymbol{\gamma}_1, \infty)$	0.2394	0.2349	1.9 %	0.12 dB
$(\boldsymbol{\gamma}_2, 4)$	1.9499	1.8630	4.5 %	4.4 dB
$(\boldsymbol{\gamma}_2, 16)$	3.1917	3.1199	2.3 %	0.5 dB
$(\boldsymbol{\gamma}_2, \infty)$	3.5950	3.5871	0.2 %	0.03 dB

Three constellation sizes, $M = 4, 16, \infty$ are considered. Let $\boldsymbol{\gamma}_1 = \{P_1|h_i|^2\}_{i=1}^{B}$ and $\boldsymbol{\gamma}_2 = \{P_2|h_i|^2\}_{i=1}^{B}$. Then, $\langle\boldsymbol{\gamma}_1\rangle = -10$ dB and $\langle\boldsymbol{\gamma}_2\rangle = 13$ dB. Since MWF and the optimal UPAT depend on $(\boldsymbol{\gamma}, M)$, there are 6 different cases, i.e., $M = 4, 16, \infty$ for each $\boldsymbol{\gamma}$.

The power allocations of MWF and the optimal UPAT are shown in Fig. 1 and Fig. 2 in case $\boldsymbol{\gamma} = \boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma} = \boldsymbol{\gamma}_2$, respectively. For each $\boldsymbol{\gamma}$, we also present the power allocations of waterfilling that does not depend on M. We note the following observations. First, all the power allocation schemes activate (or assign nonzero power to) a small portion of the blocks when $\boldsymbol{\gamma} = \boldsymbol{\gamma}_1$, while activating most of the blocks when $\boldsymbol{\gamma} = \boldsymbol{\gamma}_2$. When $\boldsymbol{\gamma} = \boldsymbol{\gamma}_2$, the optimal UPAT and waterfilling are similar to the UPA, regardless of M. Second, MWF is similar to waterfilling except the case of $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_2, M = 4)$, especially in the cases of $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_1, M = 16)$, $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_1, M = \infty)$, and $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_2, M = \infty)$. Third, in the case of $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_2, M = 4)$, MWF assigns power almost inversely proportionally to γ_i, which is completely different from waterfilling and the optimal UPAT.

For each case, Table I shows the instantaneous MI of MWF, $I_M(\boldsymbol{p}^{\text{opt}})$, the instantaneous MI of the optimal UPAT, $I_M(\boldsymbol{p}^{\text{UPAT}*})$, the loss ΔI_M of the optimal UPAT in instantaneous MI compared to MWF, and the power loss ΔP. We note the following observations. First, in all the cases, ΔI_M is insignificant, i.e., only 4.5% even in the case of $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_2, M = 4)$ where the power allocations are significantly different between MWF and the optimal UPAT. Second, ΔP is significant in the case of $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_2, M = 4)$, while it is insignificant in the other cases.

The results in Fig. 1, Fig. 2, and Table I can be explained by the following properties of $I_M^{\text{AW}}(\rho)$. Note that a sum of $I_M^{\text{AW}}(p_i\gamma_i)$ is the objective function in the relevant optimization problems (6) and (10).

(i) Given M, if ρ is so low that $I_M^{\text{AW}}(\rho) \ll \log_2 M$, $I_M^{\text{AW}}(\rho)$ is not much different from $\log_2(1+\rho)$. In particular, $I_M^{\text{AW}}(\rho) \approx \log_2(1+\rho)$ in the low ρ regime [2].
(ii) Given M, if ρ is so high that $I_M^{\text{AW}}(\rho) \approx \log_2 M$, $I_M^{\text{AW}}(\rho)$ slowly approaches its maximum $\log_2 M$ as ρ increases. Therefore, in this regime, $I_M^{\text{AW}}(\rho)$ does not vary much as ρ changes.

Due to property (i), if $(\boldsymbol{\gamma}, M)$ is such that $I_M^{\text{AW}}(\gamma_i) \ll \log_2 M$ for most of the blocks, e.g., $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_1, M = 16)$, $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_1, M = \infty)$, and $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_2, M = \infty)$, MWF is not much different from waterfilling, i.e., cutting off weakest blocks and then assigning more power to stronger blocks. This is because the solution to the problem (6) is waterfilling if $I_M^{\text{AW}}(p_i\gamma_i)$ is replaced by $\log_2(1+p_i\gamma_i)$. In this case, as shown in [5], [6] for Gaussian inputs, ΔI_M and ΔP are not significant.

