Mason's Gain Formula

\[ G = \frac{\text{Out}}{\text{In}} = \frac{\sum_{k=1}^{N} G_k \Delta_k}{\Delta} \]

\[ \Delta = 1 - \sum L_i + \sum L_i L_j - \sum L_i L_j L_k + \ldots \]

\[ \Delta = \text{determinant of the graph} \]
\[ N = \text{number of forward paths between In and Out} \]
\[ G_k = \text{gain of kth forward path} \]
\[ L_i = \text{loop gain of each closed loop} \]
\[ L_i L_j = \text{product of loop gains of any non-touching loops} \]
\[ \Delta_k = \text{cofactor of } \Delta \text{ for the kth forward path} \]
\[ \text{with loops touching the kth forward path removed} \]

Example:

\[ \Delta = 1 - bg - dh - fi + bgdh + bgfi + dfhi - bgdfhi \]

Path 1: \( G_1 = \text{abcdef} \), \( \Delta_1 = 1 \) (touch all loops)
Path 2: \( G_2 = \text{sjdef} \), \( \Delta_2 = 1 - bg \) (this path does not touch loop bg)

\[ G = \frac{\text{abcdef} + \text{sjdef}(1-bg)}{\Delta} \]
Another example:

![Diagram](image)

- Three loops, all touching at the first node

\[
L_1 = -\sqrt{5} \\
L_2 = -b\sqrt{5}^2 \\
L_3 = -c\sqrt{5}^3
\]

One forward path, \( G_1 = \sqrt{5}^3 \), that touches all loops

\[
G = \frac{\sqrt{5}^3}{1 + b\sqrt{5}^2 + c\sqrt{5}^3} = \frac{1}{\sqrt{5}^3 + b\sqrt{5}^2 + c}
\]

We can add paths without adding complexity

**if the paths touch all loops \( \Delta_c = 1 \) for all \( c \)**

Here, all loops touch the first summing node

so all forward paths should go through that node.

![Diagram](image)

- Dual form:

\[
G = \frac{w + \sqrt{5} + \sqrt{5}^2 + \sqrt{5}^3}{1 + b\sqrt{5}^2 + c\sqrt{5}^3}
\]

\[
G = \frac{w\sqrt{5}^3 + b\sqrt{5}^2 + c\sqrt{5}^2}{\sqrt{5}^3 + b\sqrt{5}^2 + c}
\]

**Flow reversal theorem:**

Reverse direction of all branches and exchange In and Out, the transfer function is unchanged.
Example: Microwave Circuits

Traveling waves instead of simple voltage and current

\[ \begin{align*}
\begin{array}{c}
\frac{a_i}{b_i} \\
\frac{a_i}{b_i}
\end{array}
\end{align*} \quad \text{Microwave Network} \quad \begin{align*}
\frac{a_i}{b_i} & \rightarrow b_2 \\
\frac{a_i}{b_i} & \rightarrow b_2
\end{align*} \quad \begin{bmatrix}
\frac{b_1}{b_2}
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
\frac{a_i}{b_i}
\end{bmatrix}
\]

port 1 \quad \text{port 2}

\[ b_1 = S_{11} a_1 + S_{12} a_2 \]
\[ b_2 = S_{21} a_1 + S_{22} a_2 \]

Ideal amplifier:

\[ S_{11} = S_{22} = S_{12} = 0, \quad S_{21} = G \]

\[ \begin{align*}
\frac{a_i}{b_i} & \rightarrow b_2 \\
\frac{a_i}{b_i} & \rightarrow \frac{b_i}{c}
\end{align*} \]

General case:

Sources and loads have reflection coefficients:

\[ \begin{align*}
\Delta = 1 - S_{11} \Gamma_s - S_{22} \Gamma_L - S_{21} \Gamma_s S_{12} \Gamma_L \\
+ S_{11} \Gamma_s S_{22} \Gamma_L
\end{align*} \]

