More on Time Delay

In
\[ \text{delay } T \]
Out
\[ e^{-sT} \]

model \( e^{-sT} \) by \( H(s) = \frac{9_s + 9.5 + 4.25^2 + \ldots}{b_0 + b_1s + b_2s^2 + \ldots} \)

we also want \( |H(s)| = 1 \Rightarrow \text{constant amplitude} \)

start with \( e^{-sT} = \frac{1 - ks}{1 + ks} \), what is \( k \)?

remember, \( e^{-sT} = 1 - sT + \left(\frac{sT}{2}\right)^2 - \ldots \) Taylor series

\[
\frac{1 - ks}{1 + ks} = 1 - sT + \left(\frac{sT}{2}\right)^2
\]

\[
1 - ks = (1 - sT + \left(\frac{sT}{2}\right)^2) \times (1 + ks)
\]

\[
1 - ks = 1 - sT + \frac{s^2T^2}{2} + ks - ks^2 + \frac{k^2T^2}{2}
\]

5 terms: \( 2k - T = 0 \Rightarrow k = \frac{T}{2} \)

5^2 terms: \( \frac{T^2}{2} - kT = 0 \Rightarrow k = \frac{T}{2} \)

5^3 terms: \( \frac{k^2T^2}{2} = 0 \Rightarrow k = 0 \) (don't use this result)

so \( k = \frac{T}{2} \) and \( e^{-sT} = \frac{1 - ks}{1 + ks} \) This is called 1st order Padé approximation

2nd order Padé approximation:

\[
e^{-sT} = \frac{1 - \frac{sT}{2} + \frac{s^2T^2}{12}}{1 + \frac{sT}{2} + \frac{s^2T^2}{12}}
\]

Apply this to feedback case

\[
\frac{\text{Out}}{\text{In}} = \frac{C(s)H(s)}{1 + C(s)H(s)T(s)k}
\]

\[
\begin{aligned}
\text{controller} & \\
\text{H(s)} & \\
\text{[+] & T(s)} & \\
\text{[+] & C(s)} & \\
\text{delay} & \\
\text{Out} & \\
\text{In} &
\end{aligned}
\]
Is the system still stable?

Poles of \( \frac{\text{out}}{\text{in}} \) must be in LHP. (Look for zeros of \( 1 + G \cdot H \cdot T \cdot K \))

\( 1 + G(5)H(5)T(5)K = 0 \) does the delay make the system unstable?

Using the car example from last lecture:

\[ G(5) = \frac{1}{m} \quad m = 1000 \text{ kg} \quad b = 50 \text{ Ns/m} \quad C = \frac{10^{-3}}{5 + 5 \times 10^{-2}} \]

\[ H(5) = h \quad \text{(to be determined)} \]

\( K = 50 \) (chosen to make the final velocity the same as the car alone)

\[ 1 + \frac{10^{-3}}{5 + 5 \times 10^{-2}} h \cdot 50 \cdot \frac{\frac{3}{2} \frac{e}{2} - s}{\frac{3}{2} \frac{e}{2} + s} = 0 \]

\( (5 + 5 \times 10^{-2})(\frac{3}{2} + s) + 5 \times 10^{-3} h (\frac{3}{2} - s) = 0 \)

\( s^2 + s (5 \times 10^{-2} + \frac{3}{2} - 5 \times 10^{-3} h) + (5 \times 10^{-2} \cdot \frac{3}{2} + 5 \times 10^{-3} h \frac{2}{e}) = 0 \)

\( s^2 + (\frac{3}{2} - (h-1)5 \times 10^{-2}) s + \frac{(h+1)10^{-1}}{2} = 0 \)

This is a 2nd order system

\[ s^2 + 2 \omega_0 s + \omega_0^2 = 0 \]

If \( \omega < 0 \), the system is unstable

\( \frac{2}{\omega} - (h-1)5 \times 10^{-2} > 0 \) for system to be stable

This puts limits on the gain of the system.