Due to property (ii), on the contrary, if $(\boldsymbol{\gamma}, M)$ is such that $I_M^{\text{AW}}(\gamma_i) \approx \log_2 M$ for most of the blocks, e.g., $(\boldsymbol{\gamma} = \boldsymbol{\gamma}_2, M = 4)$, MWF tends to assign power inversely proportionally to γ_i [2]. Intuitively, this is because with relatively small power, stronger blocks can still provide per-block MI $I_M^{\text{AW}}(p_i\gamma_i)$ close to $\log_2 M$. By assigning more power to weaker blocks, MWF increases per-block MI of weaker blocks, while minimally decreasing per-block MI of stronger blocks. This is how MWF maximizes the average of per-block MI, i.e., the instantaneous MI. In contrast, the optimal UPAT just activates most of the blocks due to the uniform power constraint and therefore the optimal UPAT tends to be the UPA. Interestingly, although the power allocations of MWF and the optimal UPAT are quite different from each other, ΔI_M is not significant since the

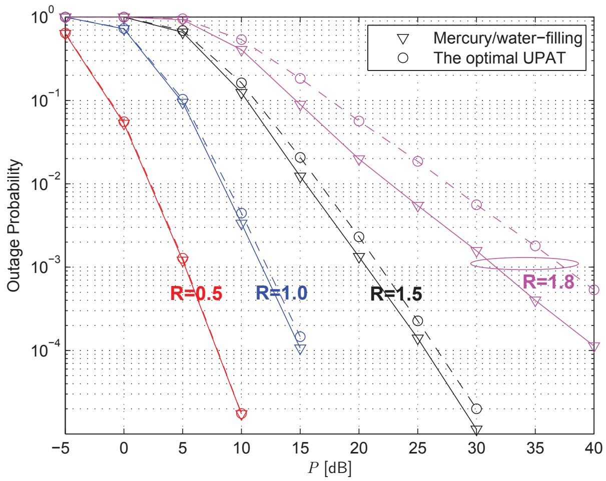

Fig. 3. Outage performance of the optimal power allocation (mercury/water-filling) and the optimal UPAT for different target transmission rates ($R = 0.5$, 1.0, 1.5, 1.8) when the number of blocks is 4 ($B = 4$) and $M = 4$ (QPSK).

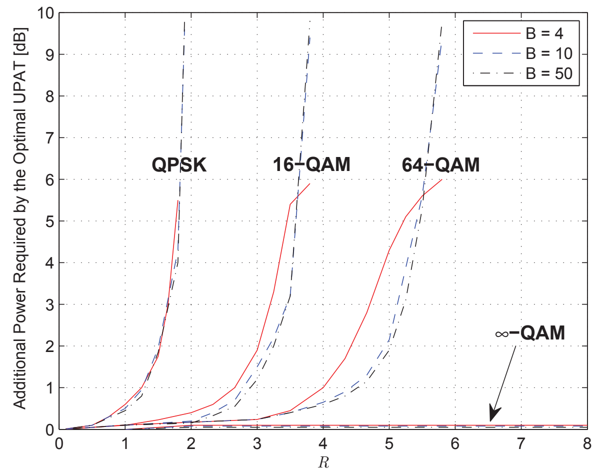

Fig. 4. UPAT suboptimality for various (M, R, B): each curve indicates additional P required for the optimal UPAT at outage rate 10^{-3} to achieve the same outage probability as mercury/water-filling.

instantaneous MI of the optimal UPAT is close to $\log_2 M$ and there is not much room to improve. However, even though ΔI_M is small, ΔP can be significant. This is because the optimal UPAT is similar to the UPA and therefore if P increases, a large portion of the increased power is assigned to the blocks that could provide per-block MI close to $\log_2 M$ without the increased power, which makes increasing P very inefficient in terms of increasing the instantaneous MI. Therefore, a large ΔP is needed to increase the instantaneous MI by a small amount.