3 loops

\[ \frac{b_2}{b_5} = \frac{S_{21}}{1 - S_{11} \Gamma_s - S_{22} \Gamma_L - (S_{21} S_{12} - S_{11} S_{22}) \Gamma_L \Gamma_s} \]

one forward path

\[ \frac{b_1}{b_5} = \frac{S_{11} (1 - S_{22} \Gamma_L) + S_{21} \Gamma_s S_{12} \Gamma_L}{1 - S_{11} \Gamma_s - S_{22} \Gamma_L - (S_{21} S_{12} - S_{11} S_{22}) \Gamma_L \Gamma_s} \]

two non-touching loops
We can also easily include a length of transmission line on an existing circuit:

Application to Ladder Networks:

\[
\begin{align*}
V_2 &= \frac{I_1 - I_2}{sC_1}, \\
V_3 &= \frac{I_2 - I_3}{sC_2}, \\
\end{align*}
\]

Equations:

\[
\begin{align*}
I_1 &= \frac{V_2 - V_1}{sL_1}, \\
I_2 &= \frac{V_2 - V_3}{sL_2}, \\
I_3 &= \frac{V_3 - V_0}{sL_3}, \\
\end{align*}
\]

\[
\begin{align*}
L_1 &= \frac{s^2C_1}{L_1}, \quad L_2 = \frac{s^2}{L_2C_1}, \quad \ldots.
\end{align*}
\]
Only one forward path touching all loops

\[ G(s) = \frac{1}{\frac{1}{sL_1} + \frac{1}{sC_1} + \frac{1}{sL_2} + \frac{1}{sL_2} + \frac{1}{sL_3} + \frac{1}{sC_3}} = -L_1L_3L_5 \]

\[ \Delta = 1 - L_1L_2L_3 - L_4 - L_5 + 2L_3 + 2L_4 + 2L_5 \\
+ L_2L_4 + L_2L_5 + L_3L_5 - 2L_3L_5 \]

\[ H = \frac{-L_1L_3L_5}{\Delta} \]

\[ \text{Lab #6 System Analysis and Design} \]

1. Dual Integrator Analysis

- Find overshoot based on \( s \) (analytical)
- Find step response (numerical), blocks for \( \square, \square, \square \)

2. Dual Integrator Realization

- Try minimizing the number of op-amps by absorbing the sum function into the integrator where possible

3. Simulink is very good for nonlinear systems and mixed analog/digital systems, so it's good for oscillators

4. We discussed the effect of time delay but didn't try it.

New is your chance

5. Signal flow graph of ladder network

We can adjust the sign of variables to minimize the number of inverters
Op-amp implementation of LC ladder network
Called "Laplace ladder" circuit, used in IC filters
Realize the equations of a passive circuit so that the active circuit has the stability and sensitivity of passive circuit

Sensitivity = how accurately we must specify component values

Example: \[ H(s) = \frac{1}{1 + sCR + s^2LC} \]
\[ \omega_0 = \frac{1}{\sqrt{LC}}, \quad \frac{2\pi}{\omega_0} = CR \]
\[ \delta = \frac{1}{2} \sqrt{\frac{\delta_\omega}{\omega}} \]

\[ \frac{\delta\omega}{\omega_0} = \frac{\delta L}{L} + \frac{\delta C}{C} \]

\[ \text{fractional error in } \omega_0 \quad \text{fractional error in } L \quad \text{fractional error in } C \]

We want \( \delta L \) and all others \( \leq 1 \)

Calculate \( S \) using logs,
\[ \omega_0^2 = \frac{1}{LC} \rightarrow 2 \log \omega_0 = -\log L - \log C \]

Take \( \frac{\delta}{\omega_0} \) of whole equation
\[ \frac{2}{\omega_0} \frac{\delta\omega}{\delta L} = -\frac{1}{2} \]
\[ \frac{\delta\omega}{\omega_0} = -\frac{1}{2} \frac{\delta L}{L} \quad \text{so } \delta L = -\frac{1}{2} \]