If \( h \) is too big, the system becomes unstable

In this case, \( h < 1 + \frac{2}{5 \times 10^{-2} \omega} \) for stability

If \( \omega = 1 \) sec, \( h < 41 \)
So far in this class we have covered:

- Filters, 2nd order behavior
- Behavior of 2nd order systems
- Feedback in circuits, and stability criterion
- General applications of feedback
  - make things more stable
  - make things more responsive
  - make things less stable (oscillators)

![Feedback Diagram]

Negative feedback usually to make something stable
Positive feedback usually to make something unstable

(but not all positive feedback results in an oscillator)

e.g. opamp:

\[ \text{Vout} = G/s (\text{Vin} + \text{Vout}) \]
\[ \text{Vout} (1 - G/s) = \text{Vin} G/s \]
\[ \frac{\text{Vout}}{\text{Vin}} = \frac{G}{s - G} \]

Pole at \( s = G \)

\[ G \times \infty \text{ unstable, but not an oscillator} \]

\[ \text{Laplace transform } \frac{1}{s-a} \rightarrow e^{at} \]

Output grows indefinitely with infinitesimal input (grows)

A good oscillator needs the poles to be right on the imaginary axis!
**Signal Flow Graphs**

- Elements are **nodes** and **branches**
- Nodes carry values (Vin, Vout, Vx, ...)
- Branches have transmittances (e.g., Vout = H Vin)
  - They are directed → there is an input and an output

\[
V_i \rightarrow [T] \rightarrow V_2 = TV_i \quad \text{or} \quad V_i \rightarrow [T] \rightarrow V_2 = TV_i
\]

- If two or more branches terminate in a node, the values sum

\[
V_1 \rightarrow [T_1] \rightarrow V_{out} = TV_1 + T_2 V_2
\]

**Explicit sum**

\[
V_1 \rightarrow [T_1] \rightarrow V_0 = TV_1 + T_2 V_2
\]

- Any number of branches leaving a node distribute the same value

\[
\frac{T_1}{T_2} \rightarrow T_1 V_1 \quad \frac{T_1}{T_2} \rightarrow T_2 V_1
\]

These are particularly useful for op-amp circuits because op-amps are good buffers (one-way circuits)

Consider a circuit like this:

![Circuit Diagram]

\[ Vin \rightarrow [G] V_{in} \rightarrow [\frac{1}{3} z] \rightarrow [\frac{1}{3} z] \rightarrow V_{out} \]
\( V_{in} - V_{out} - 2C V_{out} = s^2 V_{out} \)
\( V_{in} = V_{out} \left( 1 + 2C s + s^2 \right) \)

\[ \frac{V_{out}}{V_{in}} = \frac{1}{1 + 2C s + s^2} \rightarrow \text{low-pass filter} \]

We can also take other output points:

\( \frac{V_x}{V_{in}} = \frac{s^2 \omega^2}{1 + 2C s + s^2} : \text{HPF} \)
\( \frac{V_y}{V_{in}} = \frac{C \omega}{1 + 2C s + s^2} : \text{BP} \)
\( \frac{V_z}{V_{in}} = \frac{1}{1 + 2C s + s^2} : \text{LPF} \)

**General case: Biquad ("biquadrate")**

\( \frac{V_{out}}{V_{in}} = \frac{9 s^2 + 6 s \omega + \omega^2}{1 + 2C s + s^2} \)

Can build this with integrators and summing circuits.

\( \frac{V_{out}}{V_{in}} = -\frac{1}{sC} \)

\( \frac{V_{out}}{V_{in}} = -\frac{1}{sC} \)

\( V_{out} = -R_f \left( \frac{V_1 + V_2 + V_3}{R_1 + R_2 + R_3} \right) \)
This would give: \[
\frac{V_{out}}{V_{in}} = \frac{a + bs \omega + cs^2 \omega^2}{1 + 2c_1s \omega + s^2 \omega^2}
\]

We can also draw this as a signal flow graph:

Can also extend the biquad concept:

\[
\frac{V_{out}}{V_{in}} = \frac{1}{s^n + b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}
\]