IV. Near-Optimality of the Optimal UPAT

In this section, we discuss the outage performance loss from the optimal UPAT compared to MWF. We first show that the optimal UPAT performs close to MWF as long as the constellation size is sufficiently large. We then show that the constellation size should be sufficiently large for both MWF and the optimal UPAT in order not to suffer from a huge performance degradation.

A. The Optimal UPAT is Near-Optimal If $\log_2 M \gg R$

Fig. 3 shows the outage probability of MWF and the optimal UPAT when $M = 4$ (QPSK), $R = 0.5, 1.0, 1.5, 1.8$, and $B = 4$. We observe that when $R = 0.5$, the optimal UPAT performs close to MWF. But, as R increases, the performance loss from the optimal UPAT increases. In particular, the loss is significant when $R \approx \log_2 M$, e.g., at 10^{-3} outage rate, the loss is about 1 dB and 6 dB for $R = 1.5$ and $R = 1.8$, respectively.

The performance loss of the optimal UPAT for various values of M, R, and B is summarized in Fig. 4. Each curve in the figure indicates the additional P required for the optimal UPAT to achieve the same outage probability 10^{-3} as MWF. The results clearly show that the loss due to the optimal UPAT is marginal when $R \ll \log_2 M$, increases with R, and becomes significant when $R \approx \log_2 M$.

Intuitively, the above results can be explained as follows. By the definition of the outage probability (4), when $R \ll \log_2 M$, the outage events occur when $I_M^{\text{AW}}(\gamma_i) \ll \log_2 M$ for most of the blocks. As discussed in the previous section, for $\boldsymbol{\gamma}$ realizations such that $I_M^{\text{AW}}(\gamma_i) \ll \log_2 M$ for most of the blocks, MWF is not much different from waterfilling and the additional power required for the optimal UPAT to achieve the same instantaneous MI as MWF is not significant.

When $R \approx \log_2 M$, in contrast, the outage events occur unless $I_M^{\text{AW}}(\gamma_i) \approx \log_2 M$ for most of the blocks. For $\boldsymbol{\gamma}$ realizations such that $I_M^{\text{AW}}(\gamma_i) \approx \log M$ for most of the blocks, the optimal power allocation tends to be inversely proportional to γ_i, while the optimal UPAT tends to be the UPA. In this case, the additional power required for the optimal UPAT is significant, as discussed in the previous section.

An asymptotic behavior of these observations is proved by the following proposition[3].

Proposition 1:

$$\lim_{R\to 0} \frac{P_M^{\text{out}}(\boldsymbol{p}^{\text{UPAT}*}(\boldsymbol{\gamma}; M), P, R)}{P_M^{\text{out}}(\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma}; M), P, R)} = 1.$$

B. A System Should Operate in the Regime Where $\log_2 M \gg R$

The following property of the outage probability is important for subsequent discussion.

Lemma 1: $P_M^{\text{out}}(\boldsymbol{p}(\boldsymbol{\gamma}; M), P, R)$ is a decreasing function of the constellation size M.

Proof: Since $I_M^{\text{AW}}(\rho)$ is an increasing function of M [13], [14], $I_M(\boldsymbol{\gamma}, \boldsymbol{p}(\boldsymbol{\gamma}; M))$ is an increasing function of M. The proof follows by (4). ■

Lemma 1 implies that the outage probability with ∞-QAM inputs not only provides an analytical insight into the asymptotic behavior for large M but also serves as a lower bound for the outage probability of any finite M.

[3]The proof is omitted due to space limitations.

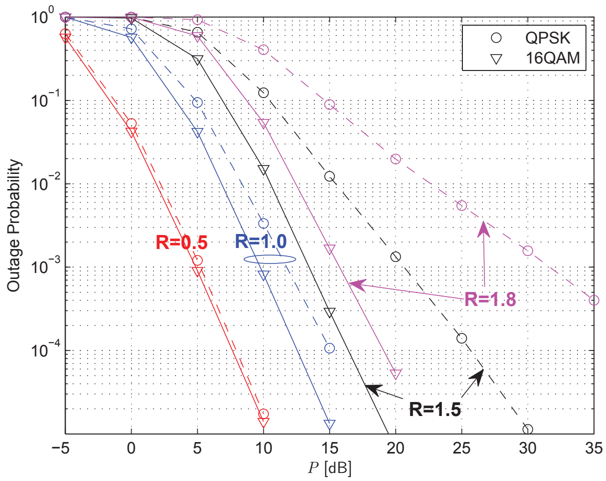

Fig. 5. Outage performance gain of 16-QAM over QPSK with the optimal power allocation (mercury/water-filling) when $B = 4$.

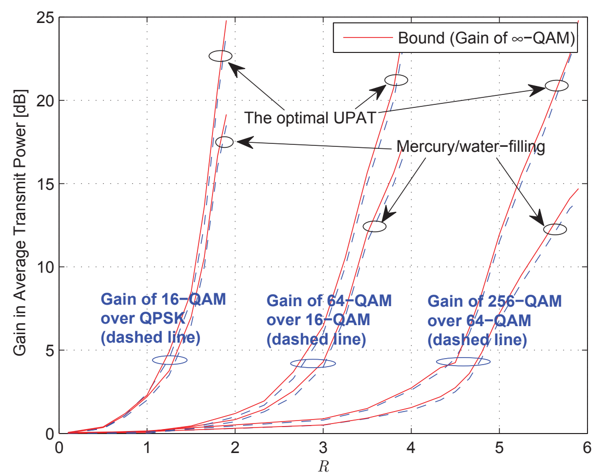

Fig. 6. Gain from a larger constellation size in average power P at 10^{-3} outage probability when $B = 4$.

Now, we show through numerical results that the condition $\log_2 M \gg R$ is a necessary prerequisite for a system to perform well. Fig. 5 shows the performance gain of 16-QAM over QPSK with MWF when $B = 4$. We observe that when $R = 0.5$, QPSK and 16-QAM exhibit a similar performance. But, as R increases, the gain of 16-QAM increases. In particular, when R is close to the maximum achievable rate of QPSK (i.e., $\log_2 M = 2$), the gain is significant, e.g., 16 dB gain at 10^{-3} outage probability when $R = 1.8$.

Fig. 6 summarizes the gain from using a larger M for various values of M and R when $B = 4$. The dashed line indicates the gain in P from using one-step larger M at 10^{-3} outage probability. The solid line indicates the gain of ∞-QAM, which bounds the gain from using any larger (finite) M. We note the following observations. If $\log_2 M \gg R$, the gain from using a larger M is marginal for both MWF and the optimal UPAT. In contrast, if $\log_2 M \approx R$, the performance of them can be significantly improved by even one-step larger M. In other words, both MWF and the optimal UPAT perform poorly in this regime, due to having too small M.

Intuitively, these results can be explained as follows. If $R \ll \log_2 M$, the outage events occur when $I_M^{\text{AW}}(\gamma_i) \ll \log_2 M$ for most of the blocks. For $\boldsymbol{\gamma}$ realizations such that $I_M^{\text{AW}}(\gamma_i) \ll \log_2 M$ for most of the blocks, increasing M does not significantly increase the instantaneous MI, since $I_M^{\text{AW}}(\gamma_i) \approx I_{M'}^{\text{AW}}(\gamma_i)$, $M < M'$, for most of the blocks. Therefore, increasing M does not significantly decrease the outage probability. In contrast, if $R \approx \log_2 M$, the outage events occur unless $I_M^{\text{AW}}(\gamma_i) \approx \log_2 M$ for most of the blocks. For $\boldsymbol{\gamma}$ realizations such that $I_M^{\text{AW}}(\gamma_i) \approx \log_2 M$ for most of the blocks, increasing M can significantly increase the instantaneous MI, since $I_M^{\text{AW}}(\gamma_i) \ll I_{M'}^{\text{AW}}(\gamma_i)$, $M < M'$, for most of the blocks. Therefore, increasing M can substantially decrease the outage probability.

C. Summary

If $\log_2 M \gg R$ the optimal UPAT performs close to MWF; otherwise, the suboptimality of the optimal UPAT can be significant. However, if $\log_2 M \approx R$, both MWF and the optimal UPAT perform poorly and therefore a system should avoid operating in this regime. In conclusion, for a given target transmission rate R, the optimal UPAT is near-optimal as long as the constellation size M is chosen appropriately not to limit the performance.